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Contents

Contents of Volume I

Preface

The volume contains an exposition of the recent development of the two main themes of the book, Lipschitz extension problems for maps between metric spaces and smooth extension-trace problems for functions on closed subsets of $\mathbb{R}^n$. The basic facts used for their study are presented in Volume I while the current volume consists only of detailed motivations and formulations within the corresponding proofs. The reader may find in the introduction of each chapter below a detailed description of the material of that chapter. Here we restrict ourselves to a discussion of the main features of the two parts forming the volume.

The first of the aforementioned topics is presented in Part 3 consisting of Chapters 6-8. Chapter 6 is mainly devoted to the Lang-Schlichenmaier theory relating the existence of the corresponding Lipschitz extensions to two important concepts of Lipschitz topology, *Nagata dimension* and *Lipschitz connectedness*, see Section 4.2 of Volume I. This culminates with an explicit construction of Lipschitz extension operators acting between large classes of metric spaces. If the target space in question is Banach, the corresponding operator becomes linear, i.e., it provides the simultaneous Lipschitz extension for all subsets of the domain. However, the extension constants in the latter case are either unspecified or have only coarse estimates of how they depend on the basic parameters.

In Chapter 7, we present two other methods for the Lipschitz simultaneous extension problem which give the corresponding results with extension constants close to optimal. One of them, due to Lee and Naor, exploits a probabilistic argument that first appeared in a computer science context, see, e.g., Section 4.1 of Volume I. A second, due to the authors of this book, is based on geometric analysis methods; this method is constructive and covers an essentially wider class of spaces.

Finally, Chapter 8 studies different relations between linear and nonlinear Lipschitz extensions and the corresponding extension constants. In particular, we study the influence of snowflake metric transforms on the existence of Lipschitz extensions, present an explicit formula relating the linear Lipschitz extension constants with the corresponding nonlinear ones and construct examples of metric spaces with Lipschitz extension constants that are finite for the nonlinear case but infinite for the linear case.

Finally, Part 4 of Volume II consisting of Chapters 9 and 10 is devoted to the smooth extension problems for functions on closed subsets of $\mathbb{R}^n$.

The first section of Chapter 9 presents the Yu. Brudnyi–Shvartsman characterization of the trace spaces to closed subsets for Lipschitz spaces of higher order and for the associated jet spaces. The proofs are strongly based on concepts and methods of Local Approximation Theory, see Section 2.3 and Appendices F, G of Volume I for some basic facts of the theory. This approach used throughout the chapter gives solutions to similar problems for Morrey-Campanato and higher order Lipschitz spaces and for a wide class of closed subsets of $\mathbb{R}^n$. This class, in particular, includes Ahlfors s-regular sets with $s > n-1$, some self-similar fractals with separation conditions, see Section 4.2 of Volume I, and the closure of *uniform* (or $\varepsilon - \delta$) domains, see Section 2.4 of Volume I.

Chapter 10 discusses two extension-trace problems going back to the classical 1934 Whitney papers. The first one asks how to distinguish traces to a closed set of functions from the space $C^k(\mathbb{R}^n)$ or the likes from those of other continuous functions on this set (*Whitney's trace problem*). The solution may be essentially simplified if one reduces the problem to the case of subsets containing only a fixed number of points depending only on parameters of the space in question (e.g., subsets of at most $3 \cdot 2^{n-1}$ points for $C^{1,1}(\mathbb{R}^n)$). This reduction is the direct consequence of the general Yu. Brudnyi-Shvartsman *finiteness principle* proved up to now only in some special cases, see subsections 10.3.2, 10.4.1 and 10.5.1 of this chapter.

The second Whitney problem asks about the existence of a linear continuous extension operator for the trace of a smoothness space into the space itself. The problem is solved constructively in several cases, see subsections 10.2.4, 10.2.5, 10.4.2 and 10.5.2 of this chapter.

All the above mentioned results are carried out in accordance with the geometric analysis approach of this book combining Lipschitz selection theorems of Sections 5.4 and 5.5 of Volume I with the local approximation extension-trace criteria of Sections 9.3 and 9.4 of this volume.

A completely new approach to the Whitney problems for the spaces $C^\ell(\mathbb{R}^n)$ and $C^{\ell,\omega}(\mathbb{R}^n)$ was proposed by Ch. Fefferman in a series of papers starting in 2003 and continuing to the present. We present a detailed account of his breakthrough results with a rather sketchy description of his methods in Sections 10.3 and 10.4. A deeper insight into Fefferman's methods is, from our point of view, the most actual problem in this area. Clarifying these very complicated and lengthy proofs and the ideas behind them surely will lead to important progress in the area.

Basic Terms and Notation

Set-theoretic operations

$\in$ membership

$\cup$ union

$\cap$ intersection

$\setminus$ set theoretic difference

$\subset$ embedding (not necessarily proper)

$\oplus$ direct sum (also known as direct or cartesian product)

Sets and subsets

Let $T \subset S$ be sets.

- $T := \{x \in S\,;\, \mathcal{P}\}$: all elements of the subset T have the property $\mathcal{P}$

  Note the *figured brackets* and *semicolon* designed for the notation of sets. Hereafter the symbol $:=$ means that the left-hand side is defined or denoted by the right-hand side.

- $T^c := \{x \in S\,;\, x \notin T\}$: complement of T in S; it is usually clear from the context with respect to which larger set S the complement is taken

- 2^S: collection of subsets in S

- Let $\mathcal{F} := \{S_j\}_{j\in J} \subset 2^S$ be a *family* (indexed set). Then

$$\cup\,\mathcal{F} := \bigcup_{j\in J} S_j, \qquad \cap\,\mathcal{F} := \bigcap_{j\in J} S_j,$$

  $\sqcup\,\mathcal{F}$: disjoint union (($S_j \cap S_{j'} = \emptyset$ if $j \neq j'$ in this case)

- $\mathcal{F}$ is a *cover* of S if $\cup\mathcal{F} = S$

- a cover $\mathcal{F}'$ is a *refinement* of $\mathcal{F}$ if every $S' \in \mathcal{F}'$ is a subset of some $S \in \mathcal{F}$

Functions

Let S, S' be sets and $T \subset S$, $T' \subset S'$.

- $f : S \to S'$: function (map, transform) acting from S into S'
- $x \mapsto f(x)$: the alternative notation of f whenever S, S' are clear from the context
- $\operatorname{Im} f := \{f(x) \in S' \,;\, x \in S\}$: image (range) of f
- $f(S)$: the alternative notation of $\operatorname{Im} f$
- $f^{-1}(T') := \{x \in S \,;\, f(x) \in T'\}$: coimage of $T' \subset S'$
- $f^{-1} : \operatorname{Im} f \to 2^S$: inverse to f given by $x' \mapsto f^{-1}(\{x'\})$
- $f|_T : T \to S'$: trace (restriction) of f to $T \subset S$

Let $F : S \to 2^{S'} \setminus \{\emptyset\}$ be a set-valued (multivalued) map.

- $f : S \to S'$: *selection* of F if $f(x) \in F(x)$ for all x
- $\operatorname{card} : 2^S \to [0, +\infty]$: cardinality (number of points)
- $\operatorname{ord} \mathcal{F}$: *order (multiplicity)* of $\mathcal{F}$, i.e., $\operatorname{ord} \mathcal{F} := \sup\limits_{x \in S}\big(\operatorname{card}\{j \in J\} \,;\, S_j \ni x\}\big)$
- $\mathbf{1}_T : S \to \{0, 1\}$: indicator (characteristic function) of $T \subset S$

Numbers and related vector spaces

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$: natural, integer, rational and real numbers
- $\mathbb{Z}_+, \mathbb{R}_+$: nonnegative integers and real numbers
- (a, b), $[a, b]$: open and closed intervals with endpoints $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$
- $\{e_j\}_{1 \le j \le n}$: the standard basis of $\mathbb{R}^n$
- $\langle x, y \rangle := \sum\limits_{i=1}^{n} x_i y_i$: the standard scalar product (sometimes also denoted by $x \cdot y$)
- $x \mapsto \|x\|_p := \left\{\sum\limits_{i=1}^{n} |x_i|^p\right\}^{1/p}$: ℓ_p-norm (quasinorm if $0 < p < 1$)
- $\ell_p^n := (\mathbb{R}^n \,,\, \|\cdot\|_p)$

Subsets of $\mathbb{R}^n$

- $\mathbb{Z}^n := \{x \in \mathbb{R}^n \,;\, x_i \in \mathbb{Z} \text{ for all } i\}$
- $\mathbb{Z}_+^n := \{x \in \mathbb{Z}^n \,;\, x_i \ge 0 \text{ for all } i\}$

- α, β, γ: elements of $\mathbb{Z}_+^n$
- $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \,;\, \|x\|_2 = 1\}$: the unit sphere
- $(x, y), [x, y]$: open and closed intervals with endpoints $x, y \in \mathbb{R}^n$
- $Lin(\mathbb{R}^n)$, $\mathrm{Aff}(\mathbb{R}^n)$: the sets of linear and affine subspaces in $\mathbb{R}^n$
- $\mathcal{C}(\mathbb{R}^n)$: the set of nonempty bounded convex sets in $\mathbb{R}^n$
- hull: *linear hull* (span, envelope)
- aff: *affine hull*
- conv: *convex hull*
- $Q_r(x)$ (briefly Q, Q' etc.): closed cube (ℓ_∞^n ball) in $\mathbb{R}^n$ of center x and radius $r > 0$
- c_Q, r_Q: the center and radius of a cube Q
- $\mathcal{K}(\mathbb{R}^n)$: the set of closed cubes in $\mathbb{R}^n$
- $\mathcal{K}_S := \{Q_r(x) \in \mathcal{K}(\mathbb{R}^n)\,;\, x \in S \text{ and } 0 < r \leq 2\,\mathrm{diam}\, S\}$
- $\mathcal{W}_S$: Whitney cover of S^c for a closed subset $S \subset \mathbb{R}^n$

Polynomials, derivatives, differences

Let $\alpha, \beta \in \mathbb{Z}_+^n$.

- $x \mapsto x^\alpha := \prod_{i=1}^{n} x_i^{\alpha_i},\ x \in \mathbb{R}^n$: α-monomial (stipulation: $0^0 := 1$)
- $|\alpha| := \sum_{i=1}^{n} \alpha_i, \quad \alpha! := \prod_{i=1}^{n} \alpha_i!, \quad \binom{\alpha}{\beta} := \dfrac{\alpha!}{(\alpha-\beta)!\beta!}$
- $\mathcal{P}_{k,n}$: the space of polynomials in $x \in \mathbb{R}^n$ of degree k, the linear hull of α-monomials with $|\alpha| \leq k$
- $D_i := \dfrac{\partial}{\partial x_i},\ 1 \leq i \leq n$: the i-th partial derivative
- $D_x := \sum_{i=1}^{n} x_i D_i$: derivative in direction $x \in \mathbb{S}^{n-1}$
- $\nabla := (D_1, \dots, D_n)$: gradient
- $D^\alpha := \prod_{i=1}^{n} D_i^{\alpha_i}$: mixed α-derivative

Let f be k-times differentiable at $x \in \mathbb{R}^n$.

- $T_x^k f := \sum_{|\alpha| \le k} \frac{(\cdot - x)^\alpha}{\alpha!} D^\alpha f(x)$: Taylor's polynomial at x of degree k

Let $f : \mathbb{R}^n \to \mathbb{R}$ and $h \in \mathbb{R}^n$.

- $\tau_h f := f(\cdot + h)$: h-shift
- $\Delta_h^k := (\tau_h - 1)^k = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \tau_{jh}$: k-difference of step h
- $\Delta_h^\alpha := \prod_{i=1}^{n} \Delta_{h_i e_i}^{\alpha_i}$: (mixed) α-difference

Topological and metric spaces

Let S be a subset of a Hausdorff topological space.

- $\bar{S}$: closure, S^o: interior, $\partial S := \bar{S} \cap \bar{S^c}$: boundary
- $(\mathcal{M}, d)$: metric space with underlying set $\mathcal{M}$ and metric d (briefly, $\mathcal{M}$ whenever d is clear from the context)

  Throughout the book $\mathcal{M}$ is assumed to be nontrivial, i.e., $\operatorname{card} \mathcal{M} > 1$.
- m, m' etc.: points of $\mathcal{M}$
- $S \subset (\mathcal{M}, d)$ (briefly, $S \subset \mathcal{M}$): a metric subspace of $(\mathcal{M}, d)$
- $(\mathcal{M}, m_0, d)$: punctured metric space ($m_0 \in \mathcal{M}$)
- $S \subset (\mathcal{M}, m_0, d)$: a metric subspace of the punctured metric space (i.e., $m_0 \in S$)
- $B_r(m_0)$, $\bar{B}_r(m_0)$: open and closed balls in $\mathcal{M}$ of *center* x_0 and radius $r > 0$;

  $\bar{B}_r(m_0) := \{m \in \mathcal{M}\,;\, d(m, m_0) \le r\}$ does not, in general, coincide with the closure $\overline{B_r(m_0)}$ of $B_r(m_0) := \{m \in \mathcal{M}\,;\, d(m, m_0) < r\}$

Let S, S' be subsets of $(\mathcal{M}, d)$.

- $S_\varepsilon := \cup\{B_\varepsilon(m) \subset \mathcal{M}\,;\, m \in S\}$: ε-neghborhood of S
- $d(m, S) := \inf\{d(m, m')\,;\, m' \in S\}$: distance from m to S
- $Pr_S : \mathcal{M} \to 2^S$: metric projection onto S, i.e.,

$$Pr_S(m) := \{m' \in S\,;\, d(m, m') = d(m, S)\}$$

- $d(S, S') := \inf\{d(m, m')\,;\, (m, m') \in S \oplus S'\}$: distance between S and S'

- $d_{\mathcal{H}}(S,S') := \inf\{\varepsilon > 0\,;\, S \subset S'_\varepsilon,\, S' \subset S_\varepsilon\}$: Hausdorff distance between S and S'

Lipschitz functions

Let $f : (\mathcal{M}, d) \to (\mathcal{M}', d')$.

- $L(f; \mathcal{M}, \mathcal{M}') := \sup\limits_{m_1 \neq m_2} \dfrac{d'(f(m_1), f)m_2))}{d(m_1, m_2)}$: Lipschitz constant (briefly, $L(f)$)
- $|f|_{Lip(\mathcal{M},\mathcal{M}')}$: alternative notation for $L(f)$
- f is *C-Lipschitz*, if $L(f) \le C$ and f is *Lipschitz* if $L(f)$ is finite
- f is *C-bi-Lipschitz embedding*, if f^{-1} exists and its *distortion* $D(f)$ satisfies $D(f) := \max\{L(f), L(f^{-1})\} \le C$
- f is a *C-isometry* (isometry for $C = 1$) if f is a bijection with $D(f) \le C$
- f is a *bi-Lipschitz homeomorphism* if f is a C-isometry for some C

Continuous and Lipschitz spaces

Let $\mathcal{M}, \mathcal{M}'$ be metric spaces.

- $C(\mathcal{M})$: the space of real continuous functions
- $C_u(\mathcal{M})$: the space of real uniformly continuous functions
- $C_b(\mathcal{M})$: the space of real bounded continuous functions equipped with the uniform norm
- $Lip(\mathcal{M}, \mathcal{M}')$: the space of Lipschitz maps from $\mathcal{M}$ into $\mathcal{M}'$ equipped with the seminorm $f \mapsto L(f)$
- $Lip(\mathcal{M}) := Lip(\mathcal{M}, \mathbb{R})$
- $Lip(\mathcal{M}, m_0, \mathbb{R}^n) := \{f \in Lip(\mathcal{M}, \mathbb{R}^n)\,;\, f(m_0) = 0\}$ (briefly, $Lip_0(\mathcal{M}, \mathbb{R}^n)$ if the choice of m_0 is clear)

Let $G \subset \mathbb{R}^n$ be a *domain* (open connected set) and ω belongs to the class of *k-majorant* Ω_k:

- $t \mapsto \omega_k(t; f)_G$, $t > 0$: k-modulus of continuity of $f : G \to \mathbb{R}$ (the subindex G is omitted for $G = \mathbb{R}^n$)
- $\dot{\Lambda}^{k,\omega}(G)$: the "homogeneous" space of k-Lipschitz functions on G equipped with the seminorm $f \mapsto |f|_{\Lambda^{k,\omega}(G)} := \sup\limits_{t>0} \dfrac{\omega_k(t; f)_G}{\omega(t)}$
- $\Lambda^{k,\omega}(G) \subset \dot{\Lambda}^{k,\omega}(G)$: the "nonhomogeneous" space of k-Lipschitz functions on G equipped with the norm $f \mapsto \|f\|_{\Lambda^{k,\omega}(G)} := \sup\limits_G |f| + |f|_{\Lambda^{k,\omega}(G)}$

- $S \mapsto E_k(S; f)$, $S \subset \mathbb{R}^n$: (local) best approximation of f by polynomials of degree $k-1$
- $\dot{\mathcal{E}}^{k,\omega}(S)$: the space of real functions on $S \subset \mathbb{R}^n$ equipped with the seminorm $f \mapsto \sup \left\{ \dfrac{E_k(S \cap Q; f)}{\omega(r_Q)} \, ; \, Q \in \mathcal{K}_S \right\}$

Spaces of differentiable and smooth functions

- $C^\ell(G)$: the space of ℓ-times continuously differentiable real functions on a domain G
- $\dot{C}_b^\ell(G)$: the subspace of $C^\ell(G)$ defined by finiteness of the seminorm

$$f \mapsto |f|_{C_b^\ell(G)} := \max_{|\alpha|=\ell} \sup_G |D^\alpha f|$$

- $\dot{C}_u^\ell(G)$: the subspace of $\dot{C}_b^\ell(G)$ consisting of functions with uniformly continuous higher derivatives
- $C_b^\ell(G)$: the subspace of $\dot{C}_b^\ell(G)$ defined by finiteness of the norm

$$f \mapsto \|f\|_{C_b^\ell(G)} := \sup_G |f| + |f|_{C_b^\ell(G)}$$

- $C^\ell \dot{\Lambda}^{k,\omega}(G)$: the subspace of $C^\ell(G)$ consisting of functions whose higher derivatives belong to $\dot{\Lambda}^{k,\omega}(G)$ equipped by the seminorm

$$f \mapsto |f|_{C^\ell \Lambda^{k,\omega}(G)} := \max_{|\alpha|=\ell} |D^\alpha f|_{\Lambda^{k,\omega}(G)}$$

- $C^\ell \Lambda^{k,\omega}(G)$: the subspace of $C^k \dot{\Lambda}^{k,\omega}(G)$ defined by the finiteness of the norm

$$f \mapsto \|f\|_{C^\ell \Lambda^{k,\omega}(G)} := \sup_G |f| + |f|_{C^\ell \Lambda^{k,\omega}(G)}$$

- $J^\ell \dot{\Lambda}^{k,\omega}(G)$ the space of ℓ-jets $\vec{f} := \{f_\alpha\}_{|\alpha| \le \ell}$ on G defined by finiteness of the seminorm

$$\vec{f} \mapsto |\vec{f}|_{J^\ell \Lambda^{k,\omega}(G)} := \max_{|\alpha|=\ell} |f_\alpha|_{\Lambda^{k,\omega}(G)}$$

- $J^\ell \Lambda^{k,\omega}(G)$: the subspace of $J^\ell \dot{\Lambda}^{k,\omega}(G)$ defined by finiteness of the norm

$$\vec{f} \mapsto \|f\|_{J^\ell \lambda^{k,\omega}(G)} := \max_{|\alpha| \le \ell} |f_\alpha| + |\vec{f}|_{J^\ell \Lambda^{k,\omega}(G)}$$

Let X be one of the above introduced functions spaces on $\mathbb{R}^n$; let $S \subset \mathbb{R}^n$ be closed.

- $X|_S := \{f : S \to \mathbb{R}\,;\ f = g|_S$ for some $g \in X\}$: the trace of X to S equipped with the *trace seminorm*

$$f \mapsto |f|_{X|_S} := \inf\{|g|_X\ ;\ g|_S = f\}$$

  if X is seminormed and the analogous trace norm if X is normed

Extension constants

Let $\mathcal{M}, \mathcal{M}'$ be metric spaces and S be a metric subspace of $\mathcal{M}$.

- $L_{ext}(f)$: Lipschitz extension constant for $f : S \to \mathcal{M}'$ (the trace norm of f in $Lip(\mathcal{M}, \mathcal{M}')|_S$)
- $\Lambda(S, \mathcal{M}; \mathcal{M}') := \sup\left\{\dfrac{L_{ext}(f)}{L(f)}\ ;\ f \in Lip(S, \mathcal{M}')\right\}$: (local) Lipschitz extension constant
- $\Lambda(\mathcal{M}, \mathcal{M}') := \sup\{\Lambda(S, \mathcal{M}, \mathcal{M}')\,;\ S \subset \mathcal{M}\}$: (global) Lipschitz extension constant

Let X be a Banach space.

- $Ext(S, X)$: the space of all bounded linear extension operators (*simultaneous extensions*) from $Lip(S, X)$ into $Lip(\mathcal{M}, X)$
- $\lambda(S, \mathcal{M}; X) := \inf\{\|E\|\,;\ E \in Ext(S, X)\}$: (local) linear Lipschitz extension constant
- $\lambda(\mathcal{M}, X) := \sup\{\lambda(S, \mathcal{M}; X)\,;\ S \subset \mathcal{M}\}$: (global) linear Lipschitz extension constant
- $\lambda(\mathcal{M}) := \lambda(\mathcal{M}, \mathbb{R})$

Let X be one of the above introduced spaces of differentiable or smooth functions on $\mathbb{R}^n$ (*smoothness spaces*) and Σ be a class of subsets in $\mathbb{R}^n$.

- $\delta_N(f; S; X) := \sup\{|f|_{X|_{S'}}\,;\ S' \subset S$ and $\operatorname{card} S' \le N\}$: (the seminorm here and below is replaced by the norm if X is normed)
- $\mathcal{F}_\Sigma(X)$: the *finiteness constant* of X with respect to Σ (the minimal N for which $X|_S$ coincides with linear space $\{g \in C(S)\,;\ \delta_N(g; S; X) < \infty\}$)
- $\mathcal{FP}(\Sigma)$: the class of all smoothness spaces with $\mathcal{F}_\Sigma(X) < \infty$
- $\gamma_\Sigma(X)$: the extension constant for X with respect to Σ equals

$$\sup\{|g|_{X|_S}\,;\ \delta_N(g; S; X) \le 1,\ S \in \Sigma\}$$

- $\mathcal{FP}_u(\Sigma)$: the subclass of $\mathcal{FP}(\Sigma)$ with $\gamma_\Sigma(X) < \infty$

For Σ being the class of all nonempty closed subsets the symbol Σ in all these notations is omitted.

Part III

Lipschitz Extensions from Subsets of Metric Spaces

Chapter 6

Extensions of Lipschitz Maps

The main part of this chapter deals with the theory of Lang and Schlichenmaier [LSchl-2005] devoted to Lipschitz extensions of maps acting between metric spaces. The theory gives a unified approach to most previously established results of this kind which now become consequences of the main extension theorem established in [LSchl-2005]. All of these results are not sharp in the sense that the extensions do not preserve Lipschitz constants; in particular, the classical Kirszbraun and Valentine results cannot be proved in this way. Moreover, the estimates of Lipschitz extension constants in the main theorem contain unspecified quantities whose dependence on the basic parameters are either implicit or too rough. More precise estimates require, for every special case, new approaches and methods. Sparse results of this kind are discussed in the final part of the chapter.

The theory in question exploits two basic concepts, *Lipschitz connectedness* and *Whitney cover*. The former, a generalization of the well-known topological concept of n-connectedness, is discussed in detail in Section 6.1. It is shown that, for several classes of metric spaces including Banach, convex geodesic and Hadamard spaces and compact Riemannian manifolds, topological connectedness implies the corresponding Lipschitz one.

Whitney covers of open subsets of metric spaces, similarly to the classical case of open sets in $\mathbb{R}^n$, are the basic geometric tools for constructing Lipschitz extensions of maps whose target spaces are Lipschitz connected. It is proved in Section 6.2 that such covers exist for an open subset such that either its Nagata dimension or that of its complement is finite. Let us recall that we have already proved in Section 4.2 finiteness of the Nagata dimension for wide classes of metric spaces, e.g., for Gromov hyperbolic spaces of bounded geometry and for doubling metric spaces.

Using the results of the previous sections we then prove in Section 6.3 the main extension theorem for Lipschitz maps. Then, in Section 6.4, we derive from it several previously established extension results of this kind.

Finally, we prove in Section 6.5 rather precise two-sided estimates of Lipschitz

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constants for $\mathbb{H}^n$, discuss a sharp extension result for the Alexandrov spaces (the Lang–Schroeder theorem [LSch-1997]) and prove two such results for $\mathbb{R}$-trees and ultrametric spaces.

6.1 Lipschitz n-connectedness

Let us recall that a Hausdorff topological space is *n-connected*, $n \in \mathbb{Z}_+$, if it is arcwise connected and, for $n \geq 1$, each of its homotopy groups of order $1 \leq \ell \leq n$ is trivial.

For our aim, the following reformulation is more suitable.

Definition 6.1. A Hausdorff topological space $\mathcal{T}$ is said to be n-connected if, for each $\ell \in \{0, 1, \dots, n\}$, every continuous map from the unit sphere $\mathbb{S}^\ell \subset \mathbb{R}^{\ell+1}$ into $\mathcal{T}$ admits a continuous extension into the unit ball bounded by $\mathbb{S}^\ell$.

Note that 0-boundedness simply means arcwise connectedness, since $\mathbb{S}^0 = \{-1, 1\}$, and the extended map is a curve.

Remark 6.2. In this definition, balls and their boundaries may and will be replaced in the sequel by the unit ℓ-cubes $\mathbb{K}^\ell := [0,1]^\ell$ and their boundaries

$$\partial\mathbb{K}^\ell := \left\{x \in \mathbb{K}^\ell\,;\ \text{one of } x_i = 1 \text{ or } \right\}.$$

Example 6.3. (a) Since the fundamental (= first homotopy) group of the standard n-torus is nontrivial, the torus is 0-connected, and is not n-connected for $n \geq 1$.

(b) The unit sphere $\mathbb{S}^n$ is $(n-1)$-connected.

In fact, by Hurewicz's Theorem 1.9 of Volume I, every continuous map $\mathbb{S}^\ell \to \mathbb{S}^n$ for $\ell \leq n-1$ admits a continuous extension to the $(\ell+1)$-ball bounded by $\mathbb{S}^\ell$. On the other hand, for the identity map $\mathbb{S}^n \to \mathbb{S}^n$ there are no such extensions, see, e.g., Volume I, Section 2.6 where the Brouwer fixed point theorem is presented in this form.

A metric equivalent of the above notion is given by

Definition 6.4. A metric space $\mathcal{M}$ is Lipschitz n-connected if there is a constant $\mu \geq 1$ such that for all $\ell \in \{0, 1, \dots, n\}$ every c-Lipschitz map from $\partial\mathbb{K}^{\ell+1}$ into $\mathcal{M}$ admits a μc-Lipschitz extension to $\mathbb{K}^{\ell+1}$.

Clearly, every Lipschitz n-connected metric space is n-connected. In general, the converse is not true, but it holds for the classes of metric spaces presented below.

Proposition 6.5. *A Banach space is Lipschitz n-connected for every $n \geq 0$.*

Proof. Fix $\ell \in \mathbb{Z}_+$. Let f be a c-Lipschitz map from $\partial\mathbb{K}^{\ell+1}$ into a Banach space X. We use the Whitney extension operator E which for this case has the form

$$Ef = \begin{cases} f & \text{on } \partial\mathbb{K}^{\ell+1}, \\ \sum\limits_{Q} f(x_Q)\varphi_Q & \text{on } \mathbb{R}^{\ell+1}\backslash\partial\mathbb{K}^{\ell+1}, \end{cases}$$

see, e.g., Volume I, Section 2.2. Here Q runs over the Whitney cover of $(\partial\mathbb{K}^{\ell+1})^c$, and $\{\varphi_Q\}$ is a Lipschitz partition of unity subordinate to this cover, and x_Q is a point in $\partial\mathbb{K}^{\ell+1}$ closest to Q.

Since there is no difference in proofs for vector-valued and scalar cases, the special case of Whitney's theorem asserts that Ef is $\mu(\ell)c$-Lipschitz. □

The next important class of Lipschitz n-connected metric spaces for every $n \in \mathbb{Z}_+$ consists of a class of geodesic spaces with convex metrics introduced by Buseman [Bu-1955].

Definition 6.6. The metric of a geodesic metric space $(\mathcal{M}, d)$ is said to be convex if, for every pair of geodesics $\gamma_1, \gamma_2 : [0, 1] \to \mathcal{M}$ parametrized proportionally to their length, the following is true for all $t \in [0, 1]$:

$$d\big(\gamma_1(t), \gamma_2(t)\big) \le (1 - t)d\big(\gamma_1(0), \gamma_2(0)\big) + td\big(\gamma_1(1), \gamma_2(1)\big). \tag{6.1}$$

It is easily seen that a space subject to this definition is uniquely geodesic, i.e., every pair of its points is joined by a unique geodesic. In particular, the metric of a Banach space is convex in the Buseman sense if and only if this space is strictly convex [1].

Another important class of geodesic spaces with convex metrics consists of *Hadamard spaces* whose definition and properties we now briefly discuss.

Definition 6.7. (a) A geodesic metric space is called a CAT(0) space if it has nonpositive curvature in the Aleksandrov sense (see Volume I, Definition 3.109 for $\kappa = 0$).

(b) A complete CAT(0) space is called a Hadamard space.

The class of Hadamard spaces consists of many important geometric objects, in particular, Hilbert spaces, Hadamard manifolds (complete simply connected Riemannian manifolds of nonpositive sectional curvature) and a few complexes of piecewise constant curvature, e.g., Tits Euclidean and hyperbolic buildings, see [BH-1999, pp. 342–343] and references therein for definitions and properties of the latter objects.

The set of examples may be essentially enlarged using completeness of the class in question under several metric constructions described in subsection 3.1.8 of Volume I. In particular, the following fact, whose proof may be found, e.g., in [BH-1999, Chs. II.1 and II.11], is true.

[1] i.e., if the equality in the triangle inequality for its norm holds only for linearly dependent pairs of points. In particular, ℓ_p with $1 < p < \infty$ is strictly convex while ℓ_1 of ℓ_∞ are not.

Theorem 6.8. *Let $\mathcal{M}_i$ be a Hadamard space and let S_i be its convex subspace*[2]*, $i = 1, 2$. Then the following holds:*

(a) *The direct sum $\mathcal{M}_1 \oplus^{(2)} \mathcal{M}_2$ is Hadamard;*

(b) *Let S be a complete metric space isometric to each S_i, $i = 1, 2$. Then the gluing $\mathcal{M}_1 \bigsqcup_S \mathcal{M}_2$ is Hadamard.*

For the definition of the latter construction, see, e.g., in Volume I, Example 3.54 (d).

Now we explain why a CAT(0) space has a convex metric.

Proposition 6.9. *If $(\mathcal{M}, d)$ is a CAT(0) space, then its metric is convex.*

Proof. Let $\gamma_i : [0, 1] \to \mathcal{M}$ be a geodesic parametrized by arc length, $i = 0, 1$. Assume that $\gamma_0(0) = \gamma_1(0)$ and consider the geodesic triangle Δ with vertices $\gamma_0(0), \gamma_0(1), \gamma_1(1)$. Let $\overline{\Delta}$ be the comparison triangle for Δ in the Euclidean plane, whose vertices corresponding to those of Δ we denote by $\overline{\gamma_0(0)}$, $\overline{\gamma_0(1)}$ and $\overline{\gamma_1(1)}$. By the definition of $\overline{\Delta}$, its sidelengths are equal to the corresponding sidelengths of Δ.

By an elementary geometric consideration, we have for points $\overline{\gamma_i(t)} \in \overline{\Delta}$ corresponding to points $\gamma_i(t)$ of Δ, $0 \le t \le 1$,

$$\left\|\overline{\gamma_0(t)} - \overline{\gamma_1(t)}\right\| = t\left\|\overline{\gamma_0(1)} - \overline{\gamma_1(1)}\right\| = td\big(\gamma_0(1), \gamma_1(1)\big).$$

Now we use the inequality (A_0^+) of Definition 3.109 of Volume I for spaces of nonpositive curvature to get

$$d\big(\gamma_0(t), \gamma_1(t)\big) \le \left\|\overline{\gamma_0(t)} - \overline{\gamma_1(t)}\right\|.$$

Combining this inequality with the previous equality we obtain the inequality $d\big(\gamma_0(t), \gamma_1(t)\big) \le td\big(\gamma_0(1), \gamma_1(1)\big)$ which proves convexity inequality (6.1) for this special case.

In the general case, we introduce an arc length parametrized geodesic $\gamma_2 : [0, 1] \to \mathcal{M}$ joining $\gamma_0(0)$ and $\gamma_1(1)$. By applying the previous case to γ_0 and γ_2 and then to γ_2 and γ_1, in the reverse order we obtain

$$\begin{aligned} d(\big(\gamma_0(t), \gamma_1(t)\big) &\le d\big(\gamma_0(t), \gamma_2(t)\big) + d\big(\gamma_2(t), \gamma_1(t)\big) \\ &\le td\big(\gamma_0(1), \gamma_2(1)\big) + (1-t)d\big(\gamma_2(0), \gamma_1(0)\big) \\ &= td\big(\gamma_0(1), \gamma_1(1)\big) + (1-t)d\big(\gamma_0(0), \gamma_1(0)\big), \end{aligned}$$

as required. □

Now we establish the desired property of Hadamard spaces.

Proposition 6.10. *A Hadamard space is Lipschitz n-connected for each $n \ge 0$.*

[2] i.e., every pair of its points is joined by a unique geodesic whose image lies in the subspace.

Proof. Let f be a c-Lipschitz map from the boundary of the cube $\mathbb{K}^{n+1}$ into a Hadamard space $(\mathcal{M}, d)$. Set $D := \operatorname{diam} f(\partial\mathbb{K}^{n+1})$. By the Lipschitz condition,

$$D \le c \operatorname{diam} \partial\mathbb{K}^{n+1} = c\sqrt{n+1}. \tag{6.2}$$

Fix a point $m^* \in f(\partial\mathbb{K}^{n+1})$. For each $x \in \partial\mathbb{K}^{n+1}$, by $\gamma_x : [0,1] \to \mathcal{M}$ we denote a unique geodesic parametrized by arc length that joins m^* with $f(x)$. Now define an extension $\bar{f}$ of f to $\mathbb{K}^{n+1}$ by setting, for a point $o + t(x - o) \in \mathbb{K}^{n+1}$, where $o := \left(\frac{1}{2}, \dots, \frac{1}{2}\right) \in \mathbb{K}^{n+1}$, $x \in \partial\mathbb{K}^{n+1}$ and $t \in [0,1]$,

$$\bar{f}(o + t(x - o)) := \gamma_x(t), \quad 0 \le t \le 1.$$

Then we have

$$\begin{aligned} d\big(\bar{f}(o + t(x - o)), \bar{f}(o + t(x' - o))\big) &= d\big(\gamma_x(t), \gamma_{x'}(t)\big) \le td\big(\gamma_x(1), \gamma_{x'}(1)\big) \\ &= td(f(x), f(x')) \le ct\|x - x'\|. \end{aligned}$$

Also, by definition,

$$d\big(\bar{f}(o + t(x - o)), \bar{f}(o + t'(x - o))\big) = |t - t'| \cdot d(m^*, f(x)) \le D \cdot |t - t'|.$$

Together with the previous inequality and (6.2) this gives

$$\begin{aligned} &d\big(\bar{f}(o + t(x - o)), \bar{f}(o + t'(x' - o))\big) \le c\sqrt{n+1} \cdot |t - t'| + ct\|x - x'\| \\ &\le \frac{3c\sqrt{n+1}}{2} \cdot |t - t'| + c\|(o + t(x - o)) - (o + t'(x' - o))\| \\ &\le c(3\sqrt{n+1} + 1)\|(o + t(x - o)) - (o + t'(x' - o))\|. \end{aligned}$$

Hence, $\bar{f} : \mathbb{K}^{n+1} \to \mathcal{M}$ is the required $\mu(n)c$-Lipschitz extension with $\mu(n) := 3\sqrt{n+1} + 1$. □

Remark 6.11. The previous result is a corollary of the following general fact.

Let $(\mathcal{M}, d)$ be a metric space with a weakly convex bicombing on $\mathcal{M}$. Let us recall, see Comments to Chapter 5 of Volume I, that $\mathcal{M}$ meets this condition if for every pair of points $m, m' \in \mathcal{M}$ there exists a geodesics $\gamma_{mm'} : [0,1] \to \mathcal{M}$ parametrized by its arclength connecting m and m' such that for every $m \in \mathcal{M}$

$$d(\gamma_{mm'}(t), \gamma_{mm''}(t)) \le Ctd(m', m'')$$

for some $C \ge 1$ and all $t \in [0,1]$. The proof of Proposition 6.10 is easily adapted to the $(\mathcal{M}, d)$ to give the following result.

Every metric space $(\mathcal{M}, d)$ with weakly convex bicombing on $\mathcal{M}$ is Lipschitz n-connected for every $n \ge 0$.

Clearly, a geodesic metric space with convex metric in the Buseman sense (in particular, Banach and Hadamard spaces) satisfies the condition of the above assertion; one more example gives the class of $\mathbb{R}$-trees, see [LaPl-2001].

Finally, we introduce the third class of Lipschitz n-connected spaces. In the formulation of the corresponding result we use a (weaker) version of the concept of spaces of bounded geometry, see Volume I, Definition 3.98. Namely, we denote by $\mathcal{G}_X(R,D)$ the class of metric spaces $(\mathcal{M},d)$ with the following property.

> *Every open ball in $\mathcal{M}$ of radius R is bi-Lipschitz homeomorphic to the unit ball of a finite-dimensional Banach space X with distortion at most D.*

The following result is, in essence, due to Lang–Schlichenmaier [LSchl-2005, Thm. 5.1].

Theorem 6.12. *Assume that a metric space is compact and belongs to $\mathcal{G}_X(R,D)$. If $\mathcal{M}$ is n-connected, then it is also Lipschitz n-connected.*

Proof. Fix an integer $\ell \in \{0,1,\dots,n\}$. We must prove that a c-Lipschitz map $f : \partial\mathbb{K}^{\ell+1} \to \mathcal{M}$ admits a μc-Lipschitz extension where a constant $\mu \geq 1$ is independent of c and f. This will be done in four steps.

We first prove that the required extension exists for c-Lipschitz maps with small c, see Lemma 6.13 below. At the second stage presented by Lemma 6.15, we divide the cube $\mathbb{K}^{\ell+1}$ into small subcubes and use the pieces of the extended Lipschitz maps on these subcubes provided by Lemma 6.13 to extend a 1-Lipschitz map $f : \partial\mathbb{K}^{\ell+1} \to \mathcal{M}$ to a Lipschitz map $\bar{f} : \mathbb{K}^{\ell+1} \to \mathcal{M}$.

Exploiting then a compactness argument we show that the Lipschitz constant for this $\bar{f}$ is bounded by a constant independent of f. Finally, we derive from here that every c-Lipschitz map $f : \partial\mathbb{K}^{\ell+1} \to \mathcal{M}$ admits the required μc-Lipschitz extension with a constant $\mu \geq 1$ independent of f and c.

Lemma 6.13. *Let $\mathcal{M} \in \mathcal{G}_X(R,D)$. Then $\mathcal{M}$ is Lipschitz n-connected in small, i.e., there exist constants $\delta > 0$ and $\mu \geq 1$ such that, for every $\ell \in \{0,1,\dots,n\}$, every c-Lipschitz map $f : \partial\mathbb{K}^{\ell+1} \to \mathcal{M}$ with $c < \delta$ admits a μc-Lipschitz extension $\bar{f}$ to $\mathbb{K}^{\ell+1}$.*

Proof. Set $\delta := \frac{1}{\sqrt{n+1}} \min\{R, \frac{1}{\mu D}\}$ where the constant $\mu = \mu(n,X) > 1$ will be introduced below. Then the image $\operatorname{Im}(f) := f(\partial\mathbb{K}^{\ell+1})$ with $c < \delta$ satisfies

$$\operatorname{diam} \operatorname{Im}(f) \leq c\sqrt{n+1} < R,$$

and therefore it lies in some ball $B_R(m) \subset \mathcal{M}$. Due to the definition of $\mathcal{G}_X(R,D)$, there exists a bi-Lipschitz homeomorphism I of $B_R(m)$ onto an open ball $B_X \subset X$ centered at the origin and of radius $1 \leq r \leq D$ such that $I(m) = 0$ and for all $m', m'' \in \mathcal{M}$,

$$d(m',m'') \leq \big\|I(m') - I(m'')\big\|_X \leq Dd(m',m''). \tag{6.3}$$

Hence the map $I \circ f : \partial\mathbb{K}^{\ell+1} \to X$ is cD-Lipschitz. Applying Proposition 6.5 we then conclude that $I \circ f$ admits a $\mu(n,X)cD$-Lipschitz extension $\bar{f} : \mathbb{K}^{\ell+1} \to X$.

The image of $\bar{f}$ therefore satisfies

$$\operatorname{diam}\operatorname{Im}(\bar{f}) \le \mu(n,X)cD\sqrt{n+1} =: \gamma,$$

and so $\operatorname{Im}(\bar{f}) \subset \gamma B_X$. Because of the choice of δ,

$$\gamma < \mu(n,X)\delta D\sqrt{n+1} \le 1,$$

and therefore $\bar{f}\circ I^{-1}$ is well defined and maps $\mathbb{K}^{\ell+1}$ into $\mathcal{M}$. By (6.3), the Lipschitz constant of this map is at most μc, where $\mu := \mu(n,X)D$.

Hence $\bar{f}\circ I^{-1}$ is the required extension of f. □

Corollary 6.14. *Let f be a c-Lipschitz map from the boundary ∂Q of a cube $Q := [a,b]^{\ell+1} \subset \mathbb{R}^{\ell+1}$ into the space $\mathcal{M}$. Assume that*

$$c < \delta(b-a).$$

Then there exists a μc-Lipschitz extension of f to Q.

Proof. Let A be an affine transform of $\mathbb{R}^{\ell+1}$ sending $\mathbb{K}^{\ell+1} = [0,1]^{\ell+1}$ onto Q. Then $f\circ H$ is a $c(b-a)$-Lipschitz map of $\partial\mathbb{K}^{\ell+1}$ into $\mathcal{M}$. By the choice of c this map admits a $\mu c(b-a)$-Lipschitz extension $\bar{f}$ to $\mathbb{K}^{\ell+1}$. Then the map $\bar{f}\circ H$ is a μc-Lipschitz extension of f to Q. □

Lemma 6.15. *A Lipschitz map $f : \partial\mathbb{K}^{\ell+1} \to \mathcal{M}$ admits a Lipschitz extension to $\mathbb{K}^{\ell+1}$.*

Proof. Let $f : \partial\mathbb{K}^{\ell+1} \to \mathcal{M}$ be c-Lipschitz. Since $\mathcal{M}$ is n-connected, there exists a *continuous* extension $f : \mathbb{K}^{\ell+1} \to \mathcal{M}$ of f. But a continuous map on a compact metric space is uniformly continuous. Hence, given $\varepsilon > 0$, there exists an integer $N \ge 2$ such that

$$d\big(g(x), g(x')\big) < \varepsilon, \tag{6.4}$$

provided that $|x - x'| < N^{-1}$. Here $|x|$ is the ℓ_∞-norm of $x \in \mathbb{R}^{\ell+1}$.

We may also assume that

$$c := L(f) \le \varepsilon N. \tag{6.5}$$

Now we equip the cube $\mathbb{K}^{\ell+1}$ with the structure of a Euclidean polyhedral complex using the subdivision of the cube into $N^{\ell+1}$ congruent subcubes. We denote by Σ_k, $0 \le k \le \ell+1$, the k-skeleton of this complex, i.e., the collection of all k-cells (k-faces) of the subcubes of the subdivision. In particular, the 0-skeleton is the uniform lattice in the cube, i.e., $\Sigma_0 = \mathbb{K}^{\ell+1} \cap (N^{-1}\mathbb{Z})^{\ell+1}$, and the $(\ell+1)$-selection is the collection of all subcubes of the subdivision.

We extend the map f to the inner points of Σ_0 by setting

$$f^{(0)} := g\big|_{\Sigma_0 \cup \partial\mathbb{K}^{\ell+1}}.$$

Hence, $f^{(0)} = f$ on the set $\Sigma_0 \cap \partial\mathbb{K}^{\ell+1}$ and, for adjacent points x, x' of $\Sigma_0 \backslash \partial\mathbb{K}^{\ell+1}$,

$$d\big(f^{(0)}(x), f^{(0)}(x')\big) = d\big(g(x), g(x')\big) < \varepsilon = \varepsilon N|x - x'|,$$

see (6.4).

Since f is c-Lipschitz and c satisfies (6.5), this inequality is also true for adjacent points in $\Sigma_0 \cap \partial\mathbb{K}^{\ell+1}$.

Hence, the map $f^{(0)} : \Sigma_0 \to \mathcal{M}$ is εN-Lipschitz (in ℓ_∞-metric).

Starting with $f^{(0)}$ we successively extend f to the k-skeletons Σ_k for $k = 1, 2, \dots, \ell + 1$ preserving the Lipschitz condition.

Suppose that $f^{(k)} : \Sigma_k \cup \partial\mathbb{K}^{\ell+1} \to \mathcal{M}$ is such that

(a) $f^{(k)} = f$ on $\Sigma_k \cap \partial\mathbb{K}^{\ell+1}$;

(b) for every $(k + 1)$-cell Q of the complex that is not contained in $\partial\mathbb{K}^{\ell+1}$ the trace $f^{(k)}\big|_{\partial Q}$ admits a Lipschitz extension to Q with the constant at most $(2\mu)^k\varepsilon N$ (with respect to ℓ_∞-metric on Q).

Note that the previously introduced map $f^{(0)}$ satisfies this condition with $k = 0$.

Now we define $f^{(k+1)} : \Sigma_{k+1} \cup \partial\mathbb{K}^{\ell+1} \to \mathcal{M}$ as follows. First let Q be a $(k + 1)$-cell that does not intersect $\partial\mathbb{K}^{\ell+1}$. Choose ε to be so small that

$$(2\mu)^k\epsilon < \delta.$$

Then by Corollary 6.14 and the induction hypothesis, the map $f^{(k)}\big|_{\partial Q}$ admits a $\mu(2\mu)^k\varepsilon N$-Lipschitz extension to Q. Then we set $f^{(k+1)}\big|_Q$ to be equal to this extension. Further, we define $f^{(k+1)}$ on one of the remaining $(k + 1)$-cells Q; each of these Q intersects $\partial\mathbb{K}^{\ell+1}$ by a facet. Since $f^{(k)}$ is c-Lipschitz on such a facet and is $(2\mu)^k\varepsilon N$-Lipschitz on other facets of Q, and, moreover, c satisfies (6.5), $f^{(k)}$ is Lipschitz with constant $\varepsilon N + (2\mu)^k\varepsilon N \le 2(2\mu)^k\varepsilon N$.

Choosing $\varepsilon > 0$ so that this Lipschitz constant becomes less than δN^{-1}, and using Corollary 6.14, we extend $f^{(k)}$ to Q with the Lipschitz constant at most $2\mu(2\mu)^k\varepsilon N$. Then we define $f^{(k+1)}|_Q$ to be equal to this extension. This completes the induction.

Now $f^{(\ell+1)}$ maps $\Sigma_{\ell+1} \cup \partial\mathbb{K}^{\ell+1} = \mathbb{K}^{\ell+1}$ into $\mathcal{M}$, coincides with f on $\partial\mathbb{K}^{\ell+1}$ and satisfies $(2\mu)^{\ell+1}\varepsilon N$-Lipschitz condition on every $(\ell + 1)$-cell. Hence, $f^{(\ell+1)}$ is a $(2\mu)^{\ell+1}\varepsilon N^2$-Lipschitz extension of f, as required. □

For the next step we need a variant of Lemma 6.13 with the parallelotope $\Pi_N := \left[0, \frac{1}{N}\right] \times \partial\mathbb{K}^{\ell+1} \subset \mathbb{R}^{\ell+1}$ substituted for the cube $\mathbb{K}^{\ell+1}$.

Lemma 6.16. *Let h be a c-Lipschitz map from the set $\{0, \frac{1}{N}\} \times \partial\mathbb{K}^{\ell+1} \subset \Pi_N$, $N \in \mathbb{N} \setminus \{1\}$, into $\mathcal{M}$. Assume that $\epsilon > 0$ is such that*

$$\begin{aligned} &c \le N\epsilon, \quad (2\mu)^\ell\epsilon < \delta \quad \text{and} \\ &d\Big(h(0, x), h\Big(\frac{1}{N}, x\Big)\Big) < \varepsilon \end{aligned} \tag{6.6}$$

for every pair of points $(0, x)$, $\left(\frac{1}{N}, x\right)$ *of* $\{0, \frac{1}{N}\} \times \partial\mathbb{K}^{\ell+1}$.

Then f *admits a* $(4\mu)^{\ell+1}\varepsilon N$*-Lipschitz extension to* Π_N.

Proof. Introduce a polydehral complex generated by the subdivision of Π_N into $2^{\ell+1}N^{\ell}$ congruent cubes, and denote by $\Sigma_k(\Pi_N)$ the k-skeleton of this complex. By (6.6), the restriction of h to $\Sigma_0(\Pi_N)$ is εN-Lipschitz. Beginning with $h^{(0)}$ and repeating the argument of Lemma 6.15, we obtain the family $\{h^{(k)}\}_{0\le k\le \ell+1}$ of maps such that

(a) $h^{(k)}$ maps $\Sigma_k(\Pi_N)$ into $\mathcal{M}$;

(b) $h^{(k)} = f$ on the set $\left(\{0, \frac{1}{N}\} \times \partial\mathbb{K}^{\ell+1}\right) \cap \Sigma_k(\Pi_N)$;

(c) $h^{(k)}$ is $(2\mu)^k\varepsilon N$-Lipschitz on subcubes of the subdivision that are not contained in $\partial\Pi_N$.

Then $h^{(\ell+1)}$ is clearly the required extension of h. □

The third step of the proof is contained in

Lemma 6.17. *A* 1*-Lipschitz map* $f : \partial\mathbb{K}^{\ell+1} \to \mathcal{M}$ *admits a* c*-Lipschitz extension to* $\mathbb{K}^{\ell+1}$ *with a constant* c *independent of* f.

Proof. Suppose, on the contrary, that for every $i \in \mathbb{N}$ there exists a 1-Lipschitz map $f_i : \partial\mathbb{K}^{\ell+1} \to \mathcal{M}$ such that each of its extensions $\bar{f}_i$ to $\mathbb{K}^{\ell+1}$ has the Lipschitz constant $L(\bar{f}_i) \ge i$. Since the domain and the target space are compact, the sequence $\{f_i\}$ is, by the Arcela–Ascoli theorem, a precompact subset of the space $C(\partial\mathbb{K}^{\ell+1}; \mathcal{M})$. Hence, we may and will assume that $\{f_i\}$ uniformly converges to a 1-Lipschitz map $f : \partial\mathbb{K}^{\ell+1} \to \mathcal{M}$. By $\bar{f} : \mathbb{K}^{\ell+1} \to \mathcal{M}$ we denote a Lipschitz extension of f which exists by Lemma 6.15.

Given $\varepsilon > 0$, choose an integer $N \in \mathbb{N} \setminus \{1\}$ such that

$$\sup_{x\in\partial\mathbb{K}^{\ell+1}} d\big(f(x), f_i(x)\big) < \varepsilon \quad \text{for all } i \ge N.$$

Diminishing ϵ and increasing N, if necessary, we assume that they satisfy inequalities (6.6).

Apply now Lemma 6.16 to a function h_i which is equal to f_i on $\{0\} \times \partial\mathbb{K}^{\ell+1}$ and to f on $\{\frac{1}{N}\} \times \partial\mathbb{K}^{\ell+1}$. Then the second inequality of (6.6) holds for h_i and, by the above mentioned lemma, h_i admits a c-Lipschitz extension $\hat{h}_i : \Pi_N \to \mathcal{M}$ with a constant c independent of i.

We use $\hat{h}_i$ to complete the proof as follows.

Let ϕ_N be a bi-Lipschitz embedding of $\mathbb{K}^{\ell+1}$ into $\prod_N \cup \left(\{\frac{1}{N}\} \times \mathbb{K}^{\ell+1}\right)$ with distortion bounded by some $c = c(\ell) > 1$ such that

$$\phi_N\big(\partial\mathbb{K}^{\ell+1}\big) = \{0\} \times \partial\mathbb{K}^{\ell+1}.$$

The restriction of ϕ_N to $\partial\mathbb{K}^{\ell+1}$ (identified with $\{0\} \times \partial\mathbb{K}^{\ell+1}$) denoted by ψ_N is a bi-Lipschitz map of $\partial\mathbb{K}^{\ell+1}$ onto $\partial\mathbb{K}^{\ell+1}$. Then there exists an extension of the converse map ψ_N^{-1} to $\mathbb{K}^{\ell+1}$ denoted by g_N whose Lipschitz constant depends only on ℓ. (Such g_N can be easily constructed using a Lipschitz retraction of a neighborhood of $\partial\mathbb{K}^{\ell+1}$ onto $\partial\mathbb{K}^{\ell+1}$ and the corresponding smooth cut-off function with support in this neighborhood.)

Now let $\bar{f}_i$ be equal to $\hat{h}_i$ on Π_N and $\bar{f}$ on $\{\frac{1}{N}\}\times\mathbb{K}^{\ell+1}$. Then $\bar{f}_i$ is a Lipschitz function on $\Pi_N \cup \left(\{\frac{1}{N}\}\times\partial\mathbb{K}^{\ell+1}\right)$ with Lipschitz constant bounded by a number independent of i. Consider the Lipschitz function $\hat{f}_i := \bar{f}_i \circ \phi_N \circ g_N$. By our construction $\hat{f}_i|_{\partial\mathbb{K}^{\ell+1}} = f_i$ and the Lipschitz constant of $\hat{f}_i$ is bounded by a number independent of i.

This contradiction proves the result. □

For the final step of the proof we assume, without loss of generality, that $\operatorname{diam}\mathcal{M} \le 1$. Let $I \ge 2$ be an integer and set $Q_I := [0,I]^{\ell+1}$. Suppose that $f : \partial Q_I \to \mathcal{M}$ is 1-Lipschitz.

We equip Q_I with the structure of a polyhedral complex whose $(\ell+1)$-cells are cubes of side length 1. The k-skeleton of this complex we denote by $\Sigma_k(Q_I)$. Then we choose an arbitrary extension of the trace $f\big|_{\partial Q_I}$ to the inner points of the lattice $\Sigma_0(Q_I)$. The extended map from $\Sigma_0(Q_I)\cup\partial Q_I$ is denoted by $f^{(0)}$. This map is 1-Lipschitz, since $\operatorname{diam}\mathcal{M}\le 1$.

Applying to f and $f^{(0)}$ the construction of Lemma 6.15 and then the result of Lemma 6.17, we construct successive extensions $f^{(k+1)} : \Sigma_k(Q_I)\cap\partial Q_I \to \mathcal{M}$ of f with constants independent of I for $k = 0,1,\dots,\ell+1$. This yields a $\bar{c}$-Lipschitz extension $\bar{f} := f^{(\ell+1)} : Q_I \to \mathcal{M}$ of f with $\bar{c}$ independent of I.

Now we complete the proof of Theorem 6.12. Let $f : \partial\mathbb{K}^{\ell+1} \to \mathcal{M}$ be a c-Lipschitz map. We must extend f to $\mathbb{K}^{\ell+1}$ with the Lipschitz constant at most μc, where $\mu > 1$ is independent of f and c.

Let first $c \ge 1$. Choose an integer I by the condition

$$c \le I \le 2c,$$

and define a function $\tilde{f} : \partial Q_I \to \mathcal{M}$ by $\tilde{f}(x) := f\left(\frac{x}{I}\right)$, $x \in \partial Q_I$. Then $\tilde{f}$ is Lipschitz with the constant $\frac{c}{I} \le 1$. By the previous result, $\tilde{f}$ admits a $\bar{c}$-Lipschitz extension $\hat{f}$ to Q_I with a constant $\bar{c}$ independent of f and I. Then the map $\bar{f} : \mathbb{K}^{\ell+1} \to \mathcal{M}$, given by $\bar{f}(x) := \hat{f}(Ix)$, $x \in \mathbb{K}^{\ell+1}$, is an extension of f with the Lipschitz constant $\le \bar{c}I \le 2\bar{c}\cdot c$.

In the remaining case $c < 1$, we replace the metric d by $\widetilde{d} := cd$. Then the diameter of the space $(\mathcal{M},\widetilde{d})$ is also bounded by 1, and the map $f : \partial\mathbb{K}^{\ell+1} \to (\mathcal{M},\widetilde{d})$ is 1-Lipschitz. By Lemma 6.17, there exists a $\bar{c}$-Lipschitz extension $\bar{f} : \mathbb{K}^{\ell+1} \to (\mathcal{M},\widetilde{d})$ of f (with $\bar{c}$ independent of f) which naturally defines the required $\bar{c}c$-Lipschitz extension of f to $\mathbb{K}^{\ell+1}$.

The theorem is proved. □

6.2 Whitney covers

Let O be a nonempty open set of a metric space $(\mathcal{M}, d)$. In the sequel, we will say that a subset $S \subset \mathcal{M}$ is *c-embedded* into O, $c > 0$, if

$$\operatorname{diam} S \leq c d(S, \mathcal{M} \backslash O). \tag{6.7}$$

Recall that for $S_i \subset \mathcal{M}$, $i = 1, 2$,

$$d(S_1, S_2) := \inf\{d(m_1, m_2)\,; m_i \in S_i\}.$$

Definition 6.18. A family of nonempty subsets $\mathcal{U} := \{U_i \subset \mathcal{M}\}_{i \in I}$ is said to be a (Whitney) (c_1, c_2, n)-cover of an open subset O, where $c_1, c_2 > 0$ and $n \in \mathbb{Z}_+$, if

(a) $\mathcal{U}$ is a cover of O;

(b) every U_i is c_1-embedded into O;

(c) every subset $S \subset \mathcal{M}$, that is c_2-embedded into O satisfies

$$\operatorname{card}\{i \in I\,; S \cap U_i \neq \emptyset\} \leq n + 1, \tag{6.8}$$

i.e., S meets at most $n+1$ members of $\mathcal{U}$.

For instance, the cover of an open set in $\mathbb{R}^N$ in the classical Whitney theorem, see Volume I, Lemma 2.14 and Corollary 2.15, is a (c_1, c_2, n)-cover with $c_1 = c_1(N)$, $c_2 = c_2(N)$ and $n := 2^N$.

Theorem 6.19. *Let S be a nonempty closed proper subspace of $(\mathcal{M}, d)$. Assume that one of the following conditions holds:*

(i) *S has the Nagata dimension at most $n-1$, $n \in \mathbb{N}$, with a constant $\nu > 0$.*

(ii) *$S := \mathcal{M}\backslash S$ has the Nagata dimension at most $n \in \mathbb{Z}_+$ with a constant $\nu > 0$.*

Then the open set $\mathcal{M}\backslash S$ admits a (c_1, c_2, n)-cover with $c_1 = c_2(\nu, n) > 1$ and $c_2 = c_2(\nu, n) \leq 1$.

Let us recall, see Volume I, Definition 4.26, that a subset $S \subset \mathcal{M}$ has the Nagata dimension at most $n \in \mathbb{Z}_+$ with a constant $\nu > 0$ if, for every $t > 0$, there exists a cover $\mathcal{U} := \{U_i\}_{i \in I}$ of S of diameter $\leq \nu t$ such that every $T \subset S$ of diameter $\leq t$ satisfies

$$\operatorname{card}\{i \in I\,; U_i \cap T \neq \emptyset\} \leq n + 1. \tag{6.9}$$

Also recall that a cover $\mathcal{U}$ with these properties is called a (ν, n, t)-*cover.*

Proof. (i) Since $\dim_N S \leq n - 1$ with the constant ν, the result of Proposition 4.28 (4) of Volume I can be formulated as

Lemma 6.20. *For every $t > 0$ there exists a $(\nu', n-1, t)$-cover of S with $\nu' = c(n, \nu)$ that is a union of families denoted by $\mathcal{U}_k$, $k = 0, 1, \dots, n-1$, such that each subset of S of diameter $\le t$ intersects at most one set in $\mathcal{U}_k$, $k = 0, 1, \dots, n-1$.*

Using this we construct the required (c_1, c_2, n)-cover of $S^c = \mathcal{M}\backslash S$ with $c_1 = c_1(\nu, n)$, $c_2 = c_2(\nu, n)$. To this end, we first represent S^c as the disjoint union of a countable family of subsets R^i, $i \in \mathbb{Z}$, setting, for $i \in \mathbb{Z}$,

$$R^i := \{m \in S^c\,; r^i \le d(m, S) < r^{i+1}\}, \tag{6.10}$$

where $r := 5 + 3\nu'$.

Then we choose for every $i \in \mathbb{Z}$ an arbitrary map $\rho_i : R^i \to S$ such that for all $m \in R^i$,

$$d\big(m, \rho_i(m)\big) < r^{i+1}.$$

Lemma 6.21. *ρ_i sends sets of diameter $\le r^{i+1}$ to sets of diameter less than $3r^{i+1}$.*

Proof. If $m, m' \in R^i$ and $d(m, m') \le r^{i+1}$, then, by the triangle inequality,

$$\begin{aligned} d\big(\rho_i(m), \rho_i(m')\big) &\le d\big(m, \rho_i(m)\big) + d(m, m') + d\big(m', \rho_i(m')\big) \\ &< r^{i+1} + r^{i+1} + r^{i+1}. \end{aligned}$$

□

Exploiting Lemma 6.20 with $t := 3r^{i+1}$, we find a $\big(\nu', n-1, 3r^{i+1}\big)$-cover of S denoted by $\mathcal{V}^i$ such that $\mathcal{V}^i = \bigcup_{k=0}^{n-1} \mathcal{V}^i_k$ and, for every k, each subset of S of diameter $\le 3r^{i+1}$ meets at most one member of $\mathcal{V}^i_k$.

Now we introduce the main part of our construction consisting of families $\{\mathcal{W}^i_k\,; i \in \mathbb{Z},\ k = 0, 1, \dots, n-1\}$, where

$$\mathcal{W}^i_k := \{\rho_i^{-1}(V)\,; V \in \mathcal{V}^i_k\}. \tag{6.11}$$

Lemma 6.22. *For every i and k, it is true that*

(a) $\operatorname{diam} \mathcal{W}^i_k \le (2 + 3\nu')r^{i+1}$;

(b) *every subset of S^c of diameter $\le r^{i+1}$ meets at most one member of $\mathcal{W}^i_k$.*

Moreover, the family

$$\mathcal{W} := \bigcup_{i,k} \mathcal{W}^i_k$$

covers S^c.

Proof. (a) If $m, m' \in \rho_i^{-1}(V)$ for some $V \in \mathcal{V}^i_k$, then

$$\begin{aligned} d(m, m') &\le d\big(m, \rho_i(m)\big) + d\big(\rho_i(m), \rho_i(m')\big) + d\big(m', \rho_i(m')\big) \\ &\le 2r^{i+1} + \operatorname{diam} V \le (2 + 3\nu')r^{i+1}. \end{aligned}$$

(b) Let $T \subset S^c$ be of diameter $\le r^{i+1}$. Then by Lemma 6.21, $\operatorname{diam} \rho_i(T) \le 3r^{i+1}$ and by the choice of $\mathcal{V}_k^i$, the set $\rho_i(T)$ intersects at most one $V \in \mathcal{V}_k^i$. Hence, T intersects at most one $W \in \mathcal{W}_k^i$. The last assertion of the lemma is a matter of definition. □

Using the cover $\mathcal{W}$ we construct the required Whitney (c_1, c_2, n)-cover of S^c with $c_1 = c_1(\nu, n)$, $c_2 = c_2(\nu, n)$ in the following way. Given $W \in \mathcal{W}_k^i$, an associated set $\widehat{W}$ is defined by

$$\widehat{W} := W \cup \{W' \in \mathcal{W}_k^{i-1}\,; d(W, W') \le r^i\}. \tag{6.12}$$

Then we set

$$\widehat{\mathcal{W}}_k^i := \{\widehat{W}\,; W \in \mathcal{W}_k^i\} \tag{6.13}$$

and define $\widetilde{\mathcal{W}}_k^i$ by

$$\widetilde{\mathcal{W}}_k^i := \{W \in \mathcal{W}_k^{i-1}\,;\ W \text{ contains no member of } \widehat{\mathcal{W}}_k^i\}. \tag{6.14}$$

Using then a partition of S^c given by

$$R_k^i := \{m \in S^c\,;\ r^{i-\frac{k}{n}} \le d(m, S) < r^{i+1-\frac{k}{n}}\},$$

where $i \in \mathbb{Z}$ and $k = 0, 1, \dots, n-1$, we finally introduce the required cover of S^c denoted by $\mathcal{Y}$ by setting

$$\mathcal{Y}_k^i := \begin{cases} \big(\widehat{\mathcal{W}}_k^i \cup \widetilde{\mathcal{W}}_k^i\big) \cap R_k^i, & \text{if } k \ge 1, \\ \mathcal{W}_0^i, & \text{if } k = 0. \end{cases} \tag{6.15}$$

and then putting

$$\mathcal{Y} := \bigcup_{i,k} \mathcal{Y}_k^i.$$

Since $\mathcal{W}$ is a cover of S^c and $\{R_k^i\}$ is its partition, $\mathcal{Y}$ is also a cover of S^c. To show that this cover satisfies Definition 6.18 we need

Lemma 6.23. *For every $i \in \mathbb{Z}$ and $k \in \{0, 1, \dots, n-1\}$ the following holds:*

(a) *There exists a constant $\alpha = \alpha(\nu, n)$ such that every $Y \in \mathcal{Y}_k^i$ is α-embedded into S^c.*

(b) *Every subset of S^c of diameter $\le r^i$ meets at most one member of $\mathcal{Y}_k^i$.*

(c) *Every subset of $R^{i-1} \cup R^i$ of diameter $\le r^i$ meets at most one member of $\mathcal{Y}_k^{i-1}$.*

Proof. (a) We show that for every $Y \in \mathcal{Y}_k^i$ the following more general inequality holds:

$$\operatorname{diam} Y \leq \alpha_1 r^{i+1} + \alpha_2 r^i \leq \alpha d(Y, S), \tag{6.16}$$

where

$$\alpha := \alpha_1 r^2 + \alpha_2 r, \ \ \alpha_1 := 2 + 3\nu' \text{ and } \alpha_2 := 6(1 + \nu').$$

The right-hand inequality is a direct consequence of the embedding $Y \subset R_k^i$ for $k \geq 1$ and that of $Y \subset R^i$ for $k = 0$. In the latter case α can be replaced by the smaller constant $\alpha_1 r + \alpha_2$.

Now we prove the left-hand inequality. If $k = 0$, i.e., $Y \in \mathcal{Y}_0^i := \mathcal{W}_0^i$, the result follows from Lemma 6.22 (a) even with a better estimate. To prove the left-hand inequality for $k \geq 1$, we must estimate $d(m', m'')$ for $m', m'' \in Y$. Since $Y \in \widehat{\mathcal{W}}_k^i \cap R_k^i$ or $Y \in \widetilde{\mathcal{W}}_k^i \cap R_k^i$, the worst case to estimate $d(m', m'')$ is: $m' \in W'$, $m'' \in W''$ where $W', W'' \in \mathcal{W}_k^{i-1}$ and $W', W'' \subset \widehat{W}$ for some $W \in \mathcal{W}_k^i$, see definitions (6.12)–(6.14). Under these conditions, given $\varepsilon > 0$, there exist points $m_1', m_1'' \in W$ and $m_2' \in W'$, $m_2'' \in W''$ such that

$$\max \{d(m_1', m_2'), d(m_1'', m_2'')\} < r^i + \varepsilon.$$

Then by the subsequent application of the triangle inequality with the inserted points m_2', m_1'', m_2'', we get

$$d(m', m'') \leq \operatorname{diam} W' + (r^i + \varepsilon) + \operatorname{diam} W + (r^i + \varepsilon) + \operatorname{diam} W''.$$

The extreme terms in the right-hand side are at most $(2 + 3\nu')r^i$ and $\operatorname{diam} W \leq (2 + 3\nu')r^{i+1}$, see Lemma 6.22 (a); therefore

$$\operatorname{diam} Y \leq \alpha_1 r^{i+1} + \alpha_2 r^i + 2\varepsilon.$$

This yields the required inequality in (6.16).

(b) Let $T \subset S^c$ be of diameter $\leq r^i$. We must show that T meets at most one member of $\mathcal{Y}_k^i$ for each $k = 0, 1, \ldots, n-1$, i.e., either at most one member of $\widetilde{W}_k^i \cap R_k^i$ or of $\widehat{W}_k^i \cap R_k^i$. For $k = 0$ the family $\mathcal{Y}_0^i$ coincides with $\mathcal{W}_0^i$, see (6.15), and the latter has the required property with $r^{i+1} > r^i$, see Lemma 6.22 (b).

For $k \geq 1$, the set T meets, by the same lemma, at most one set from $\mathcal{W}_k^{i-1}$. Assuming that such $W' \in \mathcal{W}_k^{i-1}$ exists and belongs to $\widetilde{\mathcal{W}}_k^i$, we obtain that T does not meet any member $W \in \widehat{\mathcal{W}}_k^i$, because otherwise $W' \in \widehat{W}$ in contradiction with the definition of $\widetilde{\mathcal{W}}_k^i$, see (6.14). Hence, in this case T meets at most one member of $\mathcal{Y}_k^i$.

In the remaining case we assume that T meets at most one member of $\widehat{\mathcal{W}}_k^i \cap R_k^i$. Then by Lemma 6.22 (b), T meets at most one set $W \in \mathcal{W}_k^i$ and at most one set $W' \in \mathcal{W}_k^{i-1}$. If both these sets exist, the set W' should belong to $\widehat{W}$, since

$\operatorname{diam} T = r^i$, see (6.12). We conclude from here that T may meet more than one member of $\widehat{\mathcal{W}}_k^i \cap R_k^i$ only if there exist distinct sets W_1, W_2 from $\mathcal{W}_k^i$, such that, for some $W' \in \mathcal{W}_k^{i-1}$,

$$T \cap W' \neq \emptyset \quad \text{and} \quad W' \subset \widehat{W}_1 \cap \widehat{W}_2.$$

Show that this is impossible. Indeed, otherwise, given $\varepsilon \in \left[0, \frac{1}{2}\, r^i\right]$, there exist points $m_k \in W_k$, $k = 1, 2$, and $m_1', m_2' \in W'$ such that

$$d(m_k, \widetilde{m}_k) < r^i + \varepsilon, \quad k = 1, 2.$$

By the triangle inequality and Lemma 6.22 (a), we then obtain

$$d(m_1, m_2) < 2(r^i + \varepsilon) + \operatorname{diam} W' < 2r^i + (2 + 3\nu')r^i + 2\varepsilon.$$

Since $r := 5 + 3\nu'$, see (6.10), the right-hand side equals $(4 + 3\nu')r^i + 2\varepsilon < r^{i+1}$. Hence, the set $\{m_1, m_2\}$ of diameter $\leq r^{i+1}$ meets two distinct sets W_1, W_2 from $\mathcal{W}_k^i$ in contradiction with assertion (b) of Lemma 6.22.

Thus we proved that T meets either at most one member of $\widetilde{\mathcal{W}}_k^i \cap R_k^i$ or at most one member of $\widehat{\mathcal{W}}_k^i \cap R_k^i$.

The result is established.

(c) Let T be a subset of $R^{i-1} \cup R^i$ of diameter $\leq r^i$. We must show that T meets at most one set from $\mathcal{Y}_k^{i-1}$.

For $k = 0$, this, as above, follows directly from the second equality in (6.16) and Lemma 6.23 (b).

Now let $k \geq 1$. Show first that T cannot intersect any set, say U, of the family $\widetilde{\mathcal{W}}_k^{i-1} \cap R_k^{i-1}$. In fact, if it does, then, due to (6.12)–(6.14), $U \in \mathcal{W}_k^{i-2}$ and therefore $U \subset R^{i-2}$, see (6.11). On the other hand, T is a part of $R^{i-1} \cup R^i$ separated from R^{i-2}, a contradiction.

Hence, if T intersects a set $V \in \mathcal{Y}_k^{i-1}$, then V should belong to $\widehat{\mathcal{W}}_k^{i-1}$. By the definition of the latter family, $V \in \mathcal{W}_k^{i-1}$. Since $\operatorname{diam} T \leq r^i$, Lemma 6.22 (b) implies that there exists only one such V.

The lemma is proved. □

Now we prove that if a subset T is β-embedded into S^c for some $\beta = \beta(n, \nu') \leq 1$, then it meets at most $n + 1$ members of the cover $\mathcal{Y}$. Together with assertion (a) of Lemma 6.23 this would prove that $\mathcal{Y}$ is the required Whitney (c_1, c_2, n)-cover of S^c with the constants $c_1 := \alpha$ and $c_2 := \beta$ depending only on n and ν' (hence, on n and ν).

Let $T \subset S^c$ satisfy

$$\operatorname{diam} T \leq \beta d(T, S),$$

where $\beta := \min\{1, r^{\frac{1}{2n}} - 1\}$. Choose $i \in \mathbb{Z}_+$ and $k \in \{0, 1, \ldots, n-1\}$ from the condition

$$r^{i - \frac{k+1}{n}} \leq d(T, S) < r^{i - \frac{k}{n}}. \tag{6.17}$$

Due to the choice of β, this implies

$$\operatorname{diam} T < \min\Big\{r^i, r^{i-\frac{(k-\frac{1}{2})}{n}} - r^{i-\frac{k}{n}}\Big\}. \tag{6.18}$$

In turn, this gives, for $m \in T$,

$$d(m,S) \le \operatorname{diam} T + d(T,S) < r^{i-\frac{k-\frac{1}{2}}{n}} - r^{i-\frac{k}{n}} + r^{i-\frac{k}{n}}.$$

Hence, for $m \in T$,

$$d(m,S) < r^{i-\frac{k-\frac{1}{2}}{n}}. \tag{6.19}$$

Moreover, for this m,

$$d(m,S) \ge d(T,S) \ge r^{i-\frac{k}{n}}. \tag{6.20}$$

In particular, $r^{i-1} \le d(m,S) < r^{i+1}$ and therefore, by (6.10),

$$T \subset R^{i-1} \cup R^i. \tag{6.21}$$

Now show that (for the above chosen i and k) T intersects at most two members of $\mathcal{Y}_k := \bigcup\limits_{i\in\mathbb{Z}} \mathcal{Y}_k^j$. In fact, for $k=0$, every set from $\mathcal{Y}_0^i := \mathcal{W}_0^i$ is contained in R^i, and by (6.21), T may intersect only subsets from $\mathcal{W}_0^{i-1}$ and $\mathcal{W}_0^i$. Since $\operatorname{diam} T \le r^i$, see (6.18), T meets at most one member of $\mathcal{W}_0^{i-1}$ by Lemma 6.23 (c) and at most one member of $\mathcal{W}_0^i$ by Lemma 6.22 (c).

If now $k \ge 1$, then a subset of $\mathcal{Y}_k^i$ is contained in $R_k^i \subset R^{i-1} \cup R^i$. Hence, T can intersect only members of $\mathcal{Y}_k^{i-1}$ and $\mathcal{Y}_k^i$. In turn, assertions (b) and (c) of Lemma 6.23 imply that T meets at most one member of $\mathcal{Y}_k^{i-1}$ and at most one member of $\mathcal{Y}_k^i$. Hence, T meets at most two members of $\mathcal{Y}_k$.

Finally, show that T may meet at most one member of $\mathcal{Y}_{k'}$ for $k' \ne k$. Together with the previous assertion this gives at most $2 + (n-1) = n+1$ members of $\mathcal{Y} := \bigcup\limits_{k=0}^{n-1} \mathcal{Y}_k$, as required.

So, let T be β-embedded and i, k be chosen by (6.17) and

$$V \cap T \ne \emptyset \quad \text{for some } V \in \mathcal{Y}_{k'}^j,\ k' \ne k. \tag{6.22}$$

Show that this is possible for $k' \ge k+1$ only if $j = i$ and for $k' \le k-1$ only if $j = i-1$. Therefore, since by (6.18) and (6.21) $\operatorname{diam} T \le r^i$ and $T \subset R^{i-1} \cup R^i$, this set can meet at most one member of $\mathcal{Y}_{k'}^i$ for $k' \ge k+1$ and at most one member of $\mathcal{Y}_{k'}^{i-1}$ for $k' \le k-1$, see Lemma 6.23. This would give the required result.

Since V, as a member of $\mathcal{Y}_{k'}^j$, is contained in $R_{k'}^j$, we have, by the definition of the latter set, for each $m \in V \cap T$,

$$j - \frac{k'}{n} \le \log_r d(m,S) < j + 1 - \frac{k'}{n}.$$

Moreover, (6.19) and (6.20) yield for such m,

$$i - \frac{k+1}{n} \le \log_r d(m,S) < i - \frac{k-\frac{1}{2}}{n}\,.$$

Putting these inequalities together we obtain

$$\left\lfloor \frac{k'-k-1}{n} \right\rfloor \le j - i \le \left\lfloor \frac{k'-k+\frac{1}{2}}{n} \right\rfloor.$$

For $k' \ge k+1$ the inequality yields $j = i$ (as $k'-k \le n-1$). Otherwise, $k' \le k-1$ and the extreme sides of the inequality equal -1, i.e., $j = i-1$. This proves assertion (i) of Theorem 6.19.

(ii) Now the proof is essentially simpler, since the assumption and the assertion of Theorem 6.19 concern the same open set S^c. So, let S^c be of the Nagata dimension at most n with a constant ν. We split S^c into layers

$$R^i := \big\{m \in \mathcal{M}\,; r^i \le d(m,S) < r^{i+1}\big\}, \quad i \in \mathbb{Z}, \tag{6.23}$$

where $r := 3 + 2\nu$.

Every R^i has the Nagata dimension at most n with the same ν as for S^c; hence, there exists a cover $\mathcal{U}^i := \{U^i_k\}_{k \in K_i}$ of R^i such that:

(a) every $U^i_k \subset R^i$;

(b) $\operatorname{diam} \mathcal{U}^i_k \le \nu r^i$;

(c) every subset of R^i of diameter $\le r^i$ meets at most $n+1$ members of $\mathcal{U}^i$.

Now we replace each $\mathcal{U}^i$ by a new family $\mathcal{V}^i := \{V^i_j\}_{j \in J_i}$, as follows.

The new index set J_i is a subset of K_i given by

$$J_i := \big\{k \in K_i\,; d(U^i_k, R^{i+1}) > r^i\big\}. \tag{6.24}$$

To introduce V^i_j, we choose, for every pair $(i,k) \in \mathbb{Z} \times K_{i-1}$ such that $k \in K_{i-1} \backslash J_{i-1}$, an index $j(i,k) \in K_i$ by the condition

$$d\big(U^{i-1}_k, U^i_{j(i,k)}\big) \le r^{i-1}. \tag{6.25}$$

Lemma 6.24. *$j(i,k) \in J_i$.*

Proof. We must check that

$$d\big(U^i_{j(i,k)}, R^{i+1}\big) > r^i.$$

By the triangle inequality, (6.25) and condition (b) for $\mathcal{U}^{i-1}_k$, the left-hand side is greater than $d(U^{i-1}_k, R^{i+1}) - r^i$. Since $U^{i-1}_k \subset R^{i-1}$, the first term is at least $r^{i+1} - r^i$ and therefore the distance in question is at least $r^{i+1} - 2r^i := (1+2\nu)r^i > r^i$, as required. □

Now for $j \in J_i$, we set

$$V_j^i := \cup\{U_k^{i-1}\,; j(i,k) = j\} \cup U_j^i \tag{6.26}$$

and then put

$$\mathcal{V}^i = \{V_j^i\}_{j\in J_i} \quad \text{and} \quad \mathcal{V} := \bigcup_{i\in\mathbb{Z}} \mathcal{V}^i.$$

It is easily seen that $\mathcal{V}$ is a cover of S^c. To show that $\mathcal{V}$ is a Whitney cover we need

Lemma 6.25. (a) *Every $V \in \mathcal{V}$ is α-embedded into S^c with $\alpha := 2\nu + 1$.*

(b) *Every subset of $T \subset R^{i-1} \cup R^i$ of diameter $\le r^{i-1}$ meets at most $n+1$ members of $\mathcal{V}^{i-1} \cup \mathcal{V}^i$.*

Proof. (a) Let $i \in \mathbb{Z}$ be the maximal index such that $V \cap R^i \ne \emptyset$. For this we have, by the triangle inequality and by definition (6.26),

$$\operatorname{diam} V \le \nu r^i + 2(1+\nu)r^{i-1} = (1+2\nu)(r^i - (1+\nu)r^{i-1}) \le \alpha d(V,S).$$

(b) This result was established in the proof of Theorem 4.30 (b) of Volume I, see the text after formula (4.25) there. □

Now let T be β-embedded into S^c with $\beta := \frac{1}{r}$. Show that T meets at most $n+1$ members of $\mathcal{V}$. Together with Lemma 6.25 (a) this would imply that $\mathcal{V}$ is a (c_1, c_2, n)-cover with $c_1 := \alpha$ and $c_2 := \beta$. Choose $i \in \mathbb{Z}$ from the condition $r^{i-1} \le d(T,S) < r^i$. Then

$$\operatorname{diam} T \le \beta d(T,S) < \frac{1}{r}\, r^i = r^{i-1}$$

and, so, $T \subset R^{i-1} \cup \{m \in R^i\,;\ d(m, R^{i-1}) < r^{i-1}\}$. One easily concludes from here that if T meets some $V \in \mathcal{V}$, then $V \subset \mathcal{V}^{i-1} \cup \mathcal{V}^i$. Hence, T intersects at most $n+1$ members of $\mathcal{V}$ by Lemma 6.25 (b).

This proves Theorem 6.19 (ii). □

6.3 Main extension theorem

Let S be a nonempty closed subspace of a metric space $(\mathcal{M}, d)$ satisfying, with a constant $\nu > 0$, one of the following conditions:

$$\text{either } \dim_N S \le n-1 \text{ or } \dim_N S^c \le n.$$

Theorem 6.26. *Every c-Lipschitz map f from S into a Lipschitz $(n-1)$-connected with constant ν' metric space $(\mathcal{M}', d')$ admits a μc-Lipschitz extension $\bar{f} : \mathcal{M} \to \mathcal{M}'$ where μ depends only on n, ν and ν'.*

In case $\dim_N S^c = 0$, it is understood that $(\mathcal{M}', d')$ is assumed to be an arbitrary metric space, thus μ depends on ν only.

Proof. Due to Theorem 6.19 there exists a Whitney (α_1, α_2, n)-cover of S^c with $\alpha_1 = \alpha_1(\nu, n)$, $\alpha_2 = \alpha_2(\nu, n)$. Recall that this cover denoted by $\mathcal{U} := \{U_i\}_{i \in I}$ satisfies the condition:

$$\operatorname{diam} U_i \le \alpha_1 d(U_i, S), \quad i \in I, \tag{6.27}$$

and every subset T which is α_2-embedded into S^c satisfies

$$\operatorname{card}\big\{i \in I\,; T \cap U_i \ne \emptyset\big\} \le n + 1. \tag{6.28}$$

Using this cover we introduce a family of functions $\{\varphi_i : S^c \to \mathbb{R}_+\}_{i \in I}$ by setting, for $m \in S^c$,

$$\varphi_i(m) := \max\big\{0, \beta d(U_i, S) - d(m, U_i)\big\}, \tag{6.29}$$

where $\beta \in \big(0, \frac{1}{2}\big)$ is given by

$$\beta := \frac{\alpha_2}{2(\alpha_2 + 1)}\,. \tag{6.30}$$

Lemma 6.27. (a) *The function φ_i is 1-Lipschitz.*

(b) *For every $m \in S^c$ there exist at most $n+1$ indices $i \in I$ such that $\varphi_i(m) > 0$.*

Proof. (a) φ_i is 1-Lipschitz, since the function $m \mapsto d(m, U_i)$ is.

(b) If $\varphi_i(m) > 0$, then $d(m, U_i) < \beta d(U_i, S)$ and therefore, for some point $m_i \in U_i$,

$$d(m, m_i) < \beta d(U_i, S)\ \big(\le \beta d(m_i, S)\big).$$

Denote the set of all these points m_i, $i \in I$, by T. Then

$$\operatorname{diam} T \le 2\beta d(U_i, S) \le 2\beta\big(\operatorname{diam} T + d(T, S)\big),$$

whence $\operatorname{diam} T \le \frac{2\beta}{1-2\beta}\, d(T, S) = \alpha_2 d(T, S)$. Hence, T is α_2-embedded into S^c and therefore meets at most $n + 1$ sets U_i, see (6.28). Since the condition $\varphi_i(m) > 0$ implies that U_i intersects T, we conclude that $\varphi_i(m) > 0$ for at most $n + 1$ indices. □

Due to Lemma 6.27, the function

$$\varphi := \sum_{i \in I} \varphi_i$$

is well defined and its Lipschitz constant satisfies

$$L(\varphi) \le 2(n + 1). \tag{6.31}$$

Moreover, $\mathcal{U}$ is a cover of S^c and therefore $\varphi > 0$ on this set.

Now we define a map g from S^c into the Hilbert space $\ell_2(I)$ given by

$$g := \left(\frac{\varphi_i}{\varphi}\right)_{i\in I}. \tag{6.32}$$

Further, by Σ we denote an infinite simplex in $\ell_2(I)$ given by

$$\Sigma := \left\{(x_i)_{i\in I}\,;\sum_{i\in I} x_i = 1 \text{ and all } x_i \geq 0\right\};$$

its ith vertex is denoted by v_i, i.e., $v_i = \big(\delta_j(i)\big)_{i\in I}$, where $\delta_j(i)$ equals zero if $j \neq i$ and equals one otherwise.

Accordingly, Σ_k is the *k-skeleton* of Σ consisting of all its k-dimensional simplices (*k-cells*).

Lemma 6.28. (a) *The image of g is contained in the union of n-cells of Σ.*

(b) *Let $\sigma \in \Sigma$ be a minimal finite subsimplex containing $g(m)$. Then $\varphi_i(m) > 0$ if and only if the vertex v_i belongs to σ.*

(c) *For all m, m',*

$$\big\|g(m) - g(m')\big\|_{\ell_2(I)} \leq c(n)\frac{d(m,m')}{\varphi(m)} \tag{6.33}$$

where $c(n) := (2n+3)\sqrt{2n+2}$.

Proof. (a) By Lemma 6.27 (b), every point $g(m)$ has at most $n+1$ nonzero coordinates.

(b) If $g(m)$ belongs to a finite subsimplex σ and $\varphi_i(m) > 0$, then the i-th coordinate of $g(m)$ is strictly positive. Since $g(m)$ is a convex combination of vertices of σ, one of them should be v_i.

Conversely, if $g(m)$ and v_i belong to the minimal finite subsimplex $\sigma \in \Sigma$ containing $g(m)$, then, by the same reason, $\varphi_i(m) > 0$.

(c) For a fixed i,

$$\left|\frac{\varphi_i(m)}{\varphi(m)} - \frac{\varphi_i(m')}{\varphi(m')}\right| \leq \frac{|\varphi_i(m) - \varphi_i(m')|}{\varphi(m)} + \varphi_i(m')\left|\frac{1}{\varphi(m)} - \frac{1}{\varphi(m')}\right|$$
$$\leq \frac{1}{\varphi(m)}\Big(\big|\varphi_i(m) - \varphi_i(m')\big| + \big|\varphi(m) - \varphi(m')\big|\Big).$$

By (6.31) and Lemma 6.27 (a), this yields the upper bound $(2n+3)\frac{d(m,m')}{\varphi(m)}$ for the right-hand side, which, in turn, implies, for the norm in (6.33),

$$\big\|g(m) - g(m')\big\|_{\ell_2(I)} \leq (2n+3)\frac{d(m,m')}{\varphi(m)}\left(\sum_{i\in I'} 1\right)^{\frac{1}{2}},$$

where $\operatorname{card} I' \leq 2(n+1)$. □

Now we introduce the basic ingredient of the required extension operator, a map $h = h(f)$ from the union of n-cells of Σ to the given Lipschitz $(n-1)$-connected metric space $(\mathcal{M}', d')$. Then the extension $\bar{f} : \mathcal{M} \to \mathcal{M}'$ of the c-Lipschitz map $f : S^c \to \mathcal{M}'$ is given by the composition $h \circ g$.

We obtain h as a result of an inductive procedure beginning with a map $h^{(0)}$ from the 0-skeleton $\Sigma_0 := \{v_i\,; i \in I\}$ into $\mathcal{M}'$ given for $i \in I$ by

$$h^{(0)}(v_i) := f(m_i); \tag{6.34}$$

here m_i is an arbitrarily chosen point of the set S satisfying

$$d(m_i, U_i) \le (2-\beta)d(S, U_i). \tag{6.35}$$

Note that m_i exists as $2 - \beta > \frac{3}{2}$.

Lemma 6.29. *Let $\sigma \in \Sigma_n$ and let $\sigma^{(0)}$ be the set of its vertices. The Lipschitz constant of the trace $h^{(0)}\big|_{\sigma^{(0)}}$ satisfies*

$$L\big(h^{(0)}\big|_{\sigma^{(0)}}\big) \le \frac{\operatorname{diam} h^{(0)}(\sigma^{(0)})}{\sqrt{2}}. \tag{6.36}$$

Proof. Since the left-hand side equals

$$\max\left\{\frac{d'\big(h^{(0)}(v_i), h^{(0)}(v_j)\big)}{\|v_i - v_j\|_{\ell_2(I)}}\ ;\ v_i, v_j \in \sigma^{(0)}, \quad i \ne j\right\}$$

and the denominator equals $\sqrt{2}$, the result follows. □

Now we use Lipschitz $(n-1)$-connectedness of the space $\mathcal{M}'$ to extend $h^{(0)}$ to the required map h from the union on n-cells of Σ to $\mathcal{M}'$. In this case we use an equivalent definition of $(n-1)$-connectedness (see Definition 6.4) presented in

Lemma 6.30. *If $(\mathcal{M}', d')$ is Lipschitz $(n-1)$-connected with a constant $\nu' \ge 1$, then there exists a constant $\lambda(n) > 1$ such that every c-Lipschitz map $f : \partial\sigma \to \mathcal{M}'$, where $\sigma \in \Sigma_\ell$ and $0 \le \ell \le n-1$, admits a $\lambda(n)\nu' c$-Lipschitz extension into σ.*

Proof. For $\ell = 0$ this is the matter of definition, so let $\ell \ge 1$. It is well known (and may be easily checked by exploiting an appropriate projective transform) that there exists a bi-Lipschitz map from $\sigma \in \Sigma_\ell$ onto the unit cube $\mathbb{K}^\ell$, whose Lipschitz constant and distortion are bounded by some $\gamma(\ell) > 1$ and such that $\partial\sigma$ is mapped onto $\partial\mathbb{K}^\ell$. Hence, the c-Lipschitz map f gives rise to a $\gamma(\ell)c$-Lipschitz map from $\partial\mathbb{K}^\ell$ to $\mathcal{M}'$ and the latter, by definition, admits an extension to $\gamma(\ell)\nu' c$-Lipschitz map of $\mathbb{K}^\ell$. Returning to σ we obtain $\gamma(\ell)^2\nu' c$-extension of f to σ. □

Using this lemma, we extend the map $h^{(0)} : \Sigma_0 \to \mathcal{M}'$ to a map $h^{(1)}$ from the union of 1-cells to $\mathcal{M}'$, so that for every $\sigma \in \Sigma_1$, the Lipschitz constant of the trace $h^{(1)}\big|_\sigma$ satisfies

$$L\big(h^{(1)}\big|_\sigma\big) \le \lambda(n)\nu' L(h^{(0)}).$$

Proceeding this way, cf. the proof of Lemma 6.15, we, after n steps, obtain a map $h := h^{(n)}$ acting from the union of n-cells into $\mathcal{M}'$ such that h is an extension of $h^{(0)}$, and for every $\sigma \in \Sigma_n$,

$$L(h|_\sigma) \le \big(\lambda(n)\nu'\big)^n L(h^{(0)}).$$

Together with (6.36) this yields

$$L(h|_\sigma) \le c_1(n,\nu') \operatorname{diam} h(\sigma^{(0)}) \tag{6.37}$$

for $\sigma \in \Sigma_n$.

Using this inequality we bound the Lipschitz constant of $h|_\sigma$, $\sigma \in \Sigma_n$, as follows.

Lemma 6.31. *Let $\sigma(m)$ denote the minimal simplex of Σ_n containing a point $g(m)$, $m \in S^c$. Assume that v_i is a vertex of $\sigma(m)$. Then the following holds:*

(a) *The subset U_i of the (α_1,α_2,n)-cover $\mathcal{U}$ satisfies*

$$d(U_i,S) < 2d(m,S). \tag{6.38}$$

(b) *For the point $m_i \in S$ from* (6.35),

$$d(m,m_i) < (2+\alpha_1)d(U_i,S). \tag{6.39}$$

(c) *For the Lipschitz constant of $h\big|_{\sigma(m)}$,*

$$L\big(h\big|_{\sigma(m)}\big) \le c_2 L(f) d(m,S), \tag{6.40}$$

where $c_2 := 4(\alpha_1+2)c_1(n,\nu')$.

Proof. (a) Due to Lemma 6.28 (b), $\varphi_i(m) > 0$, and then (6.29) implies that $d(m,U_i) < \beta d(U_i,S)$. In turn, this yields

$$d(U_i,S) - d(m,S) \le d(m,U_i) < \beta d(U_i,S) < \frac{1}{2}\, d(U_i,S).$$

This gives (6.38).

(b) Since U_i is α_1-embedded, see (6.27), and $m_i \in S$ satisfies (6.35), we have

$$\begin{aligned} d(m,m_i) &\le d(m,U_i) + \operatorname{diam} U_i + d(m_i,U_i) \\ &< \beta d(U_i,S) + \alpha_1 d(U_i,S) + (2-\beta) d(U_i,S). \end{aligned}$$

This implies (6.39).

(c) Let $\sigma_0(m)$ be the set of vertices of $\sigma(m)$. By the definition of $h^{(0)}$, see (6.34),

$$\begin{aligned}\operatorname{diam} h(\sigma_0(m)) &= \max\bigl\{d'\bigl(f(m_i), f(m_k)\bigr)\,; v_i, v_k \in \sigma_0(m)\bigr\}\\ &\le L(f)\max\bigl\{d(m_i, m_k)\,; v_i, v_k \in \sigma_0(m)\bigr\}.\end{aligned}$$

But if v_i, v_k are vertices of σ, then, by (6.39) and (6.38),

$$\begin{aligned}d(m_i, m_k) \le d(m_i, m) + d(m, m_k) &< (\alpha_1 + 2)\bigl\{d(U_i, S) + d(U_k, S)\bigr\}\\ &\le 4(\alpha_1 + 2)d(m, S).\end{aligned}$$

Together with the previous inequality and (6.37) this leads to the required inequality (6.40). □

Finally, we define the desired extension of $f : S \to \mathcal{M}'$ by setting

$$\bar{f} := \begin{cases} h \circ g & \text{on } S^c,\\ f & \text{on } S.\end{cases} \tag{6.41}$$

We estimate the Lipschitz constant of $\bar{f}$ as required, i.e., we bound $d'\bigl(\bar{f}(m), \bar{f}(m')\bigr)$ by $\mu L(f)d(m, m')$ for $m \in S^c$ and $m' \in S$ or S^c.

First, let $m \in S^c$ and $m' \in S$. Choose a vertex v_i of the minimal simplex $\sigma(m)$ of Σ_n containing $g(m)$ and write for this index i,

$$d'\bigl(\bar{f}(m), \bar{f}(m')\bigr) \le d'\bigl(\bar{f}(m), f(m_i)\bigr) + d'\bigl(f(m_i), f(m')\bigr),$$

where m_i is defined in (6.35). The second summand is bounded by $L(f)d(m', m_i)$ while the first one, which is equal, by (6.41) and (6.34), to $d'\bigl(h(g(m)), h(v_i)\bigr)$, is bounded by

$$L\bigl(h|_{\sigma(m)}\bigr)\bigl\|g(m) - v_i\bigr\|_{\ell_2(I)} \le \sqrt{2}L\bigl(h|_{\sigma(m)}\bigr).$$

By Lemma 6.31, the right-hand side is at most $\sqrt{2}c_2L(f)d(m, S)$. Hence, we get

$$d'\bigl(\bar{f}(m), f(m_i)\bigr) \le \sqrt{2}c_2L(f)d(m, S), \tag{6.42}$$

provided that $\varphi_i(m) > 0$.

Since $d(m, S) \le d(m, m')$ as $m' \in S$, this and the previous estimate yield

$$d'\bigl(\bar{f}(m), \bar{f}(m')\bigr) \le L(f)\bigl\{\sqrt{2}c_2 d(m, m') + d(m_i, m')\bigr\}.$$

Recall that $m_i \in S^c$ is defined by (6.35) and that $v_i \in \sigma(m)$. Therefore inequality (6.39) can be applied to the pair m, m_i to give

$$\begin{aligned}d(m_i, m') \le d(m, m') + d(m_i, m) &\le d(m, m') + 2(\alpha_1 + 2)d(m, S)\\ &\le (1 + 2\alpha_1 + 2)d(m, m').\end{aligned}$$

Together with the previous inequality this implies the required result:

$$d'\big(\bar f(m), \bar f(m')\big) \le c_3 L(f) d(m, m'),$$

where $c_3 := \sqrt{2}c_2 + 1 + 2\alpha_1 + 2$ is a constant depending only on n, ν and ν'.

Now, let m and m' belong to S^c. As in Lemma 6.31, $\sigma(m)$ and $\sigma(m')$ denote the minimal simplices of Σ_n containing $g(m)$ and $g(m')$, respectively. Further, we denote by U_i and $U_{i'}$ the subsets of the (α_1, α_2, n)-cover $\mathcal{U}$ containing the points m and m', respectively. By the definition of $\{\varphi_i\}_{i\in I}$, see (6.29), this means, in particular, that

$$\varphi_i(m) > 0 \quad \text{and} \quad \varphi_{i'}(m') > 0. \tag{6.43}$$

The subsequent proof is divided into two parts depending on whether $\max\big\{\varphi_i(m'), \varphi_{i'}(m)\big\}$ is zero or greater than zero. First let

$$\varphi_i(m') = \varphi_{i'}(m) = 0. \tag{6.44}$$

By (6.29), this implies that

$$d(m, m') \ge d(m', U_i) \ge \beta d(U_i, S)$$

and also

$$d(m, m') \ge \beta d(U_{i'}, S).$$

Together with (6.39) this gives

$$d(m, m_i) \le (\alpha_1 + 2) d(U_i, S) \le (\alpha_1 + 2)\beta^{-1} d(m, m') \tag{6.45}$$

and the similar estimate for $d(m', m_{i'})$. Then we get

$$d'\big(\bar f(m), \bar f(m')\big) \le d'\big(\bar f(m), f(m_i)\big) + d'\big(\bar f(m'), f(m_{i'})\big) + d'\big(f(m_i), f(m_{i'})\big).$$

Due to (6.43), inequality (6.42) holds for the pairs m, m_i and $m', m_{i'}$ and therefore the first two terms in the right-hand side are bounded by

$$\sqrt{2}c_2 L(f)\big(d(m, m_i) + d(m', m_{i'})\big),$$

while the third summand is at most

$$L(f) d(m_i, m_{i'}) \le L(f)\big[\big(d(m_i, m) + d(m_{i'}, m')\big) + d(m, m')\big].$$

Hence, we conclude that

$$d'\big(\bar f(m), \bar f(m')\big) \le (1 + \sqrt{2}c_2) L(f)\big[d(m_i, m) + d(m_{i'}, m')\big] + L(f) d(m, m').$$

Combining this with (6.45) we finally obtain, in this case, the desired estimate

$$d'\big(\bar f(m), \bar f(m')\big) \le c_4 d(m, m'),$$

where $c_4 := 2(1+\sqrt{2}c_2)(\alpha_1+2)\beta^{-1}+1$ depends only on n, ν and ν'.

In the remaining case, $\max\{\varphi_i(m'), \varphi_{i'}(m)\} > 0$. Let, for definiteness,

$$\varphi_i(m') > 0.$$

Then by Lemma 6.28 (b), this inequality and (6.43) imply that v_i is a common vertex of the simplices $\sigma(m)$ and $\sigma(m')$. Since the dimensions of these simplices are at most n and $\sigma(m) \cap \sigma(m') \neq \emptyset$, they may be regarded as subsimplices of the $2n$-dimensional simplex $\left\{x \in \mathbb{R}^{2n+1}\,;\ \sum\limits_{i=1}^{2n+1} x_i = 1 \text{ and all } x_i \geq 0\right\}$. An elementary geometric consideration shows that for the points $g(m) \in \sigma(m)$ and $g(m') \in \sigma(m')$ there is a point $v \in \sigma(m) \cap \sigma(m')$ and a constant $\lambda(n) \geq 1$ such that

$$\big\|g(m) - v\big\|_{\ell_2(I)} + \big\|g(m') - v\big\|_{\ell_2(I)} \leq \lambda(n)\big\|g(m) - g(m')\big\|_{\ell_2(I)}. \tag{6.46}$$

Using this common point we then write

$$\begin{aligned} d'\big(\bar f(m), \bar f(m')\big) &\leq d'\big(h(g(m)), h(v)\big) + d'\big(h(v), h(g(m'))\big) \\ &\leq L\big(h|_{\sigma(m)}\big)\|g(m) - v\|_{\ell_2(I)} + L\big(h|_{\sigma(m')}\big)\|g(m') - v\|_{\ell_2(I)}. \end{aligned}$$

Estimating the Lipschitz constants here using (6.40) and then applying (6.46), we bound the right-hand side by

$$\lambda(n)c_2L(f)\max\big\{d(m,S), d(m',S)\big\}\|g(m) - g(m')\|_{\ell_2(I)}.$$

Further, $d(m,S) \leq d(m,m_i)$ and, since v_i belongs to $\sigma(m)$, the latter is bounded, due to (6.39), by $(\alpha_1+2)d(U_i,S)$. The vertex v_i also belongs to $\sigma(m')$ and therefore, analogously, $d(m',S) < (\alpha_1+2)d(U_i,S)$.

Collecting all these estimates we obtain

$$d'\big(\bar f(m), \bar f(m')\big) \leq c_2(\alpha_1+2)\lambda(n)d(U_i,S)\cdot\big\|g(m) - g(m')\big\|_{\ell_2(I)}.$$

Finally, the ℓ_2-norm here is at most $c(n)\frac{d(m,m')}{\varphi(m')}$ by (6.33) and, moreover, the denominator $\varphi(m') \geq \varphi_i(m') \geq \beta d(U_i,S)$ by (6.29) and the condition $\varphi_i(m') > 0$.

Hence, we have the required inequality

$$\overline{d}\big(\bar f(m), \bar f(m')\big) \leq c_5 d(m,m'),$$

where $c_5 := c_2(\alpha_1+2)\lambda(n)c(n)\beta^{-1}$ depends only on n, ν, ν'.

Theorem 6.26 is proved. □

Finally, we present a version of the theorem just proved for vector-valued functions.

Let us recall that $\mathrm{Lip}(\mathcal{M}, X)$ is the space of maps f from a metric space $(\mathcal{M}, d)$ into a Banach space $\big(X, \|\cdot\|\big)$ equipped with seminorm

$$L(f) := \sup\left\{\frac{\|f(m) - f(m')\|}{d(m,m')}\ ;\ m, m' \in \mathcal{M},\quad m \neq m'\right\}.$$

Theorem 6.32. *Let S be a nonempty closed subspace of $\mathcal{M}$. Assume that either* $\dim_N S \le n-1$ *or* $\dim_N S^c \le n$ *with constant* $\nu > 0$ *in both cases. Then there exists a linear extension operator from* $\mathrm{Lip}(S,X)$ *into* $\mathrm{Lip}(\mathcal{M},X)$ *whose norm is bounded by a constant depending only on n and ν.*

Proof. Due to Proposition 6.5, a Banach space is n-connected for every $n \ge 0$. Moreover, within the proof of the proposition, a linear extension operator from $\mathrm{Lip}(\partial\mathbb{K}^\ell, X)$ into $\mathrm{Lip}(\mathbb{K}^\ell, X)$ with norm bounded by $c(\ell) > 1$ was introduced. Further, it was shown in Lemma 6.30 that this implies the existence of a similar linear extension operator, say T_ℓ, from $\mathrm{Lip}(\partial\sigma, X)$ into $\mathrm{Lip}(\sigma, X)$, where σ is an ℓ-cell of the infinite simplex Σ. Finally, the operator $f \mapsto h^0(f)$ defined by (6.34) is linear for vector-valued functions f.

Since the operator $h : f \mapsto h(f)$ constructed in the proof of Theorem 6.26 is now obtained by the subsequent application of the operators T_ℓ, $\ell = 1, \dots, n$, to $h^{(0)}$, the operator h is also linear.

The remaining part of the proof of Theorem 6.26 shows that the operator h maps $\mathrm{Lip}(S,\mathcal{M})$ into $\mathrm{Lip}(\mathcal{M},X)$ and its norm is bounded by a constant $c \ge 1$ depending only on n, ν and $\max\limits_{1\le\ell\le n} c(\ell)$, i.e., on n and ν. □

6.4 Corollaries of the main extension theorem

We describe several important consequences of the basic extension results presented in Section 6.3 concerning the problems posed in Sections 1.7 and 1.11 of Chapter 1 of Volume I. For convenience of the reader we recall several basic concepts introduced in these sections.

We say that a pair $(\mathcal{M}, \mathcal{M}')$ has the *Lipschitz extension property*, (briefly, belongs to the class $\mathcal{LE}$) if there is a constant $c \ge 1$ such that for every $S \subset \mathcal{M}$ and every Lipschitz map $f : S \to \mathcal{M}'$ there exists a $cL(f)$-Lipschitz extension $\bar{f} : \mathcal{M} \to \mathcal{M}'$ of f.

As in formula (1.27) of Volume I we set for this pair

$$\Lambda(\mathcal{M}, \mathcal{M}') := \inf c. \tag{6.47}$$

Definition 6.33. $\mathcal{M}'$ is said to be a universal Lipschitz target space (briefly, ULT-space), if

$$\Lambda(\mathcal{M}, \mathcal{M}') < \infty \tag{6.48}$$

for every $\mathcal{M}$.

It was proved in Proposition 1.47 (b) of Volume I that for a ULT-space $\mathcal{M}'$,

$$\Lambda(\mathcal{M}') := \sup_{\mathcal{M}} \Lambda(\mathcal{M}, \mathcal{M}') < \infty,$$

while Proposition 1.48 of this volume states that $\mathcal{M}' \in \mathcal{U}LT$ if and only if it is an absolute Lipschitz retract.

To formulate the subsequent results it will be useful to introduce

Definition 6.34. (a) Given $\mathcal{M}$, the class $\mathcal{T}(\mathcal{M})$ consists of all $\mathcal{M}'$ such that

$$\Lambda(\mathcal{M}, \mathcal{M}') < \infty.$$

(b) Given $\mathcal{M}'$, the class $\mathcal{U}(\mathcal{M}')$ consists of all $\mathcal{M}$ such that the above inequality holds.

Using concepts introduced we will present several important consequences of Theorem 6.26.

Corollary 6.35. *A Hadamard space of finite Nagata dimension is a ULT-space.*

Proof. Let $\dim_N \mathcal{M} \le n < \infty$ for a Hadamard space $\mathcal{M}$. By Proposition 6.10 $\mathcal{M}$ is Lipschitz n-connected. This implies that $\mathcal{M}$ is an absolute Lipschitz retract. In fact, let $\mathcal{M}$ sit isometrically in a metric space $\widetilde{\mathcal{M}}$. Then the identity map from $\mathcal{M}$ regarded as a subspace of $\widetilde{\mathcal{M}}$ into $\mathcal{M}$ admits a c-Lipschitz extension to $\widetilde{\mathcal{M}}$, see Theorem 6.26. This extension is clearly the required retract of $\widetilde{\mathcal{M}}$ onto $\mathcal{M}$ whose Lipschitz constant is independent of $\widetilde{\mathcal{M}}$. □

Special cases of this corollary were first obtained by Lang, Pavlović and Schroeder [LPSch-2000] along with a result that does not follow from Theorem 6.26. We formulate the results of these authors and briefly discuss the method of the proof.

Theorem 6.36. *The spaces of the following three classes are universal Lipschitz target spaces.*

(a) 2-*dimensional Hadamard manifolds.*

(b) *Gromov hyperbolic Hadamard n-manifolds with $n \ge 2$ whose sectional curvature κ is bounded by $-b^2 \le \kappa \le 0$.*

(c) *Homogeneous Hadamard n-manifolds with $n \ge 2$ (i.e., 1-connected n-dimensional Hadamard manifolds having the transitive group of isometries).*

The results are consequences of Corollary 6.35, since the spaces of all these classes have finite Nagata dimension. For the last two cases this fact was proved by Lang and Schlichenmaier [LSchl-2005] and for the case (a) by Schlichenmaier [Sch-2005].

However all three results of Theorem 6.36 are proved by another method which we now briefly discuss.

Let S be a subset of a metric space $(\mathcal{M}, d)$ and f be a c-Lipschitz map from S into a Hadamard n-manifold X. We must extend, under the conditions of Theorem 6.36 on X, the map f to a μc-Lipschitz map $\bar{f} : \mathcal{M} \to X$ where μ depends only on X. To this end one associates to every $m \in \mathcal{M}$ a closed convex set $A(m) \subset X$ given by

$$A(m) := \bigcap_{m' \in S} \overline{B}_{r(m,m')}(f(m'))$$

where the radii of the closed balls in X are given by

$$r(m, m') := \alpha \cdot c \cdot d(m, m'), \quad \alpha > 0.$$

Fact 1. If $\alpha \geq \sqrt{2}$, then $A(m) \neq \emptyset$.

Fact 2. If $\alpha > \sqrt{2}$, then the Hausdorff distance between the associated sets satisfies

$$d_{\mathcal{H}}(A(m), A(m')) \leq \frac{\alpha^2}{\sqrt{\alpha^2 - 2}} \cdot c \cdot d(m, m'). \tag{6.49}$$

Let $(\mathcal{C}(X), d_{\mathcal{H}})$ be the metric space of closed convex subsets of the Hadamard manifold X equipped with the Hausdorff metric. Assume for a while that the following is true.

Fact 3. There exists a Lipschitz map $\phi : (\mathcal{C}(X), d_{\mathcal{H}}) \to X$ such that

$$\phi(\{x\}) = x \quad \text{for all } x \in X.$$

Since for $m \in S$ the set $A(m) := \{f(m)\}$, composition $\phi \circ A$ is an extension of f to $\mathcal{M}$ whose Lipschitz constant is bounded by $L(\phi)\lambda(\alpha, n)c$ where $\lambda(\alpha, n)$ is the constant in (6.49).

The construction of the map ϕ is, in general, rather involved but can be done in a relatively simple way for 2-dimensional Hadamard manifolds or for the classical hyperbolic spaces $\mathbb{H}^n$. We consider the latter case in the next section and show that $\Lambda(\mathbb{H}^n) \leq 2\sqrt{2}\, n$.

In the case of a 2-dimensional Hadamard manifold, the corresponding constant is $4\sqrt{2}$. For the reader versed in hyperbolic geometry we note that $\phi(S)$ in this case is the intersection of two lines, a geodesic ray emanating from a fixed point $\xi \in \partial_\infty X$ and tangent to S in a consistent way for all S and the horosphere based at ξ which touches S so that the corresponding open horoball does not contain S.

Theorem 6.36 (a) leads to the following apparently very difficult

Problem. *Prove that a Hadamard n-manifold is a ULT-space for $n \geq 3$ (or, at least, for $n = 3$).*

However, for infinite-dimensional Hadamard manifolds the answer to the problem is negative. Since they are defined by ℓ_2-charts, it suffices to prove that an infinite-dimensional Hilbert space is not a ULT-space. But it has already been proved in Section 1.11 of Volume I, see Corollary 1.51 (a) there.

Now we present two additional consequences of Theorem 6.26 describing, in particular, the classes $\mathcal{T}(\mathcal{M})$ and $\mathcal{U}(\mathcal{M})$ for $\mathcal{M} = \mathbb{R}^n$ or $\mathbb{H}^n$, see Definition 6.34.

Corollary 6.37. *Let $\mathcal{M}_i$ be a Gromov hyperbolic Hadamard manifold of dimension n_i with sectional curvature satisfying $-b_i^2 \leq \kappa \leq 0$ for some $b_i > 0$, $1 \leq i \leq l$. Then the following is true.*

(a) *The class* $\mathcal{T}(\oplus_{i=1}^{l}\mathcal{M}_i)$ *of their direct* 2*-sum equals the class of all Lipschitz* $(n-1)$*-connected metric spaces where* $n := \sum_{i=1}^{l} n_i$.

(b) *The class* $\mathcal{U}(\mathcal{M})$ *of a Lipschitz* $(n-1)$*-connected metric space* $\mathcal{M}$ *satisfies*

$$\mathcal{T}(\oplus_{i=1}^{l}\mathcal{M}_i) \subset \mathcal{U}(\mathcal{M}).$$

Proof. (a) Let $\mathcal{M}$ be Lipschitz $(n-1)$-connected. We must prove that

$$\Lambda(\mathcal{M}, \oplus_{i=1}^{l}\mathcal{M}_i) < \infty. \tag{6.50}$$

By Theorems 4.30, 4.39 and 4.40 of Volume I

$$\dim_N(\oplus_{i=1}^{l}\mathcal{M}_i) \le \sum_{i=1}^{l} \dim_N \mathcal{M}_i = \sum_{i=1}^{l} \dim \mathcal{M}_i =: n.$$

Applying Theorem 6.26 (b) we obtain the required result.

Now, let (6.50) hold for a metric space $\mathcal{M}$. Show that $\mathcal{M}$ is Lipschitz $(n-1)$-connected. To this end we note that every $\mathcal{M}_i$ contains a bi-Lipschitz copy of the unit cube $\mathbb{K}^s$ with $1 \le s \le n_i$. In fact, if $\mathcal{M}_i$ is 0-hyperbolic, i.e., $\mathcal{M}_i$ is a complete $\mathbb{R}$-tree, then $\dim \mathcal{M}_i = 1$ and every its edge is isometric to the closed interval of $\mathbb{R}$. Otherwise, $\mathcal{M}_i$ is δ_i-hyperbolic with $\delta_i > 0$, i.e., $\mathcal{M}_i$ is a (pinched) complete n_i-dimensional Riemannian manifold. Thus, $\mathcal{M}_i$ contains diffeomorphic copies of the Euclidean unit s-balls with $1 \le s \le n_i$ and therefore contains bi-Lipschitz copies of $\mathbb{K}^s$ for these s.

Hence, $\oplus_{i=1}^{l}\mathcal{M}_i$ contains a bi-Lipschitz copy of $\mathbb{K}^s$ for every $1 \le s \le n$.

We then derive from here and (6.50) that every Lipschitz map from $\partial\mathbb{K}^s$ to $\mathcal{M}$ admits a Lipschitz extension to $\mathbb{K}^s$ for $s \le n$, i.e., $\mathcal{M}$ is Lipschitz $(n-1)$-connected, see Definition 6.4.

(b) Now let $\mathcal{M}$ be a Lipschitz $(n-1)$-connected metric space. By the definition of $\mathcal{U}(\mathcal{M})$ we must show that for every $\mathcal{M}' \in \mathcal{T}(\oplus_{i=1}^{l}\mathcal{M}_i)$,

$$\Lambda(\mathcal{M}, \mathcal{M}') < \infty.$$

This clearly follows from the above estimate of $\dim_N(\oplus_{i=1}^{l}\mathcal{M}_i)$ and Theorem 6.26. □

Finally, we present a consequence of Theorem 6.32 on simultaneous Lipschitz extensions. Likewise in the nonlinear case we introduce simultaneous Lipschitz extension constants for a metric space $\mathcal{M}$ and a Banach space X by setting

$$\lambda(\mathcal{M}, X) := \sup\{\lambda(S, \mathcal{M}, X) \ ; \ S \subset \mathcal{M}\} \tag{6.51}$$

where, in turn,

$$\lambda(S, \mathcal{M}, X) := \inf ||E|| \tag{6.52}$$

and E runs over all linear bounded extension operators from $\mathrm{Lip}(S,X)$ into $\mathrm{Lip}(\mathcal{M},X)$.

The class of all these operators is denoted by $\mathrm{Ext}(S,\mathcal{M},X)$. We simplify the notation by writing $\lambda(\mathcal{M})$, $\lambda(\mathcal{M},S)$ etc. for $X=\mathbb{R}$.

As we will show in Chapter 8 there exists even two-dimensional smooth manifolds of bounded geometry for which $\mathrm{Ext}(S,\mathcal{M})=\emptyset$ for some S. In such a case, the extension constants are assumed to be $+\infty$.

Corollary 6.38. *Let $\mathcal{M}_i$ be one of the following spaces:*

(a) *an $\mathbb{R}$-tree ;*

(b) *a doubling metric space;*

(c) *a Gromov hyperbolic spaces of bounded geometry.*

Then there is a constant $c>1$ depending only on $\mathcal{M}_i$, $1\le i\le l$, such that for $\mathcal{M}:=\oplus^{(p)}_{1\le i\le l}\mathcal{M}_i$,

$$\lambda(\mathcal{M},X)\le c. \tag{6.53}$$

Proof. Due to Theorems 4.30, 4.31, 4.34 and 4.39 of Volume I, $\mathcal{M}$ is of finite Nagata dimension. Then the result follows from Theorem 6.32. □

Remark 6.39. (a) The proof presented here gives, for the constant c, an estimate exponentially depending on the basic parameters of the spaces $\mathcal{M}_i$ such as topological dimension, doubling constant etc. To obtain estimates of c close to the optimal one requires another approach which will be discussed in Chapter 7. For instance, for $\mathcal{M}$ being the direct sum of n $\mathbb{R}$-trees we obtain that $c_1\sqrt{n}\le\lambda(\mathcal{M},X)\le c_2 n$ for any dual Banach space X where c_1,c_2 are numerical constants while for $\mathcal{M}=\mathbb{H}^n$ this approach gives $c_1 n^{\frac{1}{8}}\le\lambda(\mathcal{M},X)\le c_2 n^{\frac{5}{2}}$ for some numerical constants c_1,c_2, see Corollaries 7.41, 7.39 and 8.12.

(b) In fact, the result of Corollary 6.38 can be amplified as follows. Given a metric space $\widetilde{M}$ and its subspace S isometric to a subspace of the direct sum $\mathcal{M}$ there exists a linear continuous extension operator from $\mathrm{Lip}(S,X)$ into $\mathrm{Lip}(\widetilde{M},X)$ whose norm is bounded by the same constant $c=c(\mathcal{M})$ of Corollary 6.38.

Actually, S can be regarded as a subspace of $\mathcal{M}$ and therefore $\dim_N S\le\dim_N\mathcal{M}<\infty$ by Theorem 4.30 (b) of Volume I. It remains to apply to the pair $S,\widetilde{M}$ Theorem 6.32.

In Section 7.4, this result will be proved by another method giving, in particular, effective estimates for the corresponding extension constants.

6.5 Nonlinear Lipschitz extension constants

The Lang-Schlichenmaier theory estimates the corresponding Lipschitz extension constants by unspecified quantities whose dependence on the basic parameters is not revealed. The problem of effective estimates of extension constants is solved only for several special classes presented here. For the time being a general theory yielding effective estimates of the corresponding extension constants has been developed only for linear (simultaneous) Lipschitz extensions, see Chapters 7 and 8. Because of this the proofs of the separate results of this section are based on a wide range of ideas and methods. We present proofs for several results most related to the main line of the present book and survey others.

6.5.1 The classical spaceforms of nonpositive curvature

As it was proved before $\mathbb{R}^n$ and $\mathbb{H}^n$ are universal Lipschitz target spaces. In this part, the quantitative version of these facts will be presented using the extension constant $\Lambda(\mathcal{M})$ introduced, e.g., in Definition 6.33. The first result had already been presented in Section 1.11 of Volume I, see Corollary 1.50 (b) there. It asserts that

$$\Lambda(\mathbb{R}^n) = \frac{n\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n+1}{2})} \sim \sqrt{\frac{2n}{\pi}} \quad \text{as} \quad n \to \infty. \tag{6.54}$$

For the subsequent estimate of $\Lambda(\mathbb{H}^n)$ we require a lower bound for the Lipschitz constant of pair $(C(\mathbb{S}^{n-1}), \mathbb{R}^n)$, see (6.47) for the definition of the constant.

Proposition 6.40. $\Lambda(C(\mathbb{S}^{n-1}), \mathbb{R}^n) = \Lambda(\mathbb{R}^n) > \sqrt{\frac{2n}{\pi}}$.

Proof. By definition, $\Lambda(C(\mathbb{S}^{n-1}), \mathbb{R}^n) \leq \Lambda(\mathbb{R}^n)$. Thus we have to prove the converse inequality.

Consider subspace $H_1(\mathbb{S}^{n-1}) := \{\ell_a : x \mapsto x \cdot a,\ x \in \mathbb{S}^{n-1};\ a \in \mathbb{R}^n\} \subset C(\mathbb{S}^{n-1})$ of linear functionals on $\mathbb{R}^n$ restricted to $\mathbb{S}^{n-1}$.

Note that

$$\|\ell_a\|_{C(\mathbb{S}^{n-1})} := \max_{x \in \mathbb{S}^{n-1}} |x \cdot a| = \|a\|_2,$$

that is, $H_1(\mathbb{S}^{n-1})$ is linearly isometric to the Euclidean space $\mathbb{R}^n$ and we can write

$$\Lambda(C(\mathbb{S}^{n-1}), \mathbb{R}^n) = \Lambda(C(\mathbb{S}^{n-1}), H_1(\mathbb{S}^{n-1})).$$

Let $I : H_1(\mathbb{S}^{n-1}) \to H_1(\mathbb{S}^{n-1})$ be the identity (1-Lipschitz) map. Assume that $\widetilde{I} : C(\mathbb{S}^{n-1}) \to H_1(\mathbb{S}^{n-1})$ is a γ-Lipschitz extension of I to $C(\mathbb{S}^{n-1})$, i.e.,

$$L(\widetilde{I}) = \gamma L(I) = \gamma.$$

To prove the desired statement we should show that $\gamma \geq \Lambda(\mathbb{R}^n)$. To this end we use the linearization argument of Theorem 1.49 of Volume I, to turn $\widetilde{I}$ into a linear projection T of $C(\mathbb{S}^{n-1})$ onto $H_1(\mathbb{S}^{n-1})$ with $\|T\| \leq \gamma L(I) = \gamma$.

Hence, we get

$$\gamma \geq \|T\| \geq \inf \|P\|,$$

where P runs over all linear projections from $C(\mathbb{S}^{n-1})$ onto $H_1(\mathbb{S}^{n-1})$. This infimum, in turn, equals the projection constant of $H_1(\mathbb{S}^{n-1})$, hence, of $\mathbb{R}^n$, see Daugavet [Dau-1968, Thm. 2]. Together with Corollary 1.50 (b) of Volume I this implies

$$\gamma = L(\widetilde{I}) \geq \Pi(\mathbb{R}^n) = \Lambda(\mathbb{R}^n) = \frac{n\Gamma(\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{n+1}{2})}.$$

Finally, using the known properties of the Γ-function, see, e.g., Rutovitz [Rut-1965], one can show that the right-hand side here is greater than $\sqrt{\frac{2n}{\pi}}$ as required. □

In order to obtain the lower bound for the extension constant of $\mathbb{H}^n$ we need the following fact.

Let $\mathbb{B}_n$ be the open unit ball of $\mathbb{R}^n$ centered at 0.

Corollary 6.41. *There exist a bounded metric space $\mathcal{M}$, its subspace S and a 1-Lipschitz map $f : S \to \mathbb{B}_n$ such that every Lipschitz extension $\bar{f} : \mathcal{M} \to \mathbb{R}^n$ of f has a Lipschitz constant greater than or equal to $\Lambda(\mathbb{R}^n)$.*

Proof. Let B_n be the open unit ball of $H_1(\mathbb{S}^{n-1})$ centered at 0. Consider the isometry $I_n\big|_{B_n} : B_n \to \mathbb{B}_n$, the restriction of the above defined isometry $I_n : H_1(\mathbb{S}^{n-1}) \to \mathbb{R}^n$. We set $\mathcal{M} := B_1(C(\mathbb{S}^{n-1}))$, where $B_r(C(\mathbb{S}^{n-1}))$ is the open ball of $C(\mathbb{S}^{n-1})$ of radius r centered at 0, $S := B_n$ and $f := I_n\big|_{B_n}$. Assume, on the contrary, that there is a Lipschitz extension $\bar{f} : \mathcal{M} \to \mathbb{R}^n$ of f such that

$$L(\bar{f}) < \Lambda(\mathbb{R}^n)\,. \tag{6.55}$$

We define a map $J_k : B_{\gamma^k}(C(\mathbb{S}^{n-1})) \to \mathbb{R}^n$, $k \in \mathbb{N}$,

$$J_k(h) := \gamma^k \bar{f}(\gamma^{-k} h).$$

This map satisfies the conditions

(*a*) $L(J_k) = L(\bar{f})$;

(*b*) $J_k\big|_{\gamma^k B_n} = I_n\big|_{\gamma^k B_n}$.

Let P be a countable dense subset of $C(\mathbb{S}^{n-1})$ such that $P \cap H_1(\mathbb{S}^{n-1})$ is dense in $H_1(\mathbb{S}^{n-1})$. Using the above remark and Cantor's diagonal process we find an infinite subsequence of integers $\mathbb{N}_0 \subset \mathbb{N}$ such that for every $f \in P$ there exists the limit

$$\lim_{k \in \mathbb{N}_0} J_k(f) =: J_\infty(f).$$

Because of pointwise convergence, we derive from (a) that the map $J_\infty : P \to \mathbb{R}^n$ is Lipschitz with constant at most $L(\bar{f})$.

Let $\overline{J}_\infty$ be an extension of J_∞ by continuity to $C(\mathbb{S}^{n-1})$. Then by (b), for every $h \in H_1(\mathbb{S}^{n-1})$,

$$\overline{J}_\infty(h) = I_n(h),$$

i.e., $\overline{J}_\infty$ is a Lipschitz extension of I_n with Lipschitz constant less than $\Lambda(\mathbb{R}^n)$. This contradicts Proposition 6.40. □

We exploit the construction of Corollary 6.41 to obtain a lower estimate of $\Lambda(\mathbb{H}^n)$ while an upper estimate is obtained by the adaptation of the Lang-Pavlović-Schroeder proof [LPSch-2000] to this setting.

Theorem 6.42.

$$\sqrt{\frac{2n}{\pi}} < \Lambda(\mathbb{H}^n) \leq 2\sqrt{2}\, n.$$

Proof. To establish the left-hand inequality, we use a standard fact of Differential Geometry asserting that for any point of an n-dimensional Riemannian manifold equipped with the corresponding length metric and every $\varepsilon > 0$ there is a neighborhood $(1+\varepsilon)$-isometric to an open ball of $\mathbb{R}^n$.

Let us fix a point $m \in \mathbb{H}^n$ and its neighborhood U_ε $(1+\varepsilon)$-isometric to an open ball $B_{r_\varepsilon}(0) \subset \mathbb{R}^n$ for some $0 < r_\varepsilon < 1$. By $\phi_\varepsilon : B_{r_\varepsilon}(0) \to U_\varepsilon$ we denote the corresponding bi-Lipschitz map sending 0 to m. Making a suitable rescaling of the metric on $B_1(C(\mathbb{S}^{n-1}))$ and of the map from B_n into $\mathbb{B}_n$ given by Corollary 6.41, and denoting these new objects by $\mathcal{M}$, $S \subset \mathcal{M}$ and f_ε, respectively, we obtain the map $f_\varepsilon : S \to B_{r'_\varepsilon}(0) \subset \mathbb{R}^n$, where $r'_\varepsilon := r_\varepsilon \left(\frac{2nc}{\pi}\right)^{-\frac{1}{2}}$ for some $c > 1$ independent of ε, such that $L(f_\varepsilon) = r'_\varepsilon$ and every Lipschitz extension $\tilde{f} : \mathcal{M} \to \mathbb{R}^n$ of f_ε has a Lipschitz constant greater than r_ε. Also, by the definition there is $m^* \in S$ such that $f_\varepsilon(m^*) = 0$ and $d(m', m^*) < r'_\varepsilon$ for all $m' \in \mathcal{M}$.

Let us consider the Lipschitz map $F_\varepsilon := \phi_\varepsilon \circ f_\varepsilon : S \to \mathbb{H}^n$. Clearly $L(F_\varepsilon) \leq (1+\varepsilon) r'_\varepsilon$. We claim that for a sufficiently small $\varepsilon > 0$ every Lipschitz extension $\bar{F} : \mathcal{M} \to \mathbb{H}^n$ of F_ε has Lipschitz constant greater than $\frac{(1+\varepsilon) r_\varepsilon}{\sqrt{c}}$. This would give the required left-hand inequality of the theorem.

Assume, on the contrary, that there exist a decreasing to zero sequence of positive numbers $\{\varepsilon_k\}_{k\in\mathbb{N}}$ and Lipschitz extensions $\bar{F}_{\varepsilon_k} : \mathcal{M} \to \mathbb{H}^n$ of F_{ε_k} such that $L(\bar{F}_{\varepsilon_k}) \leq \frac{(1+\varepsilon_k) r_{\varepsilon_k}}{\sqrt{c}}$, $k \in \mathbb{N}$. Since $m \in F_{\varepsilon_k}(S) \subset U_{\varepsilon_k}$ and for all $m' \in \mathcal{M}$,

$$d_{\mathbb{H}^n}(\bar{F}_{\varepsilon_k}(m'), m) \leq \frac{(1+\varepsilon_k) r_{\varepsilon_k}}{\sqrt{c}} d(m', m^*) \leq \frac{(1+\varepsilon_k) r_{\varepsilon_k}}{\sqrt{c}} r'_{\varepsilon_k},$$

for all sufficiently large k the image $\bar{F}_{\varepsilon_k}(\mathcal{M}) \subset B_{\frac{r_{\varepsilon_k}}{1+\varepsilon_k}}(m) \subset U_{\varepsilon_k}$. Hence, for such k the map $\bar{f}_{\varepsilon_k} := \phi_{\varepsilon_k}^{-1} \circ \bar{F}_{\varepsilon_k} : \mathcal{M} \to B_{r_{\varepsilon_k}}(0) \subset \mathbb{R}^n$ is correctly defined and is Lipschitz with the constant $L(\bar{f}_{\varepsilon_k}) \leq \frac{(1+\varepsilon_k)^2 r_{\varepsilon_k}}{\sqrt{c}} < r_{\varepsilon_k}$ and $\bar{f}_{\varepsilon_k}|_S = f_{\varepsilon_k}$. This contradicts the choice of $\mathcal{M}$ and f_{ε_k}.

The proof of the left-hand inequality of the theorem is done.

We now prove the right-hand inequality. Let $f : S \to \mathbb{H}^n$ be a c-Lipschitz map from a metric subspace S of a metric space $(\mathcal{M}, d)$. We associate to every $m \in \mathcal{M}$ the set

$$A(m) := \bigcap_{m' \in S} \overline{B}_{r(m,m')}(f(m')) \subset \mathbb{H}^n,$$

where

$$r(m, m') := 2c\, d(m, m').$$

A ball of $\mathbb{H}^n$ is convex (i.e., contains every geodesic segment with endpoints lying in the ball). Moreover, $\mathbb{H}^n$ is uniquely geodesic and therefore intersections of convex sets are convex. Hence, $A(m)$ is closed and convex.

Lemma 6.43. *$A(m) \neq \emptyset$ for all m.*

Proof. The Helly index of $\mathbb{H}^n$ is $n + 1$ (see the proof of Valentine's Theorem 1.38 or Appendix C of Volume I). Hence, it suffices to prove for every subset $\{m_1, \ldots, m_k\} \subset S$ with $k \leq n + 1$ that

$$\bigcap_{i=1}^{k} \overline{B}_{r(m,m_i)}(f(m_i)) \neq \emptyset. \tag{6.56}$$

To this end, we choose m_j so that for all $i \leq k$,

$$d(m, m_j) \leq d(m, m_i)$$

and show that $f(m_j)$ belongs to every ball in (6.56). In fact,

$$\begin{aligned} d_{\mathbb{H}^n}(f(m_i), f(m_j)) &\leq cd(m_i, m_j) \leq c(d(m, m_i) + d(m, m_j)) \\ &\leq 2cd(m, m_i) =: r(m, m_i). \end{aligned}$$

The result is established. □

Proposition 6.44. *For all $m, m' \in \mathcal{M}$ the Hausdorff distance between the sets $A(m)$ and $A(m')$ satisfies*

$$d_{\mathcal{H}}(A(m), A(m')) \leq 2\sqrt{2}c\, d(m, m').$$

Proof. For $\emptyset \neq F \subset S$ we write

$$A_F(m) := \bigcap_{m' \in F} \overline{B}_{r(m,m')}(f(m')) \subset \mathbb{H}^n.$$

It suffices to prove that for a fixed point $\hat{m} \in A(m')$,

$$d_{\mathbb{H}^n}(\hat{m}, A_F(m)) \leq 2\sqrt{2}c\, d(m, m') \tag{6.57}$$

provided that F is a finite subset of the cardinality at most $n+1$. Actually, if (6.57) holds, then

$$A_F(m) \bigcap \overline{B}_{\sqrt{2}r(m,m')}(\hat{m}) \neq \emptyset.$$

By the Helly theorem for $\mathbb{H}^n$ this implies that $A(m) \cap \overline{B}_{\sqrt{2}r(m,m')}(\hat{m})$ is not empty or, equivalently, that

$$d_{\mathbb{H}^n}(\hat{m}, A(m)) \leq 2\sqrt{2}c\, d(m, m')$$

for every $\hat{m} \in A(m')$.

Hence, $A(m')$ is contained in the ε-neighborhood $[A(m)]_\varepsilon$ of $A(m)$ with $\varepsilon := 2\sqrt{2}c\, d(m, m')$. By symmetry this implies

$$\begin{aligned} &d_{\mathcal{H}}(A(m), A(m')) \\ &:= \inf\{\varepsilon \geq 0\ ;\ A(m) \subset [A(m')]_\varepsilon \quad \text{and } A(m') \subset [A(m)]_\varepsilon\} \leq 2\sqrt{2}c\, d(m, m'), \end{aligned}$$

as required.

It remains to prove (6.57). Thus let $F \subset S$, card $F \leq n+1$, and a point $\hat{m}$ belongs to $A(m')$ but does not belong to $A(m)$. Let $\tilde{m}$ be a point of $A_F(m)$ closest to $\hat{m}$:

$$d_{\mathbb{H}^n}(\tilde{m}, \hat{m}) = d_{\mathbb{H}^n}(\hat{m}, A_F(m)), \quad \tilde{m} \in A_F(m). \tag{6.58}$$

Then, by the definition of $A_F(m)$, there are points $m_1, \ldots, m_k$ in F, $1 \leq k \leq n+1$, such that

$$d(\tilde{m}, f(m_i)) = 2cd(m, m_i) \quad \text{for } i = 1, \ldots, k, \tag{6.59}$$

while for the remaining points $m' \in F$,

$$d_{\mathbb{H}^n}(\tilde{m}, f(m')) < 2cd(m, m'). \tag{6.60}$$

If $f(m_i) = \tilde{m}$ for some $i \leq k$, then from $\hat{m} \in A(m')$ and (6.58) we get

$$\begin{aligned} d_{\mathbb{H}^n}(\hat{m}, \tilde{m}) &= d_{\mathbb{H}^n}(\hat{m}, f(m_i)) \leq 2cd(m', m_i) \\ &\leq 2c\{d(m, m_i) + d(m, m')\} = 2cd(m, m'). \end{aligned}$$

This clearly implies (6.57) as

$$d_{\mathbb{H}^n}(\hat{m}, A_F(m)) = d_{\mathbb{H}^n}(\hat{m}, \tilde{m}) \leq 2cd(m, m') \leq 2\sqrt{2}c\, d(m, m').$$

Hence, we may and will assume that

$$f(m_i) \neq \tilde{m} \quad \text{for } i = 1, \ldots, k.$$

Now let $T_{\tilde{m}}\mathbb{H}^n$ be the tangent space of $\mathbb{H}^n$ at $\tilde{m}$. It is a Euclidean space with scalar product and norm denoted by $x \cdot y$ and $||x||$. Let $u_0, u_1, \ldots, u_k$ be the unit vectors tangent to the (unique) geodesic segments joining $\tilde{m}$ with $\hat{m}$, $f(m_1), \ldots, f(m_k)$, respectively. These vectors belong to the unit sphere in $T_{\tilde{m}}\mathbb{H}^n$ centered at $0_{\tilde{m}}$.

Lemma 6.45. *For all $i, j \in \{1, \ldots, k\}$,*

$$u_i \cdot u_j \geq \frac{1}{2}. \tag{6.61}$$

Proof. Set $d_i := d(m, m_i)$. Then, by the Lipschitz condition, for $1 \leq i, j \leq k$,

$$d_{\mathbb{H}^n}(f(m_i), f(m_j)) \leq c(d_i + d_j).$$

On the other hand, the cosine inequality[3] for the triangle $\Delta \subset \mathbb{H}^n$ with vertices $\tilde{m}, f(m_i), f(m_j)$, $1 \leq i, j \leq k$, and (6.59) yield

$$\begin{aligned}
&[d_{\mathbb{H}^n}(f(m_i), f(m_j))]^2 \\
&\geq [d_{\mathbb{H}^n}(\tilde{m}, f(m_i))]^2 + [d_{\mathbb{H}^n}(\tilde{m}, f(m_j))]^2 - 2d_{\mathbb{H}^n}(\tilde{m}, f(m_i))d_{\mathbb{H}^n}(\tilde{m}, f(m_j)) \cdot (u_i \cdot u_j) \\
&= (2cd_i)^2 + (2cd_j)^2 - 2(2c)^2 d_i d_j (u_i \cdot u_j).
\end{aligned}$$

Combining this with the previous squared inequality we get (6.61). □

We now need

Lemma 6.46. *There exists $j \in \{1, \ldots, k\}$ such that*

$$u_0 \cdot u_j \leq -\frac{1}{\sqrt{2}}. \tag{6.62}$$

Proof. The origin $0_{\tilde{m}} \in T_{\tilde{m}}\mathbb{H}^n$ belongs to the convex hull $\operatorname{conv}\{u_0, \ldots, u_k\}$. For otherwise by the Hahn-Banach separation theorem there exists a unit vector, say e, such that

$$e \cdot u_i > 0 \quad \text{for } i = 0, 1, \ldots, k.$$

The inequality $e \cdot u_i > 0$ means that the geodesic ray issued from $\tilde{m}$ in the direction e intersects the ball $\overline{B}_{r(m,m_i)}(f(m_i))$ by a nontrivial geodesic segment. Hence, this ray intersects also $\cap_{i=1}^k \overline{B}_{r(m,m_i)}(f(m_i))$ by a nontrivial geodesic segment $[\tilde{m}, \tilde{m}']$ in the direction e. The inequality $u_0 \cdot e > 0$ means that the angle between these vectors is less than $\frac{\pi}{2}$. Then we can find points $m_1 \in [\tilde{m}, \tilde{m}']$, $m_2 \in [\tilde{m}, \hat{m}]$ such that the triangle $\Delta(\tilde{m}, m_1, m_2)$ is isosceles with the smaller side $[m_1, m_2]$, and m_1 is so close to $\tilde{m}$ that $d(m_1, f(m')) < r(m, m')$ for all $m' \in F \setminus \{m_1, \ldots, m_k\}$, see (6.60). Then m_1 belongs to $A_F(m)$ and its distance to $\hat{m}$ satisfies

$$\begin{aligned}
d_{\mathbb{H}^n}(m_1, \hat{m}) &\leq d_{\mathbb{H}^n}(m_1, m_2) + d_{\mathbb{H}^n}(m_2, \hat{m}) \\
&< d_{\mathbb{H}^n}(\tilde{m}, m_2) + d_{\mathbb{H}^n}(m_2, \hat{m}) = d_{\mathbb{H}^n}(\tilde{m}, \hat{m}).
\end{aligned}$$

Since $\tilde{m}$ is a closest to $\hat{m}$ point of $A_F(m)$, this leads to a contradiction.

---

[3] The cosine identity for the comparison triangle $\overline{\Delta} \subset \mathbb{R}^2$ for Δ, see Volume I, Definition 3.109, yields the cosine inequality for Δ, since the angles of Δ are less than the corresponding angles of $\overline{\Delta}$.

Note that this argument also shows that

$$\min_{1\le i\le k} u_0 \cdot u_i \le 0. \tag{6.63}$$

Thus, $0_{\tilde{m}} \in \operatorname{conv}\{u_0, \dots, u_k\}$ and there exist nonnegative numbers μ_i, $i = 0, 1, \dots, k$, such that $\sum_{i=0}^{k} \mu_i = 1$ and

$$\sum_{i=0}^{k} \mu_i u_i = 0_{\tilde{m}}.$$

Squaring we get from here and (6.61)

$$\begin{aligned} 0 &= \sum_{0\le i,j\le k} \mu_i\mu_j(u_i \cdot u_j) \\ &\ge \sum_{i=0}^{k} \mu_i^2 + 2\cdot\frac{1}{2}\sum_{1\le i<j\le k} \mu_i\mu_j + 2\sum_{i=1}^{k}\mu_0\mu_i p \end{aligned} \tag{6.64}$$

where $p := \min\{u_0 \cdot u_i \ ; \ 1 \le i \le k\}$. From here we get

$$0 \ge \mu_0^2 + \frac{1}{2}(1-\mu_0)^2 + 2p\mu_0(1-\mu_0) = \left(\frac{3}{2} - 2p\right)\mu_0^2 - 2\left(\frac{1}{2} - p\right)\mu_0 + \frac{1}{2}.$$

Regarding the right-hand side as a quadratic polynomial in μ_0 and noting that it tends to $+\infty$ together with μ_0 by (6.63), we conclude from the last inequality that this polynomial should have two real roots, i.e., that its discriminant satisfies

$$\left(\frac{1}{2} - p\right)^2 - \left(\frac{3}{4} - p\right) \ge 0,$$

whence $p^2 \ge \frac{1}{2}$. Since $p \le 0$, this proves (6.62). □

Finally, we show that the desired inequality (6.57) is true, i.e., that

$$d_{\mathbb{H}^n}(\hat{m}, \tilde{m}) \le 2\sqrt{2}c\, d(m, m').$$

To this end we consider a Euclidean comparison triangle for the triangle in $\mathbb{H}^n$ with the vertices $\tilde{m}, \hat{m}$ and $f(m_j)$ where m_j satisfies (6.62). We denote by $(\tilde{m})$, $(\hat{m})$, $(f(m_j))$ the corresponding vertices of the comparison triangle and set

$$a := d_{\mathbb{H}^n}(\hat{m}, \tilde{m}), \quad b := d_{\mathbb{H}^n}(\tilde{m}, f(m_j)), \quad c := d_{\mathbb{H}^n}(\hat{m}, f(m_j)).$$

Denote by γ the angle of the comparison triangle at $(\tilde{m})$. Then $\cos\gamma \le u_0 \cdot u_j \le -\frac{1}{\sqrt{2}}$, see (6.62), i.e., $\gamma \ge \frac{3\pi}{4}$. Hence the comparison triangle is obtuse and therefore the projection of $(\tilde{m})$ onto the straight line determined by the side

$[(\tilde{m}), (f(m_j))]$ lies outside of this side. The direct triangle obtained has a leg of length $a\sin(\gamma - \frac{\pi}{2}) + b$ and the hypotenuse c. This implies that

$$a \le \frac{c-b}{\sin(\gamma - \frac{\pi}{2})} = -\frac{c-b}{\cos\gamma} \le \sqrt{2}(c-b).$$

Further, by the choice of $\hat{m}$ it belongs to $A(m') \subset \overline{B}_{r(m',m_j)}(f(m_j))$ and, moreover, $d(\tilde{m}, f(m_j)) = 2cd(m, m_j)\ (:= r(m, m_j))$, see (6.59). Together with the previous inequality this yields

$$\begin{aligned} a &:= d_{\mathbb{H}^n}(\hat{m}, \tilde{m}) \le \sqrt{2}(d_{\mathbb{H}^n}(\hat{m}, f(m_j)) - d_{\mathbb{H}^n}(\tilde{m}, f(m_j)) \\ &\le \sqrt{2}(r(m', m_j) - r(m, m_j)) \le 2\sqrt{2}c(d(m', m_j) - d(m, m_j)) \le 2\sqrt{2}c\, d(m, m'). \end{aligned}$$

This proves (6.57) and Proposition 6.44. □

The final part of the proof works with the metric space $(\mathcal{K}(\mathbb{H}^n), d_{\mathcal{H}})$ of compact subsets in $\mathbb{H}^n$ equipped with the Hausdorff metric.

Proposition 6.47. *There exists a Lipschitz map $\sigma : (\mathcal{K}(\mathbb{H}^n), d_{\mathcal{H}}) \to \mathbb{H}^n$ such that*

(i) *$\sigma(\{x\}) = x$ for all $x \in \mathbb{H}^n$;*

(ii) *the Lipschitz constant of σ is bounded by n.*

Proof. Recall that $\mathbb{H}^n$ is identified with the upper half-space model with the underlying set $\{x \in \mathbb{R}^n\ ;\ x_n > 0\}$ and the Riemannian metric $x_n^{-2}\sum_{i=1}^n dx_i^2$. For a compact set $S \subset \mathbb{H}^n$ by $H(S)$ one denotes the upper horizontal hyperplane in $\mathbb{H}^n$ tangent to S. Let $\pi_S : \mathbb{H}^n \to H(S)$ be the orthogonal projection. Then we define the required map $\sigma : (\mathcal{K}(\mathbb{H}^n), d_{\mathcal{H}}) \to \mathbb{H}^n$ at S by the formula

$$\sigma(S) := \sum_{i=1}^{n-1} h_{\pi_S(S)}(e_i)e_i \tag{6.65}$$

where $\{e_1, \dots, e_{n-1}\}$ is the standard basis in $\mathbb{R}^{n-1}$ naturally identified with $H(S)$ and $h_{\pi_S(S)}$ is the support function of the set $\pi_S(S)$, see Volume I, formula (5.91) for its definition. Clearly, $\sigma(\{x\}) = x$ for all $x \in \mathbb{H}^n$. Now, using the fact the the projection onto $H(S)$ is 1-Lipschitz in the hyperbolic metric let us show that $\sigma : (\mathcal{K}(\mathbb{H}^n), d_{\mathcal{H}}) \to \mathbb{H}^n$ is n-Lipschitz.

Let $S_0, S_1 \subset \mathbb{H}^n$ be compact subsets. Assume that $H(S_0)$ is over $H(S_1)$. In what follows we set $\widehat{S}_i := \pi_{S_0}(\widehat{S}_i)$, $i = 0, 1$ and use the inequality

$$d_{\mathbb{H}^n}(\sigma(S_0), \sigma(S_1)) \le d_{\mathbb{H}^n}(\sigma(S_0), \pi_{S_0}(\sigma(S_1))) + d_{\mathbb{H}^n}(\pi_{S_0}(\sigma(S_1)), \sigma(S_1)). \tag{6.66}$$

Lemma 6.48.

$$d_{\mathbb{H}^n}(\sigma(S_0), \pi_{S_0}(\sigma(S_1))) \le (n-1)d_{\mathcal{H}}(S_0, S_1).$$

Proof. As before we naturally identify $H(S_0)$ with $\mathbb{R}^{n-1}$. By d_{n-1} we denote the Euclidean metric on $\mathbb{R}^{n-1}$ and by $\hat{d}_{\mathcal{H}}$ the corresponding Hausdorff metric on the space of compact subsets of $\mathbb{R}^{n-1}$. Let $H(S_0) := \{x \in \mathbb{R}^n_+ \ ; \ x_n = k\}$ for some $k > 0$. Then by the definition of the hyperbolic metric on $\mathbb{H}^n$ we have

$$\begin{aligned} d_{\mathbb{H}^n}(\sigma(S_0), \pi_{S_0}(\sigma(S_1))) &:= d_{\mathbb{H}^n}\left(\sum_{i=1}^{n-1} h_{\widehat{S}_0}(e_i)e_i, \sum_{i=1}^{n-1} h_{\widehat{S}_1}(e_i)e_i\right) \\ &= \cosh^{-1}\left(1 + \frac{\sum_{i=1}^{n-1}(h_{\widehat{S}_0}(e_i) - h_{\widehat{S}_1}(e_i))^2}{2k^2}\right). \end{aligned}$$

Applying inequality (5.95) of Chapter 5 of Volume I we get

$$|h_{\widehat{S}_0}(e_i) - h_{\widehat{S}_1}(e_i)| \le \hat{d}_{\mathcal{H}}(\widehat{S}_0, \widehat{S}_1), \quad 1 \le i \le n-1.$$

This and the previous inequality imply

$$d_{\mathbb{H}^n}(\sigma(S_0), \pi_{S_0}(\sigma(S_1))) \le \cosh^{-1}\left(1 + (n-1) \cdot \frac{[\hat{d}_{\mathcal{H}}(\widehat{S}_0, \widehat{S}_1)]^2}{2k^2}\right).$$

Using the fact that $\cosh^{-1}$ is an increasing concave function on $[1, \infty)$ equals 0 at 1 by the definition of the Hausdorff metric, see Volume I, formula (1.30), we obtain from the last inequality

$$\begin{aligned} &d_{\mathbb{H}^n}(\sigma(S_0), \pi_{S_0}(\sigma(S_1))) \le (n-1)\cosh^{-1}\left(1 + \frac{[\hat{d}_{\mathcal{H}}(\widehat{S}_0, \widehat{S}_1)]^2}{2k^2}\right) \\ &= (n-1)\cosh^{-1}\left(1 + \frac{1}{2k^2} \cdot \left(\max_{i=0,1}\left\{\sup\{d_{n-1}(m_i, \widehat{S}_{1-i}) \ ; \ m_i \in \widehat{S}_i\}\right\}\right)^2\right) \\ &= (n-1)\max_{i=0,1}\left\{\sup\left\{\cosh^{-1}\left(1 + \frac{1}{2k^2} \cdot \left(d_{n-1}(m_i, \widehat{S}_{1-i})\right)^2\right) \ ; \ m_i \in \widehat{S}_i\right\}\right\} \\ &= (n-1)\max_{i=0,1}\left\{\sup\left\{d_{\mathbb{H}^n}(m_i, \widehat{S}_{1-i}) \ ; \ m_i \in \widehat{S}_i\right\}\right\} =: (n-1)d_{\mathcal{H}}(\widehat{S}_0, \widehat{S}_1). \end{aligned}$$

Since $\pi_{S_0} : \mathbb{H}^n \to \mathbb{H}^n$ has Lipschitz constant 1,

$$d_{\mathcal{H}}(\widehat{S}_0, \widehat{S}_1) \le d_{\mathcal{H}}(S_0, S_1).$$

This and the previous inequality prove the lemma. □

Lemma 6.49.

$$d_{\mathbb{H}^n}(\pi_{S_0}(\sigma(S_1)), \sigma(S_1)) \le d_{\mathcal{H}}(S_0, S_1).$$

Proof. By definition $d_{\mathbb{H}^n}(\pi_{S_0}(\sigma(S_1)), \sigma(S_1))$ equals the hyperbolic distance, say D, between $H(S_1)$ and $H(S_0)$. Assume, on the contrary, that $D > d_{\mathcal{H}}(S_0, S_1)$. Let

$L(S_1) \subset \mathbb{H}^n$ be the closed strip with boundaries $H(S_1)$ and $\mathbb{R}^{n-1}$ (the boundary of $\mathbb{R}^n_+$). Then $S_1 \subset L(S_1)$ and hence for some r satisfying $d_{\mathcal{H}}(S_0, S_1) < r < D$ the (hyperbolic) r-neighborhood of $L(S_1)$ contains S_0. Since $H(S_0) \cap S_0 \neq \emptyset$, this neighborhood contains $H(S_0)$. Thus the hyperbolic distance between $H(S_0)$ and $H(S_1)$ is less than D, a contradiction. □

Now the required result of the proposition follows from (6.66) and Lemmas 6.48, 6.49. □

Now we complete the proof of Theorem 6.42 as follows. Since $A(m) = \{f(m)\}$ for all $m \in S$, the composition $\sigma \circ A$ is by (i) an extension of f to the whole of $\mathcal{M}$. Moreover, its Lipschitz constant by Proposition 6.47 satisfies

$$L(\sigma \circ A) \leq L(\sigma)L(A) \leq 2\sqrt{2}\, n,$$

as required. □

6.5.2 Lipschitz maps between Banach spaces

Modern study of the Lipschitz extension problem for these settings begins with the papers of Marcus and Pisier [MP-1984] and Johnson and Lindenstrauss [JL-1984]. In particular, Theorem 3 of the latter paper implies that for $1 \leq p < 2$,

$$\Lambda(L_p, L_2) = \infty. \tag{6.67}$$

In [JL-1984], the authors looked for confirmation of the validity of the Maurey extension theorem for linear operators [Mau-1974] in the case of Lipschitz maps. Should the answer be positive, the following inequalities would be true:

$$\Lambda(L_p, L_2) < \infty \quad \text{for } 2 \leq p < \infty.$$

The breakthrough in solving this problem was obtained in the Ball paper [B-1992] where he introduced the notion of *Markov p-type* of a metric space $\mathcal{M}$ denoted by $\mu_p(\mathcal{M})$. The basic Ball result, a nonlinear analog of Maurey's theorem, asserts

Theorem 6.50. *Let $\mathcal{M}$ be metric and X be Banach spaces. Then*

$$\Lambda(\mathcal{M}, X) \leq 6\mu_2(\mathcal{M}) \cdot \kappa_2(X)$$

where $\kappa_2(X)$ is the 2-convexity constant of X.

Let us recall, see, e.g., [LT-1979], that $\kappa_2(X)$ is the least κ in the inequality

$$2||x||^2 + \frac{2}{\kappa^2}\, ||y||^2 \leq ||x+y||^2 + ||x-y||^2$$

holding for all $x, y \in X$.

In particular, it was shown by Ball, Carlen and Lieb [BCL-1994] that

$$\kappa_2(L_p) \le \frac{1}{\sqrt{p-1}} \quad \text{for } 1 < p \le 2.$$

Moreover, Ball [B-1992] proved that $\mu_2(L_2) = 1$. These facts lead to the inequality

$$\Lambda(L_2, L_p) \le \frac{6}{\sqrt{p-1}} \quad \text{for } 1 < p \le 2.$$

The next important step is due to Naor, Peres, Schramm and Sheffield [NPSS-2006] who for the case of Banach spaces estimated $\mu_q(X)$ with $1 < q \le 2$ by the *q-smoothness constant* $\sigma_q(X)$.

Let us recall, see, e.g., [LT-1979], that $\sigma_q(X)$ is the least $\sigma > 0$ such that for every $x, y \in X$,

$$||x-y||^q + ||x+y||^q \le 2||x||^q + 2\sigma^q ||y||^q.$$

In the aforementioned paper [BCL-1994] it was shown that

$$\sigma_2(L_p) \le \sqrt{p-1} \quad \text{for } 2 \le p < \infty.$$

Together with the above formulated Ball theorem, this leads to the following

Theorem 6.51 ([NPSS-2006]). *For every two Banach spaces X, Y,*

$$\Lambda(X,Y) \le 24\sigma_2(X) \cdot \kappa_2(Y).$$

In particular, for $2 \le p < \infty$ and $1 < q \le 2$,

$$\Lambda(L_p, L_q) \le 24\sqrt{\frac{p-1}{q-1}}. \tag{6.68}$$

The authors of the result conjecture that the constant 24 is redundant. If true, this would give a generalization of the classical Kirszbraun theorem (asserting that $\Lambda(L_2, L_2) = 1$).

Remark 6.52. Earlier, Tsarkov [Ts-1999] proved that $\Lambda(X, L_2) < \infty$ for a Banach space X with $\sigma_2(X) < \infty$ (in particular, for L_p with $2 \le p < \infty$) using a direct geometric approach.

The above presented result of Johnson and Lindenstrauss, see (6.67), and the recent result of Naor [N-2001] asserting that

$$\Lambda(L_2, L_p) = \infty \quad \text{for } 2 < p < \infty,$$

show that the left-hand side in (6.68) may be infinite for $1 < q \le 2 \le p < \infty$.

There are many other unsolved problems in this field. We formulate only two of them.

Problem. *Is $\Lambda(L_2, L_1)$ finite or infinite?*

Problem. *For $1 \le p, q \le \infty$ satisfying $\Lambda(L_p, L_q) = \infty$, find the asymptotic behavior of $\Lambda(\ell_p^n, \ell_q^n)$ as $n \to \infty$.*

6.5.3 Extensions preserving Lipschitz constants

There exist only a few results extending the classical Kirszbraun and Valentine theorems presented in Chapter 1. We begin with the following far-reaching generalization of these results due to Lang and Schroeder [LSch-1997].

Theorem 6.53. *Let $(\mathcal{M}, d)$ and $(\mathcal{M}', d')$ be complete geodesic metric spaces and the latter be simply connected. Assume that for some $\kappa \in \mathbb{R}$ the Alexandrov curvature of $\mathcal{M}'$ is at most κ and that of $\mathcal{M}$ is at least κ. Let S be an arbitrary subset of $\mathcal{M}$ and $f : S \to \mathcal{M}'$ be 1-Lipschitz with* $\operatorname{diam} f(S) \le D_{\kappa/2}$ *where $D_\kappa := \frac{\pi}{2\sqrt{\kappa}}$ for $\kappa > 0$ and $D_\kappa := +\infty$ otherwise.*

Then there exists a 1-Lipschitz extension $\bar{f} : \mathcal{M} \to \mathcal{M}'$ of f.

The proof requires many technical details concerning the geometry of Alexandrov spaces (see Definition 3.109 and the consequent text in subsection 3.3.3 of Volume I for several basic facts in this field). Therefore we restrict ourselves only to brief comments.

It suffices to extend f to one point $m \in \mathcal{M} \setminus S$ and to proceed inductively. Further, by a Helly type argument (see Appendix C) it suffices to prove this for a finite set S.

As in the proof of Theorem 6.42 we set

$$A_\alpha(m) := \bigcap_{m' \in S} \overline{B}_{r(m',m)}(f(m')) \subset \mathcal{M}',$$

where $r(m, m') := \alpha\, d(m, m')$ and $\alpha \ge 0$.

Let $\alpha_0 := \inf\{\alpha \ge 0 \ ; \ A_\alpha(m) \ne \emptyset\}$. It can be shown that $\alpha_0 < \infty$ and $A_{\alpha_0}(m)$ contains a single point, say $\bar{m}$. Set $\bar{f}(m) := \bar{m}$. Then $\bar{f}$ is an optimal extension of f in the sense that its Lipschitz constant $L(\bar{f})$ is minimal among all other extensions.

Now we enumerate points of $S := \{m_1, \dots, m_n\}$ so that for some $k \ge 1$

$$d'(\bar{f}(m), f(m_i)) = L(\bar{f}) d(m, m_i)$$

for $i = 1, \dots, k$ and

$$d'(\bar{f}(m), f(m_i)) < L(\bar{f}) d(m, m_i)$$

for all $i > k$.

Comparing the set of directions from m to m_i, $1 \le i \le k$, with that from $\bar{m}$ to $f(m_i)$, $1 \le i \le k$, using an argument similar to that of Theorem 6.42 but exploiting also the known geometrical structure of the Alexandrov space $\mathcal{M}$, one can prove that $L(\bar{f}) = 1$, as required.

Remark 6.54. (a) In the case $\kappa = 0$, Theorem 6.53 is true for an arbitrary Lipschitz constant (scale either the metric of $\mathcal{M}$ or that of $\mathcal{M}'$). Hence, the Kirszbraun theorem and its generalization to Hilbert spaces, see Volume I, Corollary 1.37, are consequences of Theorem 6.53.

(b) Clearly, the Valentine Theorem 1.38 of Volume I follows from Theorem 6.53 even for the case of the pair $(\mathbb{H}^n, \mathbb{H}^m)$, $2 \le n, m$. However, the diameter restriction does not allow to derive from Theorem 6.53 the second Valentine result concerning the pair $(\mathbb{S}^n, \mathbb{S}^m)$, $1 \le m \le n$, see Volume I, Theorem 1.40 and Corollary 1.42. In this case, the restriction of Theorem 6.53 means that $f(S)$ lies in an open hemisphere.

(c) The bound for $\operatorname{diam} f(S)$ is, in general, sharp. For example, let $\mathcal{M}$ be the Euclidean sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ with the canonical length metric and $\mathcal{M}' := \mathbb{S}^{n-1} = \mathbb{S}^n \cap \mathbb{R}^n$, $n \ge 2$. Let $S \subset \mathbb{S}^n$ be the set of vertices for a regular n-simplex inscribed in $\mathbb{S}^{n-1}$ and $f : S \to \mathcal{M}'$ be the identity map. Let $x \in \mathbb{S}^n$ be one of the poles at distance $\frac{\pi}{2}$ from S. Since every hemisphere of $\mathbb{S}^{n-1}$ contains an image point of f, there is no 1-Lipschitz extension $\bar{f} : S \cup \{x\} \to \mathbb{S}^{n-1}$ of f.

In this example, $\kappa = 1$ and $\operatorname{diam} f(S) = \arccos\left(-\frac{1}{n}\right) \to \frac{\pi}{2} =: D_1$ as $n \to \infty$.

A version of Theorem 6.53 holds also for $\kappa = -\infty$ (no curvature assumptions). More precisely, the following is true.

Theorem 6.55. *Let $(\mathcal{M}, d)$ be a metric space and $(\mathcal{M}', d')$ be a complete metric $\mathbb{R}$-tree. Then every c-Lipschitz map $f : S \to \mathcal{M}'$ from a subset $S \subset \mathcal{M}$ admits a c-Lipschitz extension $\bar{f} : \mathcal{M} \to \mathcal{M}'$.*

Proof. Without loss of generality, assume that $L(f) = 1$. As before, it suffices to extend f to one additional point $m \in \mathcal{M} \setminus S$. Further, to find $\bar{f}(m)$ it suffices to show that

$$A(m) := \bigcap_{m' \in S} \overline{B}_{d(m,m')}(f(m')) \subset \mathcal{M}' \tag{6.69}$$

is nonempty and then set $\bar{f}(m) := \bar{m} \in A(m)$.

To make this we use a Helly type theorem for $\mathbb{R}$-trees.

Lemma 6.56. *Let $\mathcal{F}$ be a nonempty finite collection of bounded convex sets in an $\mathbb{R}$-tree $(\mathcal{T}, d_{\mathcal{T}})$ such that any two of these sets intersect, then $\cap \mathcal{F} \neq \emptyset$.*

Proof. Suppose first that $\mathcal{F} = \{F_1, F_2, F_3\}$. Let $t_1 \in F_2 \cap F_3$, $t_2 \in F_1 \cap F_3$ and $t_3 \in F_1 \cap F_2$. Since there are no cycles in $\mathcal{T}$, it follows that the tree paths, one joining t_1 to t_2, one joining t_2 to t_3 and one joining t_1 to t_3, intersect. This intersection point will be in $F_1 \cap F_2 \cap F_3$. Now, suppose that $\mathcal{F} = \{F_1, F_2, \dots, F_n\}$, where $n > 3$. By the previous case, we know that any two sets in the collection $\mathcal{F}' := \{F_2 \cap F_1, F_3 \cap F_1, \dots, F_n \cap F_1\}$ intersect. The induction hypothesis then implies that the intersection of all sets from $\mathcal{F}'$ is nonempty, which implies that $\cap \mathcal{F} \neq \emptyset$. □

However, to prove a result similar to Lemma 6.56 for the infinite family of balls in (6.69) one should in addition use the completeness of $\mathcal{M}'$. Let us denote by $\overline{B}_{m'} \subset \mathcal{M}'$ the ball $\overline{B}_{d(m,m')}(f(m'))$. Every two such balls, say $B_{m'}$ and $B_{m''}$, have a nonempty intersection, since the distance between their centers satisfies

$$d'\big(f(m'), f(m'')\big) \le d(m', m'') \le d(m, m') + d(m, m''),$$

i.e., does not exceed the sum of their radii. Moreover, balls of an $\mathbb{R}$-tree are clearly convex. Then by Lemma 6.56, every finite subcollection of $\{\overline{B}_{m'}\}_{m' \in S}$ has a nonempty intersection.

Let us show that this intersection is a (nonempty) closed ball. It suffices to check that the (nonempty) intersection of two closed balls, say $\overline{B}_r(m)$ and $\overline{B}_{r'}(m')$, of an $\mathbb{R}$-tree $(\mathcal{T}, d_{\mathcal{T}})$ none of which contains the other is a closed ball. A direct computation using the degenerate triangle inequality for an $\mathbb{R}$-tree shows that the intersection is a closed ball centered at a unique point c such that

$$\begin{aligned} d_{\mathcal{T}}(c, m) &= \frac{1}{2}\left[d_{\mathcal{T}}(m, m') + r - r'\right], \\ d_{\mathcal{T}}(c, m') &= \frac{1}{2}\left[d_{\mathcal{T}}(m, m') + r' - r\right]. \end{aligned} \tag{6.70}$$

Moreover, the radius of this ball equals $\frac{1}{2}[r + r' - d_{\mathcal{T}}(m, m')]$ (it may be zero).

Now let $S' \subset S$ be a finite set. By $\overline{B}(S')$ denote the closed ball $\bigcap_{m' \in S'} \overline{B}_{m'}$ and by $r(S')$ its radius. Set

$$r(S) := \inf\{r(S')\ ;\ S' \subset S \text{ is finite}\}$$

and let $\{S_j\}$ be an increasing sequence of finite subsets of S such that

$$r(S) = \lim_{j \to \infty} r(S_j).$$

Let c_j be the center of the ball $\overline{B}(S_j)$, and let $r_j := r(S_j)$ be its radius. If, first, $r(S) > 0$, then the above formula for the radius of the intersection of two balls shows that $c_j \in \overline{B}_{m'}$ for every $m' \in S$ and for every j such that $r_j < 2r(S)$. Thus, $\bigcap_{m' \in S} \overline{B}_{m'} \ne \emptyset$ in this case.

If now $r(S) = 0$, then r_j decreasingly tends to zero and therefore $\{c_j\}$ is a Cauchy sequence. Since $\mathcal{M}'$ is complete, $\lim_{j \to \infty} c_j$ exists. Moreover, the limit point belongs to every ball $\overline{B}_{m'}$, $m' \in S$, because $\overline{B}(\{m'\} \cup S_j) \ne \emptyset$ for every j. This implies that $\bigcap_{m' \in S} \overline{B}_{m'} \ne \emptyset$ and completes the proof of the theorem. □

Finally we present a sharp extension result for Lipschitz maps of ultrametric spaces.

Theorem 6.57. *Let $(\mathcal{M}, d)$ be an ultrametic space and $(\mathcal{M}', d')$ be a proper metric space. Then every c-Lipschitz map $f : S \to \mathcal{M}'$ from a subset $S \subset \mathcal{M}$ admits a c-Lipschitz extension $\bar{f} : \mathcal{M} \to \mathcal{M}'$.*

Proof. Without loss of generality we assume that $L(f) = 1$. As before, it suffices to prove that

$$A(m) := \bigcap_{m' \in S} \overline{B}_{d(m,m')}(f(m')) \subset \mathcal{M}' \tag{6.71}$$

is nonempty and then set $\bar{f}(m) := \bar{m} \in A(m)$. Further, since by our assumptions each $\overline{B}_{d(m,m')}(f(m'))$ is a compact subset of $\mathcal{M}'$, it suffices to prove that $A_{S'}(m) := \bigcap_{m' \in S'} \overline{B}_{d(m,m')}(f(m')) \neq \emptyset$ for every finite set $S' \subset S$.

To this end, let $m' \in S'$ be such that $d(m, S') = d(m, m')$. Then we have

$$f(m') \in A_{S'}(m).$$

Indeed, from the strong triangle inequality for d we obtain

$$d'(f(m'), f(m'')) \leq d(m', m'') \leq \max\{d(m, m'), d(m, m'')\} = d(m, m'')$$

for all $m'' \in S'$. Hence $f(m'') \in \overline{B}_{d(m,m'')}(f(m''))$ for all $m'' \in S'$, as required.

This completes the proof of the theorem. □

Remark 6.58. Using similar arguments one can prove the following result:

Let $(\mathcal{M}, d)$ be a separable ultrametic space and $(\mathcal{M}', d')$ be a metric space. Then for any $\varepsilon > 0$ every c-Lipschitz map $f : S \to \mathcal{M}'$ from a subset $S \subset \mathcal{M}$ admits a $c(1+\varepsilon)$-Lipschitz extension $\bar{f} : \mathcal{M} \to \mathcal{M}'$.

In the case of a proper ultrametric space, one can take here $\varepsilon = 0$.

Comments

The material of Sections 6.1–6.3, along with the discussion on the Nagata dimension in Section 4.2 of Volume I, is a detailed presentation on the Lang–Schlichenmaier theory whose rather compressed account is contained in the paper [LSchl-2005]. Yet, the notion of Whitney's cover and Theorem 6.32 on simultaneous extensions was not formulated explicitly in the paper.

A topological counterpart of the theory in question is the Hurewicz theory on continuous extensions of maps from metric spaces into n-spheres (or, more generally, n-connected topological spaces). Fragments of the latter are presented in Chapter 1 of Volume I, see Theorem 1.9 and Appendix A there. Comparison of these theories poses several interesting problems. For instance, one may seek analogs for the Nagata dimension of the separation theorems for topological dimension presented in Appendix A.

Because of its generality, the Lang–Schlichenmaier theory gives, in principle, results of a rather qualitative nature. As an example, one considers the estimate of the extension constant for maps acting from a doubling metric space with the doubling constant $\delta > 1$ to a Banach space. In this case, the estimate of the theory is $O(\delta^c)$ where c is a (large) numerical constant, while the theory presented in Chapter 7 gives $O(\log \delta)$.

The results presented in Sections 6.4–6.6 lead to the following two conjectures.

Conjecture. *Let S be a nonempty closed subspace of a metric space $\mathcal{M}$ such that either $\dim_N S \le n-1$ or $\dim_N S^c \le n$ with constant $\nu > 0$ in both cases. Let $\mathcal{M}'$ be a Lipschitz n-connected Hadamard space with the constant $\mu > 0$.*

There exist constants $c = c(\mu, \mu') > 0$ and $a = a(\mu, \mu') > 0$ such that the extension constant $\Lambda(S, \mathcal{M}, \mathcal{M}')$ is bounded by cn^a.

Conjecture. $\Lambda(\mathbb{H}^n) \approx \sqrt{n}$ *as $n \to \infty$.*

The Daugavet result used in the proof of Proposition 6.40 is contained in his now hardly available paper [Dau-1968]. Fortunately, a detailed account of this theorem can be found in the recent book [Woj-1991] by Wojtaszchyk. The Daugavet result is based on the so-called Marcinkiewicz (–Zygmund)–Berman identity [Ber-1960].

Theorem 6.57 is due to the authors.

Chapter 7

Simultaneous Lipschitz Extensions

The chapter is designed for the study of metric spaces admitting simultaneous Lipschitz extensions for each of their subspaces. The theory of such spaces developed by the authors of the book will be presented here while some of its basic ingredients (spaces of pointwise homogeneous type and of metric balls, embeddings in spaceforms etc.) have been considered in Volume I. This allows us to simplify and shorten the exposition. The theory includes methods of constructing the corresponding linear extension operators. For the time being, three different methods have been developed independently and at the same time. The first of them, due to Lang and Schlichenmaier, was presented in Chapter 6. Because of the generality of their approach working both in nonlinear and linear settings, the corresponding estimates of extension constants are far from being optimal. Two other methods are much better in this respect, giving estimates close to optimal. One of them, due to Lee and Naor, exploits a probabilistic argument based on the stochastic "padding decompositions" that first appeared in Computer Science, see, e.g., Volume I, Section 4.1. Another one, due to the authors of the present book, is constructive and covers a wider class of metric spaces.

Now we briefly describe the content of the chapter.

Section 7.1 studies the basic characteristics of the spaces in question. The most important of them, the weak finiteness property, reduces the problem of constructing linear extension operators to that for finite metric spaces. This result is then exploited in a few situations here and in the forthcoming chapter.

Section 7.2 presents a construction of a linear extension operator for Lipschitz functions over metric spaces of pointwise homogeneous type. So, these spaces have the simultaneous Lipschitz extension property, the class of all spaces possessing this property is denoted by $\mathcal{SLE}$. As a consequence we prove that combinatorial metric trees, doubling metric spaces and direct sums of arbitrary finite combi-

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nations of these spaces belong to $\mathcal{SLE}$ and we obtain effective estimates of the corresponding extension constants.

In Section 7.3, we prove that, under mild restrictions, locally doubling metric spaces containing uniform lattices belong to $\mathcal{SLE}$.

In Section 7.4, we study metric spaces having the universal simultaneous Lipschitz extension property (shortly, this class is denoted by $\mathcal{ULE}$). This means that the isometric copy of a subspace of such a space $\mathcal{M}$ sitting in another metric space $\widetilde{\mathcal{M}}$ admits a linear continuous Lipschitz extension to $\widetilde{\mathcal{M}}$ with the extension constant bounded by $c = c(\mathcal{M})$. In particular, Gromov hyperbolic spaces of bounded geometry, combinatorial metric trees, doubling metric spaces and direct sums of their arbitrary finite combinations belong to $\mathcal{ULE}$. For single trees and doubling spaces this result was firstly established by Lee and Naor using the aforementioned probabilistic approach. Though nonconstructive, the method of these authors is of considerable interest. We briefly discuss its basic features at the final part of the section.

7.1 Characterization of simultaneous Lipschitz extension spaces

7.1.1 Basic notions

Let $(\mathcal{M}, d)$ be a metric space and X be a Banach space with norm $||\cdot||$. Recall that the space $\mathrm{Lip}(\mathcal{M}, X)$ of Banach-valued Lipschitz functions $f : \mathcal{M} \to X$ is defined by the seminorm

$$L(f; X) := \sup_{m \neq m'} \frac{||f(m) - f(m')||}{d(m, m')}.$$

In the case $X = \mathbb{R}$, we denote the corresponding space by $\mathrm{Lip}(\mathcal{M})$.

Further, we recall, see Volume I, Section 1.12,

Definition 7.1. A subset $S \subset \mathcal{M}$ admits a simultaneous Lipschitz extension with respect to X if there exists a linear bounded operator $T : \mathrm{Lip}(S, X) \to \mathrm{Lip}(\mathcal{M}, X)$ such that

$$Tf|_S = f, \quad f \in \mathrm{Lip}(S, X).$$

The set of all such operators is denoted by $\mathrm{Ext}(S, \mathcal{M}, X)$ and the optimal extension constant is given by

$$\lambda(S, \mathcal{M}, X) := \inf\{||T|| \ ; \ T \in \mathrm{Ext}(S, \mathcal{M}, X)\}.$$

(We assume that $\lambda(S, \mathcal{M}, X) = \infty$ if $\mathrm{Ext}(S, \mathcal{M}, X) = \emptyset$.) In the case $X = \mathbb{R}$, we denote the corresponding set and the constant by $\mathrm{Ext}(S, \mathcal{M})$ and $\lambda(S, \mathcal{M})$.

Definition 7.2. A metric space $(\mathcal{M}, d)$ has the simultaneous Lipschitz extension property (abbreviated $\mathcal{SLE}$), if $\mathrm{Ext}(S, \mathcal{M}) \neq \emptyset$ for each subspaces $S \in \mathcal{M}$.

Finally, we define the *global linear Lipschitz extension constant*

$$\lambda(\mathcal{M}, X) := \sup\{\lambda(S, \mathcal{M}, X)\ ;\ S \subset \mathcal{M}\}.$$

In the case $X = \mathbb{R}$, we denote this constant by $\lambda(\mathcal{M})$.

Our first result connects $\lambda(\mathcal{M}, X)$ and $\lambda(\mathcal{M})$ for certain Banach spaces X. Recall that a Banach space X is said to be *constrained in its bidual* if X is the range of a norm one linear projection when canonically embedded in its bidual X^{**}. It is well known (see Dixmier [Di-1948]) that an L_1-space or a dual Banach space X (i.e., $X = Y^*$ for a Banach space Y) meets this condition.

Proposition 7.3. *If X is constrained in its bidual, then for every subspace S of a metric space $(\mathcal{M}, d)$,*

$$\lambda(S, \mathcal{M}, X) = \lambda(S, \mathcal{M}).$$

In particular, it is true that

$$\lambda(\mathcal{M}, X) = \lambda(\mathcal{M}).$$

Proof. First we show that the constant in the left-hand side is bounded by that on the right. Since the latter remains unchanged after replacing $\mathrm{Lip}(\mathcal{M})$ by its subspace $\mathrm{Lip}_0(\mathcal{M})$ (containing functions vanishing at a fixed point of $\mathcal{M}$, say, m^*), we will work with the latter space and its canonical predual $\mathcal{F}(\mathcal{M})$ introduced in subsection 4.6.3 of Volume I. Moreover, S is now a *pointed subset* of $(\mathcal{M}, m^*, d)$.

Let $\lambda(S, \mathcal{M}) < \infty$ and $\epsilon > 0$ be fixed. Take an operator $T \in \mathrm{Ext}(S, \mathcal{M})$ with

$$||T|| \leq \lambda(S, \mathcal{M}) + \epsilon.$$

The conjugate operator T^* acts from $\mathrm{Lip}_0(\mathcal{M})^* = \mathcal{F}(M)^{**}$ into $\mathrm{Lip}_0(S)^* = \mathcal{F}(S)^{**}$. It is a matter of definition to check that T^* is a *linear continuous projection* of $\mathcal{F}(M)^{**}$ onto $\mathcal{F}(S)^{**}$. Since $\mathcal{F}(\mathcal{M})$ is isometrically embedded into $\mathcal{F}(\mathcal{M})^{**}$ and $\mathcal{M}$ is isometrically embedded into $\mathcal{F}(\mathcal{M})$, see Volume I, subsection 4.6.3, we may and will regard $\mathcal{M}$ as a (metric) subspace of $\mathcal{F}(\mathcal{M})^{**}$. Then $T^*|_{\mathcal{M}}$ is well-defined and is a Lipschitz map from $\mathcal{M}$ into $\mathcal{F}(\mathcal{M})^{**}$ such that

$$L(T^*|_{\mathcal{M}}\ ;\ \mathcal{F}(S)^{**}) \leq ||T||.$$

Using these facts we construct a linear extension operator $\widehat{T} : \mathrm{Lip}_0(S, X) \to \mathrm{Lip}_0(\mathcal{M}, X)$ satisfying $||\widehat{T}|| \leq \lambda(\mathcal{M}, S) + \epsilon$. This, clearly, implies the required inequality

$$\lambda(S, \mathcal{M}, X) \leq \lambda(S, \mathcal{M}).$$

To introduce $\widehat{T}$ we fix $f \in \mathrm{Lip}_0(S, X)$. According to Theorem 4.91 of Volume I, there exists a linear operator $T_f : \mathcal{F}(S) \to X$ such that

(i) $T_f|_S = f$;

(ii) T_f linearly depends on f;

(iii) $||T_f|| \le L(f; X)$.

We then have for $T_f^{**} : \mathcal{F}(S)^{**} \to X^{**}$,

$$T_f^{**}|_S = T_f|_S = f.$$

Since X is constrained in its bidual, there exists a linear projection $P : X^{**} \to X$ of norm 1. Composing all the operators introduced we define the function

$$\widehat{T}(f) := P \circ T_f^{**} \circ (T^*|_{\mathcal{M}})$$

acting from $\mathcal{M}$ into X. By linearity of T_f in f, the operator $f \mapsto \widehat{T}(f)$ is linear. Moreover, $\widehat{T}(f)|_S = f$ and $L(\widehat{T}(f); X) \le ||T||L(f; X)$ by definition. Hence $\widehat{T} \in \mathrm{Ext}(S, \mathcal{M}, X)$ and its norm is bounded by $||T|| \le \lambda(S, \mathcal{M}) + \epsilon$, as required.

Conversely, assume that $\lambda(S, \mathcal{M}, X) < \infty$. Let $V \subset X$ be a one-dimensional subspace of X and $v \in V$ be such that $||v|| = 1$. According to the Hahn-Banach theorem there is a linear continuous functional $F_v : X \to \mathbb{R}$ of norm 1 such that $F_v(v) = 1$. We embed $\mathbb{R}$ isometrically into X by the formula $E(t) := t \cdot v, \quad t \in \mathbb{R}$. The linear operator $\widetilde{E}f := E \circ f$, maps $\mathrm{Lip}_0(S)$ isometrically into $\mathrm{Lip}_0(S, X)$. Let $T \in \mathrm{Ext}(S, \mathcal{M}, X)$ be such that $||T|| \le \lambda(S, \mathcal{M}, X) + \epsilon$. Then the operator $\widehat{T} := F_v \circ T \circ \widetilde{E}$ maps $\mathrm{Lip}_0(S)$ into $\mathrm{Lip}_0(\mathcal{M})$ and satisfies $||\widehat{T}|| \le \lambda(S, \mathcal{M}, X) + \epsilon$, and $\widehat{T} \in \mathrm{Ext}(S, \mathcal{M})$. This implies that

$$\lambda(S, \mathcal{M}) \le \lambda(S, \mathcal{M}, X).$$

Finally, if either $\lambda(S, \mathcal{M})$ or $\lambda(S, \mathcal{M}, X)$ is ∞ the arguments presented show that the remaining quantity must be ∞ as well.

This and the above two inequalities complete the proof of the proposition. □

Problem. *Is it true that $\lambda(\mathcal{M}, X) = \lambda(\mathcal{M})$ for any Banach space X?*

One of the main questions is whether the $\mathcal{SLE}$ property of $\mathcal{M}$ is equivalent to the finiteness of $\lambda(\mathcal{M})$. In general the answer is unknown. In this part, we give the positive answer to this question for some important classes of metric spaces. To formulate the result let us recall that a metric space $\mathcal{M}$ is *proper* (or boundedly compact), if every closed ball in $\mathcal{M}$ is compact. We also require the following

Definition 7.4. A metric space $\mathcal{M}$ has the weak transition property (WTP), if for some $C \ge 1$ and every finite set F and open ball B in $\mathcal{M}$ there is a C-isometry $\sigma : \mathcal{M} \to \mathcal{M}$ such that

$$B \cap \sigma(F) = \emptyset.$$

Theorem 7.5. *Assume that $\mathcal{M}$ is either proper or has the WTP. Then $\mathcal{M} \in \mathcal{SLE}$ if and only if $\lambda(\mathcal{M}) < \infty$.*

One of the main tools of the proof is the *finiteness property* of the characteristic λ established in subsection 7.1.2, asserting, in particular, that

$$\lambda(\mathcal{M}) = \sup_F \lambda(F)$$

where F runs through all finite subspaces of $\mathcal{M}$ (with the induced metric).

Proof. We begin with the case of $\mathcal{M}$ possessing the WTP. Assume that $\mathcal{M} \in \mathcal{SLE}$ but $\lambda(\mathcal{M}) = \infty$. From here and the finiteness property for $\lambda(\mathcal{M})$ proved in subsection 7.1.2 it follows that there is a sequence of finite sets F_j with $\lambda(F_j) \geq j$, $j \in \mathbb{N}$. This, in turn, leads to the inequalities

$$\inf\{||E|| \ ; \ E \in \mathrm{Ext}(F_j, \mathcal{M})\} \geq j \ , \ j \in \mathbb{N}. \tag{7.1}$$

Using the WTP of $\mathcal{M}$ we may choose an appropriate sequence of C-isometries σ_j such that for $G_j := \sigma_j(F_j)$ the following holds:

$$\mathrm{dist}(G_j, \cup_{i \neq j} G_i) \geq C \ \mathrm{diam}\, F_j \ , \ j \in \mathbb{N}. \tag{7.2}$$

For every $j \in \mathbb{N}$, fix a point $m_j^* \in G_j$. From (7.2) we derive that the operator N_j given for every $f \in \mathrm{Lip}(G_j)$ by

$$(N_j f)(m) := \begin{cases} f(m), & m \in G_j, \\ f(m_j^*), & m \in \cup_{i \neq j} G_i \end{cases} \tag{7.3}$$

belongs to $\mathrm{Ext}(G_j, G_\infty)$ where $G_\infty := \cup_{i \in \mathbb{N}} G_i$, and, moreover,

$$||N_j|| = 1. \tag{7.4}$$

In fact, if $f \in \mathrm{Lip}(G_j)$ and $m' \in G_j$, $m'' \in G_\infty \setminus G_j = \cup_{i \neq j} G_i$, then

$$\begin{aligned} |(N_j f)(m') - (N_j f)(m'')| &= |f(m') - f(m_j^*)| \\ &\leq ||f||_{\mathrm{Lip}(G_j)} \,\mathrm{diam}\, G_j \leq (C \ \mathrm{diam}\, F_j) ||f||_{\mathrm{Lip}(G_j)}. \end{aligned}$$

Together with (7.2) this leads to

$$|(N_j f)(m') - (N_j f)(m'')| \leq ||f||_{\mathrm{Lip}(G_j)} d(m', m'').$$

Since this holds trivially for all other choices of m', m'', the equality (7.4) has been established.

Since $\mathcal{M} \in \mathcal{SLE}$, there is an operator $E \in \mathrm{Ext}(G_\infty, \mathcal{M})$ with $||E|| \leq A$ for some $A > 0$. By (7.4) the operator $E_j := EN_j \in \mathrm{Ext}(G_j, \mathcal{M})$ and $||E_j|| \leq A$. Then the operator $\widetilde{E}_j$ given for $m \in \mathcal{M}$, $f \in \mathrm{Lip}(F_j)$ by the formula

$$(\widetilde{E}_j f)(m) := (E_j(f \circ \sigma_j^{-1}))(\sigma_j(m))$$

belongs to $\mathrm{Ext}(F_j, \mathcal{M})$ and its norm is bounded by $C^2 A$. Comparing with (7.1), we get for each j,

$$C^2 A \geq j \,,$$

a contradiction.

Now, let $\mathcal{M}$ be proper. In order to prove that $\lambda(\mathcal{M}) < \infty$ we need

Lemma 7.6. *For every $m \in \mathcal{M}$ there is an open ball B_m centered at $\mathcal{M}$ such that $\lambda(B_m) < \infty$.*

Proof. Assume that this assertion does not hold for some m. Then there is a sequence of balls $B_i := B_{r_i}(m)$, $i \in \mathbb{N}$, centered at m of radii r_i such that $\lim_{i\to\infty} r_i = 0$ and $\lim_{i\to\infty} \lambda(B_i) = \infty$. According to the finiteness property for $\lambda(B_i)$ this implies the existence of finite subsets $F_i \subset B_i$, $i \in \mathbb{N}$, such that

$$\inf\{||E|| \ ; \ E \in \operatorname{Ext}(F_i, B_i)\} \to \infty \quad \text{as } i \to \infty. \tag{7.5}$$

We may assume that m, the common center of all B_j, belongs to every F_j. Otherwise we simply replace F_j by $G_j := F_j \cup \{m\}$ and show that (7.5) is true with G_i substituting for F_i. In fact, let L_i be the operator given for every $f \in \operatorname{Lip}(F_i)$ by

$$(L_i f)(m') := \begin{cases} f(m_i), & \text{if } m' = m, \\ f(m'), & \text{if } m' \in F_i, \end{cases}$$

where m_i is the closest to $\mathcal{M}$ point from F_i. Then $L_i \in \operatorname{Ext}(F_i, G_i)$ and $||L_i|| \le 2$, since

$$\begin{aligned} |(L_i f)(m) - (L_i f)(m')| &= |f(m_i) - f(m')| \\ &\le ||f||_{\operatorname{Lip}(F_i)} d(m_i, m') \le 2||f||_{\operatorname{Lip}(F_i)} d(m, m'). \end{aligned}$$

If now (7.5) does not hold for $\{G_i\}$ substituted for $\{F_i\}$, then there is a sequence $E_i \in \operatorname{Ext}(G_i, B_i)$ such that $\sup_i ||E_i|| < \infty$. But then the same will be true for the norms of $\widetilde{E}_i := E_i L_i \in \operatorname{Ext}(F_i, B_i)$, $i \in \mathbb{N}$, in contradiction with (7.5).

The proof will be now finished by the following argument. Choose a subsequence $F_{i_k} \subset B_{i_k} := B_{r_{i_k}}(m)$, $k \in \mathbb{N}$, such that

$$r_{i_{k+1}} < \min\{r_{i_k}, \ \operatorname{dist}(F_{i_{k+1}} \setminus \{m\}, \cup_{s<k+1} F_{i_s} \setminus \{m\})\}.$$

Without loss of generality we assume that the sequence $\{F_i\}$ satisfies this condition, i.e.,

$$r_{i+1} < \min\{r_i, \ \operatorname{dist}(F_{i+1} \setminus \{m\}, \cup_{s<i+1} F_s \setminus \{m\})\}. \tag{7.6}$$

Set $F_\infty := \bigcup_{s\in\mathbb{N}} F_s$ and show that

$$\operatorname{Ext}(F_\infty, \mathcal{M}) = \emptyset \tag{7.7}$$

which gives the required contradiction to the $\mathcal{SLE}$ of $\mathcal{M}$.

To prove this we choose the center m as a marked point of $\mathcal{M}$. Then all F_i are subspaces of $(\mathcal{M}, m)$ and $f(m) = 0$ if $f \in \operatorname{Lip}_0(F_i)$.

Define now the operator N_i by

$$(N_i f)(m') := \begin{cases} f(m'), & \text{if } m' \in F_i, \\ 0, & \text{if } m' \in F_\infty \setminus F_i. \end{cases}$$

Then for $f \in \mathrm{Lip}_0(F_i)$ and $m' \in F_i \setminus \{m\}$ and $m'' \in F_\infty \setminus F_i$ we have

$$|(N_i f)(m') - (N_i f)(m'')| = |f(m') - f(m)| \le ||f||_{\mathrm{Lip}_0(F_i)} d(m', m).$$

Moreover, $m'' \in B_j$ for some $j \neq i$. Assume, first, that $j > i$. Then by (7.6)

$$\begin{aligned} d(m', m) &\le d(m', m'') + d(m'', m) \le d(m', m'') + r_j \\ &\le d(m', m'') + \mathrm{dist}(F_j \setminus \{m\}, F_i \setminus \{m\}) \le 2d(m', m''). \end{aligned}$$

If now $j < i$, then by (7.6) we have

$$d(m', m) \le r_i < \mathrm{dist}(F_i \setminus \{m\}, F_j \setminus \{m\}) \le d(m', m'').$$

Combining these we prove that $N_i \in \mathrm{Ext}(F_i, F_\infty)$ and $||N_i|| \le 2$.

If now (7.7) is not true, then there is an operator $E \in \mathrm{Ext}(F_\infty, \mathcal{M})$, and so every operator $\widetilde{E}_i := EN_i$ belongs to $\mathrm{Ext}(F_i, \mathcal{M})$ and $||\widetilde{E}_i|| \le 2||E||$, $i \in \mathbb{N}$, a contradiction to (7.5).

The proof is complete. □

Remark 7.7. In this proof the properness of $\mathcal{M}$ is not used.

To proceed we need

Lemma 7.8. *Let $\mathcal{U}$ be a finite open cover of a compact set $C \subset \mathcal{M}$. Then there is a partition of unity $\{\rho_U\}_{U \in \mathcal{U}}$ on C subordinate to $\mathcal{U}$ such that every ρ_U is Lipschitz with a constant depending only on the cover.*

Proof. Define $d_U : \mathcal{M} \to \mathbb{R}$ by

$$d_U(m) := \mathrm{dist}(m, \mathcal{M} \setminus U), \quad m \in \mathcal{M}.$$

This function is supported on U and has the Lipschitz constant 1; moreover, $\sum_{U \in \mathcal{U}} d_U > 0$ on C. Putting now

$$\rho_U(m) := \frac{d_U(m)}{\sum_{U \in \mathcal{U}} d_U(m)}, \quad m \in C \cap U, \quad U \in \mathcal{U},$$

we get the required partition. □

The next result implies finiteness of $\lambda(\mathcal{M})$ for compact $\mathcal{M}$.

Lemma 7.9. *For every compact subspace $C \subset \mathcal{M}$ the constant $\lambda(C)$ is finite.*

Proof. We have to show that for every $S \subset C$ there is an operator $E_S \in \mathrm{Ext}(S, C)$ such that

$$\sup_S ||E_S|| < \infty. \tag{7.8}$$

We may assume that S and C are subspaces of $(\mathcal{M}, m^*)$ so that $f(m^*) = 0$ for f belonging to $\mathrm{Lip}_0(S)$ or $\mathrm{Lip}_0(C)$. By compactness of C and Lemma 7.6 there is a finite cover $\{U_i\}_{1\le i\le n}$ of C by open balls such that for some constant $A > 0$ depending only on C we have $\lambda(U_i) < A$, $1 \le i \le n$. By the definition of λ this implies the existence of $E_i \in \mathrm{Ext}(S \cap U_i, U_i)$ with

$$||E_i|| \le A, \quad 1 \le i \le n. \tag{7.9}$$

For $f \in \mathrm{Lip}_0(S)$ one sets

$$f_i := \begin{cases} f|_{S\cap U_i}, & \text{if } S \cap U_i \ne \emptyset, \\ 0, & \text{if } S \cap U_i = \emptyset \end{cases} \tag{7.10}$$

and introduces a function f_{ij} given on $U_i \cap U_j$ by

$$f_{ij} := \begin{cases} E_i f_i - E_j f_j, & \text{if } U_i \cap U_j \ne \emptyset, \\ 0, & \text{if } U_i \cap U_j = \emptyset, \end{cases} \tag{7.11}$$

here $E_i f_i := 0$, if $f_i = 0$. Then (7.9) implies that

$$||f_{ij}||_{\mathrm{Lip}(U_i\cap U_j)} \le 2A. \tag{7.12}$$

Moreover, we get

$$f_{ij} = 0 \quad \text{on} \quad S \cap U_i \cap U_j. \tag{7.13}$$

Finally, introduce a function g_i given on $C \cap U_i$ by

$$g_i(m) := \sum_{1\le j\le n} \rho_j(m) f_{ij}(m) \tag{7.14}$$

where $\rho_j := \rho_{U_j}$, $1 \le j \le n$, is the partition of unity of Lemma 7.8.

A straightforward computation leads to the equalities:

$$g_i - g_j = f_{ij} \quad \text{on} \quad U_i \cap U_j \cap C \quad \text{and} \quad g_i|_{S\cap U_i} = 0. \tag{7.15}$$

Introduce now an operator E_S given for $f \in \mathrm{Lip}_0(S)$ and $m \in U_i \cap C$ by

$$(E_S f)(m) := (E_i f_i - g_i)(m). \tag{7.16}$$

We will show that E_S is an extension operator. In fact, if $m \in S$, then $m \in S \cap U_i$ for some $1 \le i \le n$ and by (7.15) and (7.10) we get

$$(E_S f)(m) = (E_i f_i)(m) = f(m).$$

We now show that $E_S \in \mathrm{Ext}(S, C)$ and $||E_S||$ is bounded by a constant depending only on C. To this end we denote by $\delta = \delta(C) > 0$ the *Lebesgue number* of the cover $\mathcal{U}$, see, e.g., Lemma C.6 of Volume I. So every subset of

$\cup_{i=1}^n U_i$ of diameter at most δ lies in one of the U_i. Using this we first establish the corresponding Lipschitz estimate for $m', m'' \in (\cup_{i=1}^n U_i) \cap C$ with

$$d(m', m'') \leq \delta. \tag{7.17}$$

In this case both $m', m'' \in U_{i_0}$ for some i_0. Further, (7.14)-(7.16) imply that for $m \in U_{i_0} \cap C$,

$$(E_S f)(m) = \sum_{U_i \cap U_{i_0} \neq \emptyset} (\rho_i E_i f_i)(m). \tag{7.18}$$

In this sum, each ρ_i is Lipschitz with the constant L depending only on the subspace C; moreover, $0 \leq \rho_i \leq 1$. In turn, $E_i f_i$ is Lipschitz on $U_i \cap U_{i_0}$ with constant A, see (7.9) (recall that $E_i f_i = 0$ if $S \cap U_i = \emptyset$).

If now $m \in U_i$ and $S \cap U_i \neq \emptyset$, then for arbitrary $m_i \in S \cap U_i$,

$$\begin{aligned} |(E_i f_i)(m)| &\leq |(E_i f_i)(m) - (E_i f_i)(m_i)| + |(E_i f_i)(m_i)| \\ &\leq A||f||_{\mathrm{Lip}_0(S)} d(m, m_i) + |f(m_i) - f(m^*)| \\ &\leq A||f||_{\mathrm{Lip}_0(S)} (d(m, m_i) + d(m_i, m^*)). \end{aligned}$$

This implies for all $m \in C \cap U_i$ the inequality

$$|(E_i f_i)(m)| \leq 2A \ \operatorname{diam} C \ ||f||_{\mathrm{Lip}_0(S)}. \tag{7.19}$$

Together with (7.18) this leads to the estimate

$$|(E_S f)(m') - (E_S f)(m'')| \leq An(2L \ \operatorname{diam} C + 1)||f||_{\mathrm{Lip}_0(S)} d(m', m''), \tag{7.20}$$

provided that $m', m'' \in U_{i_0} \cap C$. To prove a similar estimate for $d(m', m'') > \delta$, $m', m'' \in C$, we note that the left-hand side in (7.20) is bounded by

$$2 \sup_{m \in C} |(E_S f)(m)| \leq 4A \ \operatorname{diam} C \ ||f||_{\mathrm{Lip}_0(S)},$$

see (7.19). In turn, the right-hand side of the last inequality is less than or equal to $4\delta^{-1} A \cdot \operatorname{diam} C \cdot ||f||_{\mathrm{Lip}_0(S)} d(m', m'')$. Together this implies that E belongs to $\mathrm{Ext}(S, C)$ and that its norm is bounded by a constant depending only on C. □

Lemma 7.10. *Assume that for a family of finite spaces $\{F_i \subset (\mathcal{M}, m^*);\ i \in \mathbb{N}\}$,*

$$\sup_i \lambda(F_i, \mathcal{M}) = \infty \ . \tag{7.21}$$

Then for every closed ball $\overline{B}$ centered at m^*,

$$\sup_i \lambda(F_i \setminus \overline{B}, \mathcal{M}) = \infty \ . \tag{7.22}$$

Proof. Arguing as in the proof of Lemma 7.6, we can assume that $m^* \in F_i$ for all $i \in \mathbb{N}$. By the same reason we may and will assume that all F_i contain a fixed point $m' \in \mathcal{M} \setminus \overline{B}$. If now (7.22) is not true, then for some $A_1 > 0$ there are operators E_i^1, $i \in \mathbb{N}$, such that

$$E_i^1 \in \mathrm{Ext}(F_i \setminus \overline{B}, \mathcal{M}) \quad \text{and} \quad ||E_i^1|| \leq A_1; \tag{7.23}$$

as above we set $E_i^1 := 0$, if $F_i \setminus \overline{B} = \emptyset$.
Let $2B$ be the open ball centered at m^* and of twice the radius of B. Introduce an open cover of $\mathcal{M}$ by

$$U_1 := \mathcal{M} \setminus \overline{B} \quad \text{and} \quad U_2 := 2B, \tag{7.24}$$

and let $\{\rho_1, \rho_2\}$ be the corresponding Lipschitz partition of unity (cf. Lemma 7.8) given for $m \in \mathcal{M}$ by

$$\rho_j(m) := \frac{d_{U_j}(m)}{d_{U_1}(m) + d_{U_2}(m)}, \quad j = 1, 2.$$

By this definition

$$|\rho_j(m_1) - \rho_j(m_2)| \leq \frac{3d(m_1, m_2)}{\max_{k=1,2}\{d_{U_1}(m_k) + d_{U_2}(m_k)\}}, \quad m_1, m_2 \in \mathcal{M}. \tag{7.25}$$

Set now $H_i := F_i \cap 2B$, $i \in \mathbb{N}$. Since these are subsets of the compact set $2\overline{B}$, Lemma 7.9 gives

$$\sup_i \lambda(H_i, 2B) \leq \lambda(2\overline{B}) < \infty.$$

This, in turn, implies the existence of operators E_i^2, $i \in \mathbb{N}$, such that

$$E_i^2 \in \mathrm{Ext}(H_i, 2B) \quad \text{and} \quad ||E_i^2|| \leq A_2 \tag{7.26}$$

with A_2 independent of i.

We now follow the proof of Lemma 7.9 in which the set S and compact subspace $C \supset S$ are replaced by F_i and the (noncompact) space $\mathcal{M}$, respectively, and the cover by that in (7.24). Since

$$H_i = F_i \cap U_2 \quad \text{and} \quad F_i \setminus \overline{B} = F_i \cap U_1,$$

we can use in our derivation the operators E_i^j, $j = 1, 2$, instead of those in (7.9). By (7.23) and (7.26) inequalities similar to (7.9) hold for these operators. Then we set $f_j := f|_{F_i \cap U_j}$, and define for $f \in \mathrm{Lip}_0(F_i)$ functions $f_{12} := -f_{21}$ on $U_1 \cap U_2$ and g_1 on U_1 and g_2 on U_2 by

$$f_{12} := E_i^1 f_1 - E_i^2 f_2, \quad g_1 := \rho_2 f_{12}, \quad g_2 := \rho_1 f_{21}.$$

Finally, we introduce the required operator E_i on $\mathrm{Lip}_0(F_i)$ given for $F_i \cap U_j$, $j = 1, 2$, by

$$(E_i f)(m) := (E_i^j f_j)(m) - g_j(m).$$

As in Lemma 7.9, E_i is an operator extending functions from F_i to all of $\mathcal{M}$. To estimate the Lipschitz constant of $E_i f$ we use the McShane theorem to extend each function $E_i^j f_j$ outside U_j so that its extension $\widetilde{F}_j$ satisfies

$$||\widetilde{F}_j||_{\mathrm{Lip}(\mathcal{M})} = ||E_i^j f_j||_{\mathrm{Lip}(U_j)}.$$

Now, the definition of E_i implies that

$$E_i f := \rho_1 \widetilde{F}_1 + \rho_2 \widetilde{F}_2.$$

Then as in the proof of (7.19) we obtain for arbitrary $m \in \mathcal{M}$,

$$|\widetilde{F}_j(m)| \leq \begin{cases} A_1 ||f||_{\mathrm{Lip}_0(F_i)}(d(m, m') + d(m', m^*)), & \text{if } j = 1, \\ A_2 ||f||_{\mathrm{Lip}_0(F_i)} d(m, m^*), & \text{if } j = 2. \end{cases}$$

This implies for all $m \in \mathcal{M}$ the inequality

$$|\widetilde{F}_j(m)| \leq \; A(d(m, m^*) + d(m', m^*))||f||_{\mathrm{Lip}_0(F_i)} \tag{7.27}$$

with $A := 2 \max(A_1, A_2)$.
Together with (7.25) this leads to the estimate

$$\begin{aligned} &|(E_i f)(m_1) - (E_i f)(m_2)| \\ &\leq \left(\frac{3(d(m_1, m^*) + d(m', m^*))}{\max_{k=1,2}\{d_{U_1}(m_k) + d_{U_2}(m_k)\}} + 1 \right) 2A \, ||f||_{\mathrm{Lip}_0(F_i)} d(m_1, m_2). \end{aligned} \tag{7.28}$$

Since $\max_{k=1,2}\{d_{U_1}(m_k) + d_{U_2}(m_k)\} \geq R$, the radius of B, and

$$\lim_{d(m_1, m^*) \to \infty} \frac{d(m_1, m^*) + d(m', m^*)}{d_{U_1}(m_1) + d_{U_2}(m_1)} = 1,$$

(7.28) implies that $E_i \in \mathrm{Ext}(F_i, \mathcal{M})$ and its norm is bounded by a constant independent of i. By definition this yields

$$\sup_i \lambda(F_i, \mathcal{M}) < \infty$$

in a contradiction with (7.21). □

Now we will complete the proof of Theorem 7.5. Recall that it has been already proved for compact $\mathcal{M}$, see Lemma 7.9. So it remains to consider the case of proper $\mathcal{M}$ with

$$\operatorname{diam} \mathcal{M} = \infty. \tag{7.29}$$

In this case we will show that

$$\sup_F \lambda(F, \mathcal{M}) < \infty, \tag{7.30}$$

where F runs through all finite subsets $F \subset (\mathcal{M}, m^*)$. Since $\sup_F \lambda(F)$ is bounded by the supremum in (7.30), which, in turn, equals $\lambda(\mathcal{M})$, see Corollary 7.13, this proves the result.

Suppose on the contrary that (7.30) does not hold. Then there is a sequence of finite subsets $F_i \subset (\mathcal{M}, m^*)$, $i \in \mathbb{N}$, that satisfies the assumption of Lemma 7.10, see (7.21). We use this to construct a sequence G_i, $i \in \mathbb{N}$, such that

$$\lambda(G_i, M) \geq i - 1 \quad \text{and} \quad \operatorname{dist}(G_{i+1}, G_i) \geq \operatorname{dist}(m^*, G_i) \quad \text{for all} \quad i \in \mathbb{N}. \tag{7.31}$$

If this will be done we set $G_\infty := \cup_{i\in\mathbb{N}} G_i$, and use with minimal changes the argument of Lemma 7.6 to show that

$$\operatorname{Ext}(G_\infty, \mathcal{M}) = \emptyset.$$

Since this contradicts the $\mathcal{SLE}$ property of $\mathcal{M}$, the result will be proved.

To construct the required $\{G_i\}$, set $G_1 := F_1$ and assume that the first j terms of this sequence have already been defined. Choose the closed ball $\overline{B}$ such that

$$\operatorname{dist}(G_j, \mathcal{M} \setminus \overline{B}) \geq \operatorname{dist}(m^*, G_j)$$

(it exists because of (7.29)). Then apply Lemma 7.10 to find $F_{i(j)}$ such that

$$\lambda(F_{i(j)} \setminus \overline{B}, \mathcal{M}) \geq j - 1.$$

Setting $G_{j+1} := F_{i(j)} \setminus \overline{B}$ we obtain the next term satisfying condition (7.31).

The proof is complete. □

Problem. *Does there exist a metric space $\mathcal{M} \in \mathcal{SLE}$ but with $\lambda(\mathcal{M}) = \infty$?*

In conclusion of this subsection, we point out two other problems whose solutions may lead to a better understanding of the nature of the extension constant λ regarding it as a function from the category of metric spaces into $\mathbb{R}_+ \cup \{\infty\}$.

Problem. *Suppose $\lambda(\mathcal{M}_j) < \infty$, $j = 1, 2$. Are $\lambda(\mathcal{M}_1 \times \mathcal{M}_2)$ and $\lambda(\mathcal{M}_1 \times_S \mathcal{M}_2)$ finite?*

Here S is a common subspace of $\mathcal{M}_1$ and $\mathcal{M}_2$ and $\mathcal{M}_1 \times_S \mathcal{M}_2$ is the metric space obtaining by gluing $\mathcal{M}_1$ and $\mathcal{M}_2$ along S, see Volume I, Example 3.54 (d) for its definition.

7.1.2 Finiteness property

In order to formulate this property, we require

Definition 7.11.

(a) A sequence of metric spaces $\{\mathcal{M}_i\}_{i\in\mathbb{N}}$ is said to be finitely convergent to a metric space $\mathcal{M}$ if for every finite subspace $F \subset \mathcal{M}$ there exists a sequence of finite subspaces $\{F_i \subset \mathcal{M}_i\}_{i\in\mathbb{N}_0}$ where $\mathbb{N}_0 \subset \mathbb{N}$ such that $\lim_{\mathbb{N}_0} F_i = F$ (convergence in the Gromov-Hausdorff metric).

(b) This sequence δ-converges to $(\mathcal{M}, d)$ if every $\mathcal{M}_i$ is bi-Lipschitz homeomorphic to $\mathcal{M}$ with distortion D_i and $\lim\limits_{i\to\infty} D_i = 1$.

Due to the definition of the Gromov-Hausdorff convergence, the basic condition of Definition 7.11 (a) is equivalent to existence of bi-Lipschitz homeomorphisms of F_i onto F with distortion D_i, $i \in \mathbb{N}_0$, such that $\lim\limits_{i\to\infty} D_i = 1$, see subsection 3.1.8 and Proposition 3.55 of Volume I.

In particular, δ-convergence of $\{\mathcal{M}_i\}$ to $\mathcal{M}$ implies that this sequence finitely converges to $\mathcal{M}$.

Theorem 7.12. *Suppose that a sequence of metric spaces $\{(\mathcal{M}_i, d_i)\}_{i\in\mathbb{N}}$ finitely converges to a metric space $(\mathcal{M}, d)$. Then:*

(a) *It is true that*
$$\lambda(\mathcal{M}) \le \varlimsup_{n\to\infty} \lambda(\mathcal{M}_i).$$

(b) *If, in addition, $\{(\mathcal{M}_i, d_i)\}_{i\in\mathbb{N}}$ δ-converges to a metric subspace $S \subset \mathcal{M}$, then*
$$\lambda(\mathcal{M}) = \sup_{F\subset S} \lambda(F)$$
where F runs over all finite subspaces of S.

Choosing $S = \mathcal{M}$ we immediately obtain from part (b)

Corollary 7.13.
$$\lambda(\mathcal{M}) = \sup_{F} \lambda(F) \tag{7.32}$$
where F runs over all finite subspaces of $\mathcal{M}$.

Together with Theorem 7.12 this implies

Corollary 7.14. *Let $S \subset \mathcal{M}$ satisfy the assumptions of Theorem 7.12. Then*
$$\lambda(\mathcal{M}) = \lambda(S).$$

In the forthcoming proofs, the duality results of subsection 4.6.3 of Volume I will play a considerable role. For convenience of the reader we recall them now.

Let $(\mathcal{M}, m^*)$ be a pointed metric space and S be its subspace, i.e., S is a subspace of $\mathcal{M}$ containing m^*. By $\mathrm{Lip}_0(S)$ we denote a subspace of $\mathrm{Lip}(S)$ determined by the condition $f(m^*) = 0$.

Further, the *Lipschitz-free space* $\mathcal{F}(S)$ is the closed linear span in the dual space $\mathrm{Lip}_0(\mathcal{M})^*$ of the set of *point evaluations* (δ-functionals) $\{\delta_S(m)\}_{m\in S}$ where

$$\delta_S(m)(f) := f(m).$$

Equivalently, see Volume I, formula (4.131),

$$\mathcal{F}(S) := \overline{\text{span } \delta_S(S)} \quad (\text{closure in } \ \ell_\infty(B_S))$$

where B_S stands for the closed unit ball of $\mathrm{Lip}_0(S)$.

Then the required dual results of subsection 4.6.3 of Volume I are summarized in

Proposition 7.15. (a) *The dual space $\mathcal{F}(S)^*$ is linearly isometric to* $\mathrm{Lip}_0(\mathcal{M})$.

(b) *A map $J_S : S \to \mathcal{F}(S)$ given by*

$$J_S(m) := \delta_S(m) \tag{7.33}$$

is a (nonlinear) isometric embedding.

(c) *$\mathcal{F}(S)$ is the minimal closed linear subspace of $\ell_\infty(B_S)$ containing $J_S(S)$.*

Proof of Theorem 7.12. We begin with the case of $S = \mathcal{M}$, i.e., with Corollary 7.13.

Proposition 7.16. *Let S be a given finite subspace of $(\mathcal{M}, m^*)$. Assume that for every finite subspace $G \supset S$ there exists an extension operator $E_G \in \mathrm{Ext}(S, G)$ and*

$$A := \sup_G ||E_G|| < \infty. \tag{7.34}$$

Then there exists an operator $E \in \mathrm{Ext}(S, \mathcal{M})$ such that

$$||E|| \le A. \tag{7.35}$$

Proof. Using the map J_G, the canonical linear embedding $\kappa_G : \mathcal{F}(G) \to \mathcal{F}(G)^{**}$ and the conjugate operator E_G^* we then define a vector-valued function $\phi_G : G \to \mathcal{F}(S)^{**}$ by

$$\phi_G := E_G^* \kappa_G J_G. \tag{7.36}$$

The function is well-defined, since the conjugate to the operator $E_G \in \mathrm{Ext}(S, G)$ acts from $\mathrm{Lip}_0(G)^* = \mathcal{F}(G)^{**}$ into $\mathrm{Lip}_0(S)^* = \mathcal{F}(S)^{**}$.

Lemma 7.17. (a) *$\phi_G \in \mathrm{Lip}(G, \mathcal{F}(S)^{**})$ and its norm satisfies*

$$||\phi_G|| \le A. \tag{7.37}$$

(b) *For $m \in S$,*

$$\phi_G(m) = \kappa_G J_G(m). \tag{7.38}$$

In particular, $\phi_G(m^*) = 0$.

Proof. (a) Let $m \in G$ and $h \in \mathrm{Lip}_0(S)(= \mathcal{F}(S)^*)$. By (7.36)

$$\langle \phi_G(m),\ h\rangle = \langle \kappa_G J_G(m),\ E_G h\rangle = \langle E_G h,\ J_G(m)\rangle = (E_G h)(m).$$

This immediately implies that for $m', m'' \in G$,

$$|\langle \phi_G(m') - \phi_G(m''),\ h\rangle| \le ||E_G|| d(m', m'')||h||_{\mathrm{Lip}_0(F)},$$

and (7.37) follows.
(b) Since $(E_G h)(m) = h(m)$, $m \in S$, and $h(m^*) = 0$, the previous identity implies (7.38). □

Our next aim is to find a limit point of the family $\{\phi_G\}$. To this end we extend every ϕ_G by zero to $\mathcal{M}$ and denote this extension by $\widehat{\phi}_G$. Due to Lemma 7.17 for every $m \in \mathcal{M}$,

$$||\widehat{\psi}_G(m)||_{\mathcal{F}(S)^{**}} \le Ad(m, m^*). \tag{7.39}$$

Further, by $\overline{B}_m$ denote the closed ball $\overline{B}_{Ad(m,m^*)}(0) \subset \mathcal{F}(S)^{**}$ and set

$$\Phi := \{\psi : \mathcal{M} \to \mathcal{F}(S)^{**}\,;\ \psi(m) \in \overline{B}_m \text{ for all } m \in \mathcal{M}\}.$$

Using the map $\psi \mapsto (\psi(m))_{m\in\mathcal{M}}$ we then identify Φ with the direct product $\prod_{m\in\mathcal{M}} \overline{B}_m$ equipped with the product topology. Since $\dim \mathcal{F}(S)^{**} < \infty$, every $\overline{B}_m$ is compact. Hence, their direct product (and Φ) is also compact.

Now the collection of sets $G \supset S$ is partially ordered and therefore $\{\widehat{\phi}_G\}$ is a *net* containing by (7.39) in the compact set Φ. By compactness the net contains a subnet $\{\phi_{G_\alpha}\}_{\alpha\in A}$ such that for some $\phi \in \Phi$,

$$\lim_\alpha \widehat{\phi}_{G_\alpha} = \phi,$$

see, e.g., [Kel-1957, Ch. 5 Thm. 5]. By the definition of the product topology one also has for every $m \in \mathcal{M}$,

$$\lim_\alpha \widehat{\phi}_{G_\alpha}(m) = \phi(m) \quad (\text{convergence in } \mathcal{F}(S)^{**}). \tag{7.40}$$

We now show that ϕ is Lipschitz. Let $m', m'' \in \mathcal{M}$ be given, and $\widetilde{N}$ be a subnet of N containing those of $\widehat{\phi}_{G_\alpha}$ for which $m', m'' \subset G_\alpha$. Then by (7.36) and (7.40) one has for $h \in \mathrm{Lip}_0(S)$,

$$\begin{aligned}
&\langle \phi(m') - \phi(m''),\ h\rangle = \lim_{\widetilde{N}} \langle \phi_{G_\alpha}(m') - \phi_{G_\alpha}(m''),\ h\rangle \\
&= \lim_{\widetilde{N}} \langle E_{G_\alpha} h,\ J_{G_\alpha}(m') - J_{G_\alpha}(m'')\rangle = \lim_{\widetilde{N}} [(E_{G_\alpha} h)(m') - (E_{G_\alpha} h)(m'')].
\end{aligned}$$

Together with (7.34) this leads to the inequality

$$|\langle\phi(m') - \phi(m''),\ h\rangle| \le A||h||_{\mathrm{Lip}_0(S)} d(m', m''),$$

implying that,

$$||\phi||_{\mathrm{Lip}(\mathcal{M}, \mathcal{F}(S)^{**})} \le A. \tag{7.41}$$

Using (7.38) we also conclude that for $m \in S$,

$$\phi(m) = \kappa_S J_S(m), \quad \text{and} \quad \phi(m^*) = 0. \tag{7.42}$$

Finally, using the function ϕ we define the required extension operator $E : \mathrm{Lip}_0(S) \to \mathrm{Lip}_0(\mathcal{M})$ as follows. Let $\widetilde{\kappa}_S : \mathcal{F}(S)^* \to \mathcal{F}(S)^{***}$ be the canonical embedding (an isometry in this case). For $h \in \mathrm{Lip}_0(S) = \mathcal{F}(S)^*$ we set

$$(Eh)(m) := \langle\widetilde{\kappa}_S h,\ \phi(m)\rangle, \quad m \in \mathcal{M}. \tag{7.43}$$

Then by (7.41)

$$|(Eh)(m') - (Eh)(m'')| = |\langle\phi(m') - \phi(m''),\ h\rangle| \le A||h||_{\mathrm{Lip}_0(S)} d(m', m''),$$

and (7.35) follows.

Now by (7.42) we have for $m \in S$,

$$(Eh)(m) = \langle\phi(m),\ h\rangle = \langle\kappa_S J_S(m),\ h\rangle = h(m);$$

in particular, $(Eh)(m^*) = 0$.

The proof of the proposition is complete. □

We are now ready to prove Theorem 7.12(b) for the case of $S = \mathcal{M}$.

Since the inequality

$$A := \sup_F \lambda(F) \le \lambda(\mathcal{M}) \tag{7.44}$$

with F running through all finite subspaces of $(\mathcal{M}, m^*)$ is trivial, it remains to establish the converse. In other words, we must prove that for every $S \subset (\mathcal{M}, m^*)$ and $\epsilon > 0$ there exists an operator $E \in \mathrm{Ext}(S, \mathcal{M})$ with $||E|| \le A + \epsilon$.

In order to find the E we, as in Lemma 7.17, associate to every finite subspace $F \subset S$ a function $\phi_F : \mathcal{M} \to \mathcal{F}(S)^{**}$ and then find a limit point of the family $\{\phi_F\}$. The latter will be used to define the required extension operator E by a procedure similar to that in (7.43).

To define ϕ_F we use several linear operators first of which denoted by T_F is given as follows.

By the definition of the constant A, for each pair $F \subset G$ of finite subspaces of (S, m^*) and $\epsilon > 0$ there exists an extension operator $E_G \in \mathrm{Ext}(F, G)$ such that $||E_G|| \le A + \epsilon$. Applying then Proposition 7.16 to $\{E_G\}$ we find the required extension operator $T_F \in \mathrm{Ext}(F, \mathcal{M})$ such that

$$||T_F|| \le A + \epsilon. \tag{7.45}$$

Further, we need the following

Lemma 7.18. *For every subspace G of (S, m^*) there exists a linear isometric embedding*

$$I_G : \mathcal{F}(G) \to \kappa_S(\mathcal{F}(S)).$$

Proof. To define I_G we use the restriction operator $R_G : f \mapsto f|_G$. Due to McShane's Theorem 1.27 of Volume I, R_G regarding as a linear operator from $\mathrm{Lip}_0(S)$ into $\mathrm{Lip}_0(G)$ satisfies

$$||R_G|| = 1. \tag{7.46}$$

Then we set

$$I_G := R_G^* \kappa_G. \tag{7.47}$$

Since R_G^* maps $\mathrm{Lip}_0(G)^* = \mathcal{F}(G)^{**}$ into $\mathcal{F}(S)^{**}$ and the canonical embedding κ_G sends $\mathcal{F}(G)$ into $\mathcal{F}(G)^{**}$, the operator I_G is acting from $\mathcal{F}(G)$ into $\mathcal{F}(S)^{**}$.

Let us show that I_G ranges in $\kappa_S(\mathcal{F}(S))$. To this end we use the (nonlinear) isometric embedding $J_S : S \to \mathcal{F}(S)$, see (7.33). Let $m \in G$ and $h \in \mathrm{Lip}_0(S)$. By the definitions of I_G and J_S,

$$\langle (I_G J_S)(m)\,, h(m)\rangle = \langle ((\kappa_G J_S)(m)\,, h|_G\rangle = \langle h\,, J_S(m)\rangle = h(m).$$

This, clearly, means that

$$I_G(J_S(G)) \subset \kappa_S(J_S(S)).$$

Since $J_S(G) = J_G(G)$ and $\mathcal{F}(G)$ is the minimal closed subspace containing $J_G(G)$ (the similar is true for $\mathcal{F}(S)$ and $J_S(S)$), this embedding implies the required result:

$$I_G(\mathcal{F}(G)) \subset \kappa_S(\mathcal{F}(S)).$$

Finally, by (7.46) and (7.47),

$$\|I_G\| = \|R_G^*\| = \|R_G\| = 1. \qquad \square$$

Using the operators introduced and the canonical embeddings $\kappa_{\mathcal{M}}$ and κ_F we then define linear operators

$$P_F := T_F^* \kappa_{\mathcal{M}} \quad \text{and} \quad Q_F := I_F(\kappa_F)^{-1} P_F.$$

One notes that for a finite F the canonical embedding is invertible, since $\dim \mathcal{F}(F) < \infty$. Moreover, P_F acts from $\mathcal{F}(\mathcal{M})$ into $\mathcal{F}(F)^{**}$ and therefore Q_F maps $\mathcal{F}(\mathcal{M})$ into $\mathcal{F}(S)^{**}$. We also have, see (7.45),

$$\|Q_F\| \le \|P_F\| \le \|T_F\| \le A + \varepsilon. \tag{7.48}$$

Now, we define the desired vector function $\phi_F : \mathcal{M} \to \mathcal{F}(S)^{**}$ by setting

$$\phi_F := Q_F J_{\mathcal{M}}; \tag{7.49}$$

recall that the (nonlinear) isometric embedding $J_{\mathcal{M}} : \mathcal{M} \to \mathcal{F}(\mathcal{M})$ was introduced by (7.33).

Arguing now as in Lemma 7.17 and using (7.48) we then obtain the estimate

$$||\phi_F||_{\mathrm{Lip}(\mathcal{M},\mathcal{F}(S)^{**})} \le A + \epsilon. \tag{7.50}$$

Further, by (7.49) and (7.47) we have for $m \in F$ and $h \in \mathrm{Lip}_0(\mathcal{M})$,

$$\langle \phi_F \, , h\rangle(m) := \langle R_F^* \kappa_F (\kappa_F)^{-1} P_F J_{\mathcal{M}} \, , h\rangle(m) = \langle h|_F \, , P_F J_{\mathcal{M}}\rangle(m) = \langle T_F h \, , J_{\mathcal{M}}\rangle(m).$$

Since $T_F h$ is an extension of h from F and $J_{\mathcal{M}}(m)$ is the δ-functional at m, the last term equals $h(m)$.

Hence for $m \in F$,

$$\phi_F(m) = \kappa_{\mathcal{M}} J_{\mathcal{M}}(m); \tag{7.51}$$

in particular, $\phi_F(m^*) = 0$.

From here and (7.50) we derive that the set $\{\phi_F(m)\}$ with F running through the set of all finite subspaces of S is a subset of the closed ball $\overline{B}_m \subset \mathcal{F}(S)^{**}$ centered at 0 and of radius $(A+\epsilon)d(m, m^*)$. In the weak* topology $\overline{B}_m$ is compact. From this point the proof repeats word for word that of Proposition 7.16. Namely, we consider the set Φ of functions $\psi : \mathcal{M} \to \mathcal{F}(S)^{**}$ satisfying

$$||\psi(m)||_{\mathcal{F}(S)^{**}} \le (A + \epsilon)d(m, m^*), \quad m \in \mathcal{M}.$$

Equip $\overline{B}_m$ with the weak* topology and introduce the set $Y := \prod_{m\in\mathcal{M}} B_m$ equipped with the product topology. Then Y is compact and, so, Φ is also compact in the topology induced by the bijection $\Phi \ni \psi \mapsto (\psi(m))_{m\in\mathcal{M}} \in Y$. Therefore there exists a subnet N of the net $\{\phi_F \; ; \; (F, m^*) \subset (S, m^*) \; , \; \mathrm{card}\, F < \infty\}$ such that

$$\lim_N \phi_F = \phi$$

for some $\phi \in \Phi$. By the definition of the product topology for every $m \in \mathcal{M}$,

$$\lim_N \phi_F(m) = \phi(m) \quad \text{(convergence in the weak}^* \text{ topology of } \mathcal{F}(S)^{**}).$$

Arguing as in the proof of Proposition 7.16, see (7.41), we derive from (7.50) that

$$||\phi||_{\mathrm{Lip}(\mathcal{M},\mathcal{F}(S)^{**})} \le A + \epsilon. \tag{7.52}$$

Moreover, for the subnet N' of the net $N = \{\phi_F\}$ determined by the condition $F \ni m$ and for every $m \in S$ we obtain from (7.51),

$$\phi(m) = \lim_{N'} \phi_F(m) = \kappa_{\mathcal{M}} J_{\mathcal{M}}(m). \tag{7.53}$$

Finally, we use the canonical embedding $\widetilde{\kappa}_S : \mathcal{F}(S)^* = \mathrm{Lip}_0(S) \to \mathcal{F}(S)^{***}$ to introduce the required extension operator $E \in \mathrm{Ext}(S, \mathcal{M})$ setting for $m \in \mathcal{M}$, $h \in \mathrm{Lip}_0(S)$,

$$(Eh)(m) := \langle \widetilde{\kappa}_S h, \phi(m)\rangle.$$

Since $\phi(m) \in \mathcal{F}(S)^{**}$, this is well defined. Then for $m', m'' \in \mathcal{M}$ we get from (7.52)

$$\begin{aligned}|(Eh)(m') - (Eh)(m'')| &\le ||h||_{\mathrm{Lip}_0(S)} ||\phi(m') - \phi(m'')||_{\mathcal{F}(S)^{**}} \\ &\le (A+\epsilon)||h||_{\mathrm{Lip}_0(S)} d(m', m'').\end{aligned}$$

Moreover, by (7.53) we have for $m \in S$,

$$(Eh)(m) = \langle \widetilde{\kappa}_S h,\ \kappa_{\mathcal{M}} J_{\mathcal{M}}(m)\rangle = \langle h,\ J_{\mathcal{M}}(m)\rangle = h(m).$$

Hence, $E \in \mathrm{Ext}(S, \mathcal{M})$ and $||E|| \le A + \epsilon$. This implies the converse to (7.44), inequality

$$\lambda(\mathcal{M}) \le \sup_F \lambda(F).$$

The proof of Theorem 7.12 (b) for $S = \mathcal{M}$ is complete.

Now we prove Theorem 7.12 in the general case.

(a) Assume that a sequence of metric spaces $\{(\mathcal{M}_i, d_i)\}_{i\in\mathbb{N}}$ finitely converges to $(\mathcal{M}, d)$. Suppose first that $\lambda(\mathcal{M}) < \infty$. Then the first part of Theorem 7.12 (b) (i.e., Corollary 7.13) implies:

Given $\epsilon > 0$ there is a finite subspace $F \subset \mathcal{M}$ such that

$$\lambda(\mathcal{M}) - \epsilon \le \lambda(F) \le \lambda(\mathcal{M}). \tag{7.54}$$

Definition 7.11 and that of the Gromov-Hausdorff convergence imply that for some subsequence $\mathbb{N}_0 \subset \mathbb{N}$ there exist finite subsets $F_i \subset \mathcal{M}_i$ and bi-Lipschitz homeomorphisms of F_i onto F with distortions D_i, $i \in \mathbb{N}_0$, such that $\lim_{i\to\infty} D_i = 1$. Then for every $i \in \mathbb{N}_0$ we clearly have

$$\lambda(F) \le \widetilde{D}_i^2 \lambda(F_i) \le \widetilde{D}_i^2 \lambda(\mathcal{M}_i).$$

Passing to the limit as $i \to \infty$ we get

$$\lambda(F) \le \varlimsup_{i\to\infty} \lambda(\mathcal{M}_i).$$

This and (7.54) then yield

$$\lambda(\mathcal{M}) \le \varlimsup_{i\to\infty} \lambda(\mathcal{M}_i).$$

Assume now that $\lambda(\mathcal{M}) = \infty$. Then the first part of Theorem 7.12 (b) implies for this case:

Given $l > 0$ there is a finite subspace $F \subset \mathcal{M}$ such that

$$l \le \lambda(F). \tag{7.55}$$

Using the above argument with (7.55) instead of (7.54) we obtain for arbitrary l that

$$l \le \varlimsup_{i\to\infty} \lambda(\mathcal{M}_i).$$

This completes the proof of Theorem 7.12 (a).

(b) Assume, in addition, that $\{(\mathcal{M}_i, d_i)\}_{i\in\mathbb{N}}$ δ-converges to a subspace $S \subset \mathcal{M}$. Then Definition 7.11 (b) and the definition of $\lambda(S)$ imply that

$$\lambda(S) = \lim_{i\to\infty} \lambda(\mathcal{M}_i).$$

This together with Theorem 7.12 (a) and Corollary 7.13 give

$$\lambda(\mathcal{M}) \le \lambda(S) = \sup_F \lambda(F)$$

where F runs over all finite subspaces of S.

The proof of Theorem 7.12 has been completed. □

For the next corollary of Theorem 7.12 we require the notion of a *dilation*, a bi-Lipschitz homeomorphism of $\mathcal{M}$ of distortion 1.

Corollary 7.19. *Assume that S is a subspace of $\mathcal{M}$ such that for some dilation $\phi : \mathcal{M} \to \mathcal{M}$ we have*

(a) $S \subset \phi(S)$*;*

(b) $\cup_{j=0}^{\infty}\phi^j(S)$ *is dense in $\mathcal{M}$.*

Then

$$\lambda(\mathcal{M}) = \sup_{F\subset S} \lambda(F)$$

where F runs over all finite subspaces of S.

Proof. From condition (b) it follows that the sequence $\{(\phi^j(S), d)\}_{j\in\mathbb{N}}$ finitely converges to $(\mathcal{M}, d)$. Since ϕ is a dilation of $\mathcal{M}$, every ϕ^j, $j > 1$, is a dilation of $\mathcal{M}$ as well. Thus the sequence $\{(\phi^j(S), d)\}_{j\in\mathbb{N}}$ δ-converges to (S, d). Now the required result follows from Theorem 7.12 (b). □

For ϕ being the identity map this and Corollaries 7.13 and 7.14 imply that $\lambda(S) = \lambda(\mathcal{M})$ for a dense subset $S \subset \mathcal{M}$.

Finally, as a consequence of Theorem 7.12 we obtain a unique for now sharp result (preserving Lipschitz constants) on simultaneous Lipschitz extensions.

Corollary 7.20. *Let $(\mathcal{M}, d)$ be an ultrametric space. Then*

$$\lambda(\mathcal{M}) = 1.$$

Proof. According to Theorem 7.12 it suffices to prove that $\lambda(\mathcal{M}) = 1$ for any finite ultrametric space $(\mathcal{M}, d)$. So let $S \subset \mathcal{M}$ be a proper subset of such a space. Take a point $m' \in \mathcal{M} \setminus S$ and denote by $r(m')$ a point from S such that

$$d(m', r(m')) = d(m', S).$$

Now for each function $f \in \mathrm{Lip}(S)$ we define its extension $\widetilde{f} \in \mathrm{Lip}(S \cup \{m'\})$ by the formula

$$\widetilde{f}(m) := \begin{cases} f(m), & \text{if } m \in S, \\ f(r(m')), & \text{if } m = m'. \end{cases}$$

Then the strong triangle inequality for d implies

$$\begin{aligned} |\widetilde{f}(m') - \widetilde{f}(m)| &= |f(r(m')) - f(m)| \le L(f)d(r(m'), m) \\ &\le L(f)\max\{d(m', r(m')), d(m', m)\} = L(f)d(m', m). \end{aligned}$$

Therefore $L(\widetilde{f}\,) = L(f)$.

Applying the same extension procedure to $S \cup \{m'\}$ and points outside this set we construct by induction an extension $\widehat{f} \in \mathrm{Lip}(\mathcal{M})$ of f with $L(\widehat{f}) = L(f)$. This gives a linear extension operator $E \in \mathrm{Ext}(S, \mathcal{M})$ of norm 1. Then Theorem 7.12 implies the required result. □

Remark 7.21. (a) For the relative extension constant $\lambda(S, \mathcal{M})$ we obtain by repeating literally the proof of Corollary 7.13 that

$$\lambda(S, \mathcal{M}) = \sup_F \lambda(F, \mathcal{M}) \tag{7.56}$$

where F runs over all finite subspaces of S.

(b) Using the compactness argument of the proof of Proposition 7.16 one can also show that for every subspace $S \subset \mathcal{M}$ there exists an extension operator $E_{min} \in \mathrm{Ext}(S, \mathcal{M})$ such that

$$||E_{min}|| = \lambda(S, \mathcal{M}). \tag{7.57}$$

(c) The same argument allows to establish the following fact.

The set function $S \mapsto \lambda(S)$ defined on closed subspaces of $\mathcal{M}$ is lower semi-continuous in the Hausdorff metric.

7.2 Main extension result

We prove that the direct p-sum of spaces of pointwise homogeneous type has the simultaneous Lipschitz extension property and estimate the corresponding Lipschitz constants. For convenience of the reader we recall several basic notions introduced in subsection 3.2.5 of Volume I which will be used in the proof.

A triple $(\mathcal{M}, d, \mathcal{F})$ where $\mathcal{F} = \{\mu_m\}_{m \in \mathcal{M}}$ is a collection of Borel measures on $(\mathcal{M}, d)$ is said to be a *space of pointwise homogeneous type* (*$\mathcal{PHT}$ space*) if the following holds:

(a) $\mathcal{F}$ is *uniformly doubling*, i.e., its *doubling constant* $D(\mathcal{F})$ is finite.
Here $D(\mathcal{F}) := D(\mathcal{F}; 2)$ and the *dilation function* $D(\mathcal{F}; \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is given

for $l > 1$ by

$$D(\mathcal{F}; l) := \sup\left\{\frac{\mu_m(B_{lR}(m))}{\mu_m(B_R(m))};\ R > 0 \text{ and } m \in \mathcal{M}\right\}, \tag{7.58}$$

see Volume I, Definitions 3.87 and 3.89.

(b) $\mathcal{F}$ is *consistent with the metric* d, i.e., for each $t > 0$ and all $R > 0$ and pairs $m_1, m_2 \in \mathcal{M}$ satisfying

$$d(m_1, m_2) \le tR$$

there exists a constant $C > 0$ such that the inequality

$$|\mu_{m_1} - \mu_{m_2}|(B_R(m)) \le \frac{C\mu_m(B_R(m))}{R}\, d(m_1, m_2) \tag{7.59}$$

holds for m equals m_1 or m_2.

The optimal C in this inequality denoted by $C(\mathcal{F}; t)$ is said to be the *consistency function* of $\mathcal{F}$ while the number $C(\mathcal{F}; 1)$ is denoted by $C(\mathcal{F})$ and is called the *consistency constant* of $\mathcal{F}$.

Finally, $\mathcal{F}$ is called K-*uniform* ($K \ge 1$) if for all $m_1, m_2 \in \mathcal{M}$ and $R > 0$

$$\mu_{m_1}(B_R(m_1)) \le K\mu_{m_2}(B_R(m_2)). \tag{7.60}$$

The theorem formulated below concerns the direct p-sum of the family $(\mathcal{M}_i, d_i, \mathcal{F}_i)$, $1 \le i \le N$, i.e., a triple $(\mathcal{M}, d_p, \mathcal{F})$ where

$$\mathcal{M} := \prod_{i=1}^N \mathcal{M}_i\,, \quad d_p := \left\{\sum_{i=1}^N d_i^p\right\}^{1/p}, \quad \mathcal{F} := \{\mu_{\widetilde{m}}\}_{\widetilde{m}\in\mathcal{M}}$$

$$\text{where for } \widetilde{m} = (m_1, \dots, m_N) \in \mathcal{M}\,, \quad \mu_{\widetilde{m}} := \bigotimes_{i=1}^N \mu_{m_i}.$$

Theorem 7.22. *Let $(\mathcal{M}_i, d_i, \mathcal{F}_i)$ be $\mathcal{PHT}$ spaces such that $\mathcal{F}_i$ is K_i-uniform, $1 \le i \le N$. Then its direct p-sum $(\mathcal{M}, d_p, \mathcal{F})$ ($1 \le p \le \infty$) has the simultaneous Lipschitz extension property and the extension constant satisfies*

$$\lambda(\mathcal{M}, X) \le c_0(\log_2 D + KC + 1), \tag{7.61}$$

where X is an arbitrary Banach space,

$$K := \prod_{i=1}^N K_i, \quad D := \prod_{i=1}^N D(\mathcal{F}_i), \quad C := \left\{\sum_{i=1}^N C(\mathcal{F}_i)^q\right\}^{1/q},$$

c_0 is a numerical constant and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 7.23. (a) For the special case of a single $\mathcal{PHT}$ space the assumption of K-uniformity can be excluded. It can be also eliminated for $N > 1$ but the estimate for the extension constant becomes in this case essentially worse.

(b) The linear operator $E \in \mathrm{Ext}(S, \mathcal{M}, X)$ constructed in the theorem extends a Lipschitz function $f : S \to X$ to that on $\mathcal{M}$ assigning its values in the closed convex hull of $f(S)$. Therefore the theorem holds for a target space being a closed convex subset of a Banach space.

(c) If some of the spaces $\mathcal{M}_i$ are of homogeneous type (with doubling measures μ_i), then for these $\mathcal{M}_i$ the family $\mathcal{F}_i = \{\mu_i\}$ and $D(\mathcal{F}_i) = D(\mu_i)$, $C(\mathcal{F}_i) = 0$ and K_i may be taken to be equal to 1.

Since the proof is long and rather involved, we briefly discuss its structure and basic steps.

We begin with the construction of the desired linear extension operator for a $\mathcal{PHT}$ space $(\mathcal{M}, d, \mathcal{F})$ and its subspace S. Since $\mathrm{Lip}_0(\mathcal{M})$ is 1-complemented in $\mathrm{Lip}(\mathcal{M})$, it suffices to find E to be acting from $\mathrm{Lip}_0(S)$ into $\mathrm{Lip}_0(\mathcal{M})$.

Hereafter m^* is a fixed point of $\mathcal{M}$ and every subspace $S \subset (\mathcal{M}, m^*)$ contains m^* whereas $\mathrm{Lip}_0(\mathcal{M})$ and $\mathrm{Lip}_0(S)$ are determined by the condition $f(m^*) = 0$. Further, without loss of generality we assume that S is a proper closed subspace of $\mathcal{M}$.

To introduce the operator E we exploit the Dugundji extension construction from the proof of Borsuk-Dugundji Theorem 1.8 of Volume I to construct a linear extension operator $f \mapsto \widehat{f}$ sending $\mathrm{Lip}_0(S, X)$ into the space of locally bounded continuous functions $C_b^{loc}(\mathcal{M}, X)$. Then we use the family $\mathcal{F} = \{\mu_m\}_{m \in \mathcal{M}}$ to "smooth" $\widehat{f}$ outside of S, exploiting for this goal an average operator given for $g \in C_b^{loc}(\mathcal{M}, X)$, $m \in \mathcal{M}$ and $R > 0$ by

$$I(g; m, R) := \frac{1}{\mu_m(B_R(m))} \int_{B_R(m)} g \, d\mu_m.$$

The required extension operator is then given for $f \in \mathrm{Lip}_0(S, X)$ by

$$(Ef)(m) := I(\widehat{f}; m; d(m))$$

where we set hereafter

$$d(m) := d(m, S) \, (= \inf_{m' \in S} d(m, m')). \tag{7.62}$$

In the next, considerably harder, part of the proof we will estimate the norm of $E : \mathrm{Lip}_0(S, X) \to \mathrm{Lip}_0(\mathcal{M}, X)$ via the basic characteristics of the family $\mathcal{F}$. In the derivation we will use the dilation function $t \mapsto D(\mathcal{F}; t)$ instead of the dilation constant $D(\mathcal{F})$ to minimize the final estimate over $t > 1$. For the dilation function equivalent to $t \mapsto t^\lambda$ for some $\lambda > 0$ an argument presented below will give the estimate of the norm:

$$||E|| \le c_0(\log_2 D(\mathcal{F}) + C(\mathcal{F}) + 1);$$

the uniformity constant K may be omitted in this case.

However, in general, this approach gives a result far beyond from the required. To improve the situation we will apply this method to a $\mathcal{PHT}$ space, say $(\widehat{\mathcal{M}}, \widehat{d}, \widehat{\mathcal{F}})$, which extends the initial space $(\mathcal{M}, d, \mathcal{F})$ in the sense that $\widehat{\mathcal{M}}$ contains an isometric copy of $\mathcal{M}$. Therefore the corresponding extension constants satisfy

$$\lambda(\mathcal{M}, X) \le \lambda(\widehat{\mathcal{M}}, X). \tag{7.63}$$

As soon as a suitable bound of the right-hand side via the basic characteristics of $\widehat{\mathcal{F}}$ would be established, it remained to bound them by those of $\mathcal{F}$. For a technical reason the latter derivation will be done before estimating the norm of the operator E.

These results lead to the required estimate of the simultaneous extension constant $\lambda(\mathcal{M}, X)$.

At the final step we apply the results obtained to a $\mathcal{PHT}$ space being the direct p-sum of $\mathcal{PHT}$ spaces $(\mathcal{M}_i, d_i, \mathcal{F}_i)$, $1 \le i \le N$. Combining this with the results of Proposition 3.92 of Volume I estimating the basic characteristics of the direct p-sum by those of $(\mathcal{M}_i, d_i, \mathcal{F}_i)$ we will complete the proof.

Proof of Theorem 7.22.

A. *Extension operator*

Let $(\mathcal{M}, d, m^*)$ be a pointed metric space and S be its subspace. We should construct a linear extension operator from $\mathrm{Lip}_0(S, X)$ into $\mathrm{Lip}_0(\mathcal{M}, X)$ where X is a Banach space. To this end we first recall the Dugundji extension method, see the proof of Borsuk-Dugundji Theorem 1.8 of Volume I.

Let g be a continuous function on a closed subspace S of a metric space $(\mathcal{M}, d)$ ranged into a Banach space X. Let $\{B_m\}_{m \in S^c}$ be a cover of the open set $S^c := \mathcal{M} \setminus S$ by the open balls

$$B_m := B_{r_m}(m), \quad \text{where} \quad r_m := \frac{1}{3} d(m, S). \tag{7.64}$$

Since any metric space is paracompact, there exists a continuous partition of unity $\{p_\alpha\}_{\alpha \in A}$ subordinate to the cover $\{B_m\}$ whose supports $U_\alpha := \{m \in S^c \,;\, p_\alpha(m) > 0\}$ form a *locally finite* cover of S^c, see Volume I, Proposition 3.17.

For every $\alpha \in A$, we now pick points

$$m_1(\alpha) \in S \quad \text{and} \quad m_2(\alpha) \in U_\alpha = \operatorname{supp} p_\alpha \tag{7.65}$$

such that

$$d(m_1(\alpha), m_2(\alpha)) < 2d(m_2(\alpha), S). \tag{7.66}$$

The Dugundji extension operator $g \mapsto \widehat{g}$ is then given for $m \in \mathcal{M}$ by

$$\widehat{g}(m) := \begin{cases} g(m), & \text{if } m \in S, \\ \sum_{\alpha \in A} g(m_1(\alpha)) p_\alpha(m), & \text{if } m \in S^c. \end{cases} \tag{7.67}$$

Repeating word-for-word the arguments of the proof of Theorem 1.8 of Volume I one obtains that $\widehat{g}$ is continuous on $\widehat{M}$.

Lemma 7.24. *Let $f \in \mathrm{Lip}_0(S, X)$. Then the extended function $\widehat{f} : \mathcal{M} \to X$ satisfies for all $m, m' \in \mathcal{M}$ the inequality*

$$||\widehat{f}(m) - \widehat{f}(m')||_X \leq 7L(f)\{d(m, m') + d(m, S) + d(m', S)\}. \tag{7.68}$$

Proof. In the case $m, m' \in S$, inequality (7.68) (even with constant 1) is trivial, since $\widehat{f} = f$ on S and $d(m, S) = d(m', S) = 0$.

Let now $m \in S$ and $m' \in S^c$. We denote by V_m an open ball in the Banach space X given by the inequality

$$||\widehat{f}(m) - x||_X < (5d(m, m') + 2d(m', S))L(f), \quad x \in X. \tag{7.69}$$

Inequality (7.68) in this case clearly would follow from the inclusion

$$\widehat{f}(m') \in V_m. \tag{7.70}$$

Since $\widehat{f}(m')$ is a convex combination of the points $f(m_1(\alpha))$, $\alpha \in A_0$, where the finite set A_0 is given by

$$A_0 := \{\alpha \in A \ ; \ m' \in \mathrm{supp}\ p_\alpha\},$$

see (7.67), inclusion (7.70) follows from the condition

$$f(m_1(\alpha)) \in V_m, \quad \alpha \in A_0.$$

This, in turn, is a direct consequence of the inequality

$$d(m_1(\alpha), m) < 5d(m, m') + 2d(m', S) \tag{7.71}$$

and the fact that $f \in \mathrm{Lip}_0(S, X)$.

To prove (7.71) we choose for $\alpha \in A_0$ a point $m(\alpha) \in S^c$ so that

$$B_{m(\alpha)} \supset \mathrm{supp}\ p_\alpha \ \ (\ni m_2(\alpha)).$$

Then $m' \in B_{m(\alpha)}$, $m \in S$, and this and (7.64) imply that

$$d(m(\alpha), S) \leq d(m(\alpha), m) \leq d(m(\alpha), m') + d(m', m) \leq \frac{1}{3} d(m(\alpha), S) + d(m, m').$$

Hence,

$$d(m(\alpha), S) \le d(m(\alpha), m) \le \frac{3}{2} d(m, m'). \tag{7.72}$$

Further, $m_2(\alpha) \in B_{m(\alpha)}$ and therefore

$$d(m_2(\alpha), m) \le d(m_2(\alpha), m(\alpha)) + d(m(\alpha), m) \le \frac{1}{3} d(m(\alpha), S) + d(m(\alpha), m).$$

Combining this with the previous inequality we obtain

$$d(m_2(\alpha), m) \le 3d(m, m').$$

Finally, this, (7.66) and (7.72) together with the inequality

$$d(m_1(\alpha), m) \le d(m_1(\alpha), m_2(\alpha)) + d(m_2(\alpha), m)$$

give the required inequality (7.71).

It remains to consider the case of $m, m' \in S^c$. For the sake of definiteness, let

$$d(m', S) \le d(m, S). \tag{7.73}$$

Given $\epsilon > 0$ we pick a point $m'' \in S$ satisfying

$$d(m', m'') \le d(m', S) + \epsilon$$

and write

$$||\widehat{f}(m) - \widehat{f}(m')||_X \le ||\widehat{f}(m) - \widehat{f}(m'')||_X + ||\widehat{f}(m'') - \widehat{f}(m')||_X.$$

Since $m'' \in S$, we can apply the estimate obtained in the previous part of the proof to bound the right-hand side by

$$L(f)\{2(d(m, S) + d(m', S)) + 5(d(m, m'') + d(m', m''))\}.$$

Moreover, by the choice of m'',

$$d(m, m'') \le d(m, m') + d(m', m'') \le d(m, m') + d(m', S) + \epsilon.$$

Therefore, the sum in the curly brackets is bounded by

$$2(d(m, S) + d(m', S)) + 5d(m, m') + 10d(m', S) + 10\epsilon.$$

This and (7.73), in turn, give the required inequality (7.68).

The lemma has been proved. □

We are now ready to define the required extension operator E. It is given for $f \in \mathrm{Lip}_0(S, X)$ by

$$(Ef)(m) := \begin{cases} f(m), & \text{if } m \in S, \\ I(\widehat{f}; m, d(m)), & \text{if } m \in S^c. \end{cases} \tag{7.74}$$

Here $\widehat{f}$ is defined by (7.67), and for a continuous and locally bounded function (i.e., continuous and bounded on every bounded subset of $\mathcal{M}$) $g : \mathcal{M} \to X$ we set

$$I(g; m, R) := \frac{1}{\mu_m(B_R(m))} \int_{B_R(m)} g \, d\mu_m. \tag{7.75}$$

According to Lemma 7.24, the operator E is well defined, that is, the vector function $\widehat{f}$ is continuous and bounded on every bounded subset of $\mathcal{M}$.

The extension operator constructed will then be applied to the aforementioned extension of the $\mathcal{PHT}$ space $(\mathcal{M}, d, \mathcal{F})$ denoted now by $(\mathcal{M}_n, d_n, \mathcal{F}_n)$. Its underlying set is given by

$$\mathcal{M}_n := \mathcal{M} \times \mathbb{R}^n \tag{7.76}$$

and $\widetilde{m} = (m, x)$, $\widetilde{m}' = (m', x')$ etc. stand for its points.

Further, the metric d_n is given for $\widetilde{m} = (m, x)$, $\widetilde{m}' = (m', x')$ by

$$d_n(\widetilde{m}, \widetilde{m}') := d(m, m') + \sum_{i=1}^{n} |x_i - x'_i|. \tag{7.77}$$

The integer $n \geq 2$ will be chosen later to minimize the corresponding estimates.

We equip the space $(\mathcal{M}_n, d_n)$ with the family of measures $\mathcal{F}_n := \{\mu_{\widetilde{m}}\}_{\widetilde{m} \in \mathcal{M}_n}$ where for $\widetilde{m} = (m, x)$,

$$\mu_{\widetilde{m}} := \mu_m \otimes \lambda_n; \tag{7.78}$$

here λ_n is the Lebesgue measure on $\mathbb{R}^n$.

B. *Properties of the extended metric space*

It is easy to show that the $(\mathcal{M}_n, d_n, \mathcal{F}_n)$ is a $\mathcal{PHT}$ space but we need qualitative estimates of its basic parameters in terms of those for $(\mathcal{M}, d, \mathcal{F})$.

This goal will be achieved by several lemmas presented below. In the formulation of the first lemma $D_n(\cdot)$ denotes the dilation function for $(\mathcal{M}_n, d_n, \mathcal{F}_n)$ defined as in (7.58) with μ_m replaced by measure (7.78). We denote by $D := D(\mathcal{F}; 2)$ the doubling constant of $(\mathcal{M}, d, \mathcal{F})$.

Lemma 7.25. *Assume that n is related to the doubling constant D by*

$$n \geq \lfloor \log_2 D \rfloor + 5. \tag{7.79}$$

Then we have

$$D_n(1 + 1/n) \leq \frac{6}{5} e^4.$$

Proof. The result follows from Lemma 3.94 of Volume I where $(\mathcal{M}', d')$ is taken to be ℓ_1^n. □

Our next auxiliary result evaluates the constant $C_n(1/n) := C(\mathcal{F}_n; 1/n)$ in terms of $C(1/n) := C(\mathcal{F}; 1/n))$; recall that the consistency function $C(\mathcal{F}; \cdot)$ is defined in (7.59).

Lemma 7.26. $C_n(1/n) \le \left(1+\frac{4e}{3}\right) nC(1/n)$.

Proof. Using Fubini's theorem, we obtain

$$\mu_{\widetilde{m}}(B_R(\widetilde{m})) = \beta_n \int_0^R \mu_m(B_s(m))(R-s)^{n-1}ds \tag{7.80}$$

where β_n is the volume of the unit sphere in ℓ_1^n. Then for $\widetilde{m}_i = (m_i, x^i)$, $i = 1, 2$, we have

$$|\mu_{\widetilde{m}_1} - \mu_{\widetilde{m}_2}|(B_R(\widetilde{m}_i)) \le \beta_n \int_0^R |\mu_{m_1} - \mu_{m_2}|(B_s(m_i)) \cdot (R-s)^{n-1}ds.$$

Now let $d_n(\widetilde{m}_1, \widetilde{m}_2) \le \frac{R}{n}$. Divide the interval of integration into subintervals $[0, R/n]$ and $[R/n, R]$ and denote the corresponding integrals over these intervals by I_1 and I_2. Replacing $B_s(m_i)$ in I_1 by the larger ball $B_{s+R/n}(m_i)$ and applying (7.59) with $t = 1/n$ we obtain

$$I_1 \le C(1/n)\left(\beta_n \int_0^{R/n} \frac{\mu_{m_i}(B_{s+R/n}(m_i))}{s+R/n}(R-s)^{n-1}ds\right) d(m_1, m_2).$$

Replacing s by $t = s + R/n$ we bound the expression in the brackets by

$$\left(\beta_n \int_{R/n}^{2R/n} \mu_{m_i}(B_t(m_i))(R-t)^{n-1}dt\right) \max_{R/n \le t \le 2R/n} \frac{(R+R/n-t)^{n-1}}{t(R-t)^{n-1}}.$$

Since $[R/n, 2R/n] \subset [0, R]$ and the maximum is at most $\frac{n}{R}\left(1+\frac{1}{n-2}\right)^{n-1} < \frac{4e}{3}\frac{n}{R}$ for $n \ge 5$, this and (7.80) yield

$$I_1 \le \frac{4e}{3} C(1/n) n \frac{\mu_{\widetilde{m}_i}(B_R(\widetilde{m}_i))}{R} d(m_1, m_2).$$

For the second term we get from (7.59),

$$I_2 \le C(1/n)\left(\beta_n \int_{R/n}^{R} \frac{\mu_{m_i}(B_s(m_i))}{s}(R-s)^{n-1}ds\right) d(m_1, m_2)$$

and by (7.80) the term in the brackets is at most $\mu_{\widetilde{m}_i}(B_R(\widetilde{m}_i)) \cdot \frac{n}{R}$. Hence, we have

$$I_2 \le C(1/n) n \frac{\mu_{\widetilde{m}_i}(B_R(\widetilde{m}_i))}{R} d(m_1, m_2).$$

Further note that $d(m_1, m_2) \le d_n(\widetilde{m}_1, \widetilde{m}_2)$. Hence, we obtain finally the inequality

$$|\mu_{\widetilde{m}_1} - \mu_{\widetilde{m}_2}|(B_R(\widetilde{m}_i)) \le \left(1+\frac{4e}{3}\right) nC(1/n) \frac{\mu_{\widetilde{m}_i}(B_R(\widetilde{m}_i))}{R} d_n(\widetilde{m}_1, \widetilde{m}_2)$$

whence $C_n(1/n) \le \left(1+\frac{4e}{3}\right) nC(1/n)$. □

Lemma 7.27. *Let* $A_n := \frac{6}{5}e^4 n$ *and* $n \geq \lfloor \log_2 D \rfloor + 6$*. Then for all* $R_2 \geq R_1 > 0$ *and* $\widetilde{m} \in \mathcal{M}_n$*,*

$$\mu_{\widetilde{m}}(B_{R_2}(\widetilde{m})) - \mu_{\widetilde{m}}(B_{R_1}(\widetilde{m})) \leq A_n \frac{\mu_{\widetilde{m}}(B_{R_2}(\widetilde{m}))}{R_2}(R_2 - R_1).$$

Proof. We write $\mathcal{M}_n = \mathcal{M}_{n-1} \times \mathbb{R}$ and $\mu_{\widetilde{m}} = \mu_{\widehat{m}} \otimes \lambda_1$ where $\widehat{m} \in \mathcal{M}_{n-1} := \mathcal{M} \oplus^{(1)} \mathbb{R}^{n-1}$. Then by Fubini's theorem we have for $0 < R_1 \leq R_2$,

$$\mu_{\widetilde{m}}(B_{R_2}(\widetilde{m})) - \mu_{\widetilde{m}}(B_{R_1}(\widetilde{m})) = 2\int_{R_1}^{R_2} \mu_{\widehat{m}}(B_s(\widehat{m}))ds \leq \frac{2R_2\mu_{\widehat{m}}(B_{R_2}(\widehat{m}))}{R_2}(R_2 - R_1).$$

We claim that for arbitrary $l > 1$ and $R > 0$,

$$R\mu_{\widehat{m}}(B_R(\widehat{m})) \leq \frac{lD_{n-1}(l)}{2(l-1)}\mu_{\widetilde{m}}(B_R(\widetilde{m})). \tag{7.81}$$

Together with the previous inequality this will yield

$$\mu_{\widetilde{m}}(B_{R_2}(\widetilde{m})) - \mu_{\widetilde{m}}(B_{R_1}(\widetilde{m})) \leq \frac{lD_{n-1}(l)}{l-1} \cdot \frac{\mu_{\widetilde{m}}(B_{R_2}(\widetilde{m}))}{R_2}(R_2 - R_1).$$

Finally we choose here $l = 1 + \frac{1}{n-1}$ and use Lemma 7.25. This will give the required inequality.

It remains to establish (7.81). By the definition of the dilation function D_{n-1} for $\mathcal{M}_{n-1}$ we have

$$\mu_{\widetilde{m}}(B_{lR}(\widetilde{m})) = 2l\int_0^R \mu_{\widehat{m}}(B_{ls}(\widehat{m}))ds \leq lD_{n-1}(l)\mu_{\widetilde{m}}(B_R(\widetilde{m})).$$

On the other hand, replacing $[0, R]$ by $[l^{-1}R, R]$ we also have

$$\mu_{\widetilde{m}}(B_{lR}(\widetilde{m})) \geq 2l\mu_{\widehat{m}}(B_R(\widehat{m}))(R - l^{-1}R) = 2(l-1)R\mu_{\widehat{m}}(B_R(\widehat{m})).$$

Combining the last two inequalities we get (7.81). □

C. *Norm of the extension operator*

Let $\widehat{E}$ be the linear extension operator for the space $(\mathcal{M}_n, d_n, \mathcal{F}_n)$ constructed in part A. Hence, $\widehat{E} \in \mathrm{Ext}(S, \mathcal{M}_n, X)$ where S is an arbitrary closed proper subspace of $\mathcal{M}_n$.

Proposition 7.28. *For some numerical constant* $c_0 > 1$ *and*

$$n := \lfloor \log_2 D \rfloor + 6 \tag{7.82}$$

the following is true

$$\|\widehat{E}\| \leq c_0 \left(n + C\left(\mathcal{F}_n; \frac{1}{n}\right)\right).$$

Proof. We need several auxiliary results. In their derivations we use the following notation:

$$K_n(l) := 42(A_n + \widehat{C}_n)(l+3)D_n(l) \tag{7.83}$$

where $\widehat{C}_n := C_n(\mathcal{F}_n; \frac{1}{n})$; recall that

$$A_n := \frac{6}{5}e^4 n \quad \text{and} \quad D_n(l) := D(\mathcal{F}_n; l).$$

Proposition 7.29. *The following inequality holds for* $l := 1 + \frac{1}{n}$,

$$||\widehat{E}|| \le 56A_n + \max\left\{\frac{14(l+3)}{l-1}, K_n(l)\right\}. \tag{7.84}$$

Before our proof we derive from here Proposition 7.28. Since for $l = 1 + \frac{1}{n}$,

$$\frac{14(l+3)}{l-1} < 70n < 42A_n,$$

the maximum in (7.84) is attained at the second term. Moreover, by Lemma 7.25,

$$D\left(\mathcal{F}_n; 1 + \frac{1}{n}\right) \le \frac{6}{5}e^4,$$

whence for some numerical constant $c_1 > 1$,

$$\|\widehat{E}\| \le 56A_n + 42 \cdot \frac{6}{5}e^4\left(4 + \frac{1}{n}\right)[A_n + \widehat{C}_n] \le c_1\left(n + C\left(\mathcal{F}_n; \frac{1}{n}\right)\right),$$

as required. □

Proof of Proposition 7.29. We assume without loss of generality that

$$||f||_{\mathrm{Lip}(S,X)} = 1 \tag{7.85}$$

and simplify the forthcoming computations by introducing the following notation:

$$\begin{aligned} &R_i := d_n(\widetilde{m}_i) := d_n(\widetilde{m}_i, S),\ \mu_i := \mu_{\widetilde{m}_i},\\ &B_{ij} := B_{R_j}(\widetilde{m}_i),\ v_{ij} := \mu_i(B_{ij}),\quad 1 \le i, j \le 2. \end{aligned} \tag{7.86}$$

Assuming for definiteness that $0 < R_1 \le R_2$ we have by the triangle inequality

$$0 \le R_2 - R_1 \le d_n(\widetilde{m}_1, \widetilde{m}_2). \tag{7.87}$$

Further, due to Lemma 7.27,

$$v_{i2} - v_{i1} \le \frac{A_n v_{i2}}{R_2}(R_2 - R_1). \tag{7.88}$$

Next, using the consistency inequality for $\mathcal{F}_n$ with $t = \frac{1}{n}$, see (7.59), we also have

$$|\mu_1 - \mu_2|(B_{ij}) \le \frac{\widehat{C}_n v_{ij}}{R_j} d_n(\widetilde{m}_1, \widetilde{m}_2), \quad d_n(\widetilde{m}_1, \widetilde{m}_2) \le \frac{R}{n}. \tag{7.89}$$

Now, from inequality (7.68) applied to our setting and the triangle inequality we obtain

$$\max\{||\widetilde{f}(\widetilde{m})||_X \ ; \ \widetilde{m} \in B_{i2}\} \le 28R_2 + 7(i-1)d_n(\widetilde{m}_1, \widetilde{m}_2); \tag{7.90}$$

here $i = 1, 2$ and we set

$$\widetilde{f}(\widetilde{m}) := \widehat{f}(\widetilde{m}) - \widehat{f}(\widetilde{m}_1); \tag{7.91}$$

recall that $\widehat{f}$ is the Dugundji extension of f given by (7.67).

We now estimate $\|(\widehat{E}f)(\widetilde{m}_2) - (\widehat{E}f)(\widetilde{m}_1)\|_X$ for $\widetilde{m}_1 \in S$ and $\widetilde{m}_2 \notin S$. We begin with the evident inequality

$$||(\widehat{E}f)(\widetilde{m}_2) - (\widehat{E}f)(\widetilde{m}_1)||_X = \frac{1}{v_{22}} \left|\left| \int_{B_{22}} \widetilde{f}(\widetilde{m}) d\mu_2 \right|\right|_X \le \max_{B_{22}} ||\widetilde{f}||_X.$$

Applying (7.90) with $i = 2$ we then bound this maximum by $28R_2 + 7d_n(\widetilde{m}_1, \widetilde{m}_2)$. But $\widetilde{m}_1 \in S$ and therefore

$$R_2 = d_n(\widetilde{m}_2) \le d_n(\widetilde{m}_1, \widetilde{m}_2);$$

hence, in this case

$$||(\widehat{E}f)(\widetilde{m}_2) - (\widehat{E}f)(\widetilde{m}_1)||_X \le 35\|f\|_{\mathrm{Lip}(S,X)} d_n(\widetilde{m}_1, \widetilde{m}_2). \tag{7.92}$$

The remaining case $\widetilde{m}_1, \widetilde{m}_2 \notin S$ requires some additional auxiliary results. For their formulations we first write

$$(\widehat{E}f)(\widetilde{m}_1) - (\widehat{E}f)(\widetilde{m}_2) := D_1 + D_2, \tag{7.93}$$

where

$$\begin{aligned} D_1 &:= I(\widetilde{f}; \widetilde{m}_1, R_1) - I(\widetilde{f}; \widetilde{m}_1, R_2), \\ D_2 &:= I(\widetilde{f}; \widetilde{m}_1, R_2) - I(\widetilde{f}; \widetilde{m}_2, R_2); \end{aligned} \tag{7.94}$$

recall that the average operator I is given by (7.75).

Lemma 7.30. *We have*

$$||D_1||_X \le 56 A_n d_n(\widetilde{m}_1, \widetilde{m}_2);$$

recall that $A_n := \frac{6}{5} e^4 n$.

Proof. Using the notations introduced, see (7.94), (7.91) and (7.86), we write

$$D_1 = \frac{1}{v_{11}}\int_{B_{11}} \widetilde{f} d\mu_1 - \frac{1}{v_{12}}\int_{B_{12}} \widetilde{f} d\mu_1 = \left(\frac{1}{v_{11}} - \frac{1}{v_{12}}\right)\int_{B_{11}} \widetilde{f} d\mu_1 - \frac{1}{v_{12}}\int_{B_{12}\setminus B_{11}} \widetilde{f} d\mu_1.$$

This immediately implies that

$$||D_1||_X \leq 2 \cdot \frac{v_{12} - v_{11}}{v_{12}} \cdot \max_{B_{12}} ||\widetilde{f}||_X.$$

Applying now (7.88) and (7.87), and then (7.90) with $i = 1$ we get the desired estimate. □

To obtain a similar estimate for D_2 we will use the following two facts.

Lemma 7.31. *Assume that for a given* $l \in [1, 1 + 1/n]$,

$$d_n(\widetilde{m}_1, \widetilde{m}_2) \leq (l - 1)R_2 \tag{7.95}$$

and let for definiteness $v_{22} \leq v_{12}$. *Then for symmetric difference* $B_{12}\Delta B_{22}$ *it is true that*

$$\mu_2(B_{12}\Delta B_{22}) \leq 2(A_n + \widehat{C}_n)D_n(l)\frac{v_{12}}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2). \tag{7.96}$$

Proof. Set $R := R_2 + d_n(\widetilde{m}_1, \widetilde{m}_2)$. Then $B_{12} \cup B_{22} \subset B_R(\widetilde{m}_1) \cap B_R(\widetilde{m}_2)$ and therefore

$$\mu_2(B_{12}\Delta B_{22}) \leq (\mu_2(B_R(\widetilde{m}_1)) - \mu_2(B_{12})) + (\mu_2(B_R(\widetilde{m}_2)) - \mu_2(B_{22})). \tag{7.97}$$

The first term on the right-hand side is at most

$$|\mu_2 - \mu_1|(B_R(\widetilde{m}_1)) + |\mu_2 - \mu_1|(B_{R_2}(\widetilde{m}_1)) + (\mu_1(B_R(\widetilde{m}_1)) - \mu_1(B_{R_2}(\widetilde{m}_1))).$$

Estimating the first two terms by the consistency inequality, see (7.89), and the third by Lemma 7.27 we bound this sum by

$$\widehat{C}_n\left(\frac{\mu_1(B_R(\widetilde{m}_1))}{R} + \frac{\mu_1(B_{R_2}(\widetilde{m}_1))}{R_2}\right) d_n(\widetilde{m}_1, \widetilde{m}_2) + A_n \frac{\mu_1(B_R(\widetilde{m}_1))}{R}(R - R_2).$$

Moreover, $R_2 \leq R \leq lR_2$, see (7.95), and $R - R_2 := d_n(\widetilde{m}_1, \widetilde{m}_2)$. By the definition of the dilation function $D_n(l) := D(\mathcal{F}_n; l)$, see (7.58), we therefore have in notation (7.86),

$$\mu_2(B_R(\widetilde{m}_1)) - \mu_2(B_{12}) \leq [\widehat{C}_n(D_n(l) + 1) + A_n D_n(l)]\frac{v_{12}}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2).$$

Similarly, from Lemma 7.27 and the inequality $v_{22} \leq v_{12}$ we derive that

$$\begin{aligned}\mu_2(B_R(\widetilde{m}_2)) - \mu_2(B_{22}) &\leq A_n \frac{\mu_2(B_R(\widetilde{m}_2))}{R}(R - R_2)\\ &\leq A_n D_n(l)\frac{v_{22}}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2) \leq A_n D_n(l)\frac{v_{12}}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2).\end{aligned}$$

Combining the last two estimates with (7.97) we get the result. □

Lemma 7.32. *Under the assumptions of the previous lemma*

$$v_{12} - v_{22} \le 3(A_n + \widehat{C}_n)D_n(l)\frac{v_{12}}{R_2}d_n(\widetilde{m}_1, \widetilde{m}_2). \tag{7.98}$$

Proof. By (7.86) the left-hand side is bounded by

$$|\mu_1(B_{12}) - \mu_2(B_{12})| + \mu_2(B_{12}\Delta B_{22}).$$

Estimating these terms by (7.89) and (7.96) we get the required result. □

We now estimate D_2 from (7.94) beginning with

Lemma 7.33. *Under the assumptions of Lemma 7.31 we have*

$$||D_2||_X \le K_n(l)d_n(\widetilde{m}_1, \widetilde{m}_2);$$

recall that $K_n(l) := 42(A_n + \widehat{C}_n)D_n(l)(l+3)$.

Proof. By the definition of D_2 we have in the notation introduced

$$\begin{aligned} ||D_2||_X &:= \left|\left|\frac{1}{v_{12}}\int_{B_{12}} \widetilde{f} d\mu_1 - \frac{1}{v_{22}}\int_{B_{22}} \widetilde{f} d\mu_2\right|\right|_X \\ &\le \frac{1}{v_{12}}\int_{B_{12}} ||\widetilde{f}||_X \, d|\mu_1 - \mu_2| + \frac{1}{v_{12}}\int_{B_{12}\Delta B_{22}} ||\widetilde{f}||_X \, d\mu_2 \\ &\quad + \left|\frac{1}{v_{12}} - \frac{1}{v_{22}}\right| \int_{B_{22}} ||\widetilde{f}||_X \, d\mu_2 := J_1 + J_2 + J_3. \end{aligned}$$

Now (7.89) and (7.90) with $i = 1$ imply that

$$J_1 \le \frac{1}{v_{12}}|\mu_1 - \mu_2|(B_{12}) \sup_{B_{12}} ||\widetilde{f}||_X \le \frac{\widehat{C}_n}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2) 28R_2 = 28\widehat{C}_n d_n(\widetilde{m}_1, \widetilde{m}_2).$$

In turn, by (7.96), (7.95) and (7.90) we have

$$\begin{aligned} J_2 &\le \frac{1}{v_{12}}\mu_2(B_{12}\Delta B_{22}) \sup_{B_{12}\Delta B_{22}} ||\widetilde{f}||_X \\ &\le \frac{2(A_n + \widehat{C}_n)D_n(l)}{R_2} d_n(\widetilde{m}_1, \widetilde{m}_2)(7d_n(\widetilde{m}_1, \widetilde{m}_2) + 28R_2) \\ &\le 14(A_n + \widehat{C}_n)D_n(l)(l+3)d_n(\widetilde{m}_1, \widetilde{m}_2). \end{aligned}$$

Finally, (7.98), (7.90) and (7.95) yield

$$J_3 \le 21(A_n + \widehat{C}_n)D_n(l)(l+3)d_n(\widetilde{m}_1, \widetilde{m}_2).$$

Combining these we get the required estimate. □

It remains to consider the case of points $\widetilde{m}_1, \widetilde{m}_2 \in \mathcal{M}_n$ satisfying the inequality

$$d_n(\widetilde{m}_1, \widetilde{m}_2) > (l-1)R_2$$

converse to (7.95). In this case, definition (7.94) of D_2 and (7.90) imply that

$$||D_2||_X \le 2 \sup_{B_{12} \cup B_{22}} ||\widetilde{f}||_X \le 2(28R_2 + 7d_n(\widetilde{m}_1, \widetilde{m}_2)) \le 14\left(\frac{4}{l-1} + 1\right) d_n(\widetilde{m}_1, \widetilde{m}_2).$$

Combining this with the inequalities of Lemmas 7.30 and 7.33 and equality (7.92) we obtain the required estimate of the Lipschitz norm of the extension operator $\widehat{E}$:

$$||\widehat{E}|| \le 56A_n + \max\left(\frac{14(l+3)}{l-1}, K_n(l)\right) \tag{7.99}$$

where $K_n(l)$ is the constant in (7.83).

This proves Proposition 7.29 and, hence, Theorem 7.22 for $N = 1$. □

D. *Proof of Theorem* 7.22 *for* $N > 1$

At this stage we apply Proposition 7.29 to the space $(\mathcal{M}, d_p, \mathcal{F})$ being the direct p-product of $\mathcal{PHT}$ spaces $(\mathcal{M}_i, d_i, \mathcal{F}_i)$, $1 \le i \le N$. We change notation, denoting by $(\widehat{\mathcal{M}}, \widehat{d})$ the direct 1-sum $\mathcal{M} \oplus^{(1)} \ell_1^n$ and by $\widehat{\mathcal{F}}$ the tensor product $\mathcal{F} \otimes \{\lambda_n\}$ (previously these were denoted by $(\mathcal{M}_n, d_n)$ and $\mathcal{F}_n$). Then by Proposition 7.29,

$$\lambda(\widehat{\mathcal{M}}, X) \le c_0\left(n + C\left(\widehat{\mathcal{F}}; \frac{1}{n}\right)\right)$$

where $n := \lfloor \log_2 D(\mathcal{F}) \rfloor + 6$.

Further, we apply Proposition 3.92 of Volume I estimating $D(\mathcal{F})$ and $C(\widehat{\mathcal{F}}; t)$ through the corresponding characteristics of $\mathcal{F}_j$, $1 \le j \le N$. For the former we get from there

$$D(\mathcal{F}) \le \prod_{j=1}^{N} D(\mathcal{F}_j) =: D,$$

while for the latter we have

$$C\left(\widehat{\mathcal{F}}; \frac{1}{n}\right) \le \gamma_p\left(\frac{1}{n}\right) \prod_{j=1}^{N} K_j \left\{ C\left(\{\lambda_n\}; \frac{1}{n}\right)^q + \sum_{j=1}^{N} C\left(\mathcal{F}_j; \frac{1}{n}\right)^q \right\}^{1/q}.$$

Since $C(\{\lambda_n\}; t) = 0$ for all $t > 0$ and $C(\mathcal{F}_j; \frac{1}{n}) \le C(\mathcal{F}_j; 1) =: C(\mathcal{F}_j)$, we conclude from here that

$$C\left(\widehat{\mathcal{F}}; \frac{1}{n}\right) \le \gamma_p\left(\frac{1}{n}\right) KC$$

where we recall that

$$K := \prod_{j=1}^{N} K_j \quad \text{and} \quad C := \left\{ \sum_{j=1}^{N} C(\mathcal{F}_j)^q \right\}^{1/q}.$$

Finally, by the definition of γ_p,

$$\gamma_p\left(\frac{1}{n}\right) := \inf_{a>0} \left(\left[(1+a)^p - \frac{1}{n^p} \right]^{-\frac{1}{p}} D(\widehat{\mathcal{F}}; (1+a)) \right)$$
$$\leq \left[\left(1+\frac{1}{n}\right)^p - \frac{1}{n^p} \right]^{-\frac{1}{p}} D\left(\widehat{\mathcal{F}}; 1+\frac{1}{n}\right).$$

Since the first factor is less than $[(1+\frac{1}{n^p}) - \frac{1}{n^p}]^{-\frac{1}{p}} = 1$ and the second is at most $\frac{6}{5}e^4$, by Lemma 7.25 we finally get for some $c > 1$,

$$\lambda(\mathcal{M}, X) \leq \lambda(\widehat{\mathcal{M}}, X) \leq c(\log_2 D + KC + 1).$$

Theorem 7.22 has been established. □

It is worth noting that the inequality for $C\left(\widehat{\mathcal{F}}_n; \frac{1}{n}\right)$ given by Proposition 3.92 of Volume I gives a slightly sharper result than that of Theorem 7.22 with K replaced by $K^1 := \prod_{j=2}^{N} K_j$ for $N > 1$ and $K^1 := 1$ for $N = 1$. Since the latter will be used below, we present it as

Corollary 7.34. *Let $(\mathcal{M}, d, \mathcal{F})$ be a $\mathcal{PHT}$ space and X be Banach. Then for some constant $c_0 > 1$,*

$$\lambda(\mathcal{M}, X) \leq c_0(\log_2 D_2(\mathcal{F}) + C(\mathcal{F}) + 1).$$

The subsequent two corollaries are, in fact, variants of Theorem 7.22 obtained by applying the argument used in its proof.

Corollary 7.35. *Let $(\mathcal{M}, d, \mathcal{F})$ be a $\mathcal{PHT}$ space whose dilation function satisfies for some $\sigma \geq 0$,*

$$a := \sup_{l\geq 1} \left\{ \frac{D(\mathcal{F}; l)}{l^\sigma} \right\} < \infty.$$

Then for some $c > 1$,

$$\lambda(\mathcal{M}, X) \leq c\left(n + C\left(\mathcal{F}; \frac{1}{n}\right)\right)$$

where $n := \lfloor \log_2 a \rfloor + \sigma + 1$.

Proof. We apply Proposition 7.29 to space $(\widehat{\mathcal{M}}, \widehat{d}, \widehat{\mathcal{F}})$ where

$$(\widehat{\mathcal{M}}, \widehat{d}) := (\mathcal{M}, d) \oplus^{(1)} \ell_1^k \quad \text{and} \quad \widehat{\mathcal{F}} := \mathcal{F} \otimes \{\lambda_k\} \quad \text{and} \quad k := \lfloor \log_2 a \rfloor + \sigma + 6.$$

This gives

$$\lambda(\widehat{\mathcal{M}}, X) \leq c \left(k + C \left(\widehat{\mathcal{F}}; \frac{1}{k} \right) \right).$$

Further, by Proposition 3.92 of Volume I and Lemma 7.27,

$$\begin{aligned} C\left(\widehat{\mathcal{F}}; \frac{1}{k}\right) &\leq \gamma_1 \left(\frac{1}{k}\right) \left(C\left(\{\lambda_k\}; \frac{1}{k}\right) + C\left(\mathcal{F}; \frac{1}{k}\right) \right) \\ &\leq \frac{6}{5} e^4 C\left(\mathcal{F}; \frac{1}{k}\right) \leq \frac{6}{5} e^4 C\left(\mathcal{F}; \frac{1}{n}\right). \end{aligned}$$

The result is proved. □

We now single out a special case of the above result with a better estimate of $\lambda(\mathcal{M}, X)$. Specifically, suppose that $\mathcal{F}$ is n-homogeneous, i.e., for all balls in $\mathcal{M}$,

$$\mu_m(B_R(m)) = \gamma R^\sigma, \quad \gamma, \sigma > 0. \tag{7.100}$$

Under this assumption the following holds.

Corollary 7.36. *It is true that*

$$\lambda(\mathcal{M}, X) \leq c \left(\sigma^* + C \left(\mathcal{F}; \frac{1}{\sigma^*} \right) \right)$$

where $\sigma^* := \max\{\sigma, 1\}$ *and* $c < 311$.

Proof. In this case, we do not need the extended space trick. In fact, we exploit it only for the estimates of Lemmas 7.25 and 7.27. But the required variant of the latter for the space $(\mathcal{M}, d, \mathcal{F})$ satisfying (7.100) may be obtained by the next straightforward evaluation with $R_2 > R_1 > 0$,

$$\begin{aligned} \mu_n(B_{R_2}(m)) - \mu_m(B_{R_1}(m)) &:= \gamma(R_2^\sigma - R_1^\sigma) \leq \gamma \max\{\sigma, 1\} R_2^{\sigma-1}(R_2 - R_1) \\ &:= \sigma^* \cdot \frac{\mu_m(B_{R_2}(m))}{R_2}(R_2 - R_1). \end{aligned}$$

Further, repeating in this case the arguments of Proposition 7.29 with σ^* in place of n and $l := 1 + \frac{1}{3\sigma^*}$ and noting that $A_n := \frac{6}{5}e^4 n$ is now replaced by σ^* we get

$$\lambda(\mathcal{M}, X) \leq 56\sigma^* + \max\left\{ 42\sigma^* \left(4 + \frac{1}{3\sigma^*}\right), K_n \left(1 + \frac{1}{3\sigma^*}\right) \right\}.$$

Since under this choice of n,

$$K_n \left(1 + \frac{1}{3\sigma^*}\right) := 42 \left(\sigma^* + C\left(\mathcal{F}; \frac{1}{\sigma^*}\right) \right) \left(4 + \frac{1}{3\sigma^*}\right) \cdot D\left(\mathcal{F}; 1 + \frac{1}{3\sigma^*}\right),$$

the maximum is attained at the second term. Moreover, by (3.121),

$$D\left(\mathcal{F}; 1+\frac{1}{3\sigma^*}\right) := \left(1+\frac{1}{3\sigma^*}\right)^{\sigma} < \sqrt[3]{e}.$$

Combining these estimates we prove the result. □

Now we present several consequences of Theorem 7.22 and its corollaries beginning with the following elegant result firstly established by Lee and Naor [LN-2005] in a nonconstructive way, see Section 7.3 for a discussion of their method.

Corollary 7.37. *Let $(\mathcal{M}, d)$ be a nontrivial doubling metric space with doubling constant δ. Then for some numerical constant $c > 1$,*

$$\lambda(\mathcal{M}, X) \le c \log_2 \delta.$$

Proof. Without loss of generality we may assume that $\mathcal{M}$ is complete. Then by the Koniagin-Vol'berg Theorem 4.43 of Volume I the space $\mathcal{M}$ carries a doubling measure μ with dilation function satisfying

$$D(\mu; l) \le c(s) l^s$$

where $s > \log_2 \delta + 1$ and

$$\log_2 c(s) := (\log_2 \delta + 1)\log_2 24 + 5s \max\left\{\frac{1+s}{s - \log_2 \delta - 1}, \log_2 21\right\},$$

see Volume I, Section 4.3.

For $\mathcal{M}$ containing at least two points its doubling constant is $\delta \ge 2$. Therefore choosing $s := 2\log_2 \delta + 1$, we obtain $\log_2 c(s) < \widetilde{c} \log_2 \delta$ for a numerical constant $\widetilde{c} > 1$. Then $(\mathcal{M}, d, \{\mu\})$ is a space of homogeneous type satisfying the condition of Corollary 7.35 with $\sigma := 2\log_2 \delta + 1$, $a := \delta^{\widetilde{c}}$. Moreover, $C(\{\mu\}; t) = 0$ for all $t > 0$ and we obtain from this corollary that

$$\lambda(\mathcal{M}, X) \le c \log_2 \delta$$

where $c > 1$ is a numerical constant.

The result is proved. □

Now we estimate $\lambda(\mathcal{M}, X)$ for $\mathcal{M}$ being the direct p-sum $(1 \le p \le \infty)$ of the Riemannian manifolds $\mathbb{H}^{n_i}_{\omega_i}$, $1 \le i \le N$. Let us recall that $\mathbb{H}^n_\omega$ is defined by the Riemannian metric

$$ds^2 := \omega(x_n)^2 \cdot \sum_{i=1}^{n} dx_i^2$$

on the underlying set $H^n_+ := \{x \in \mathbb{R}^n ; x_n > 0\}$ and is regarded as a metric space (with the geodesic metric). Further, $\omega : (0, +\infty) \to \mathbb{R}_+$ is a continuous nonincreasing function satisfying the condition

$$\int_1^\infty \omega(s) ds = \infty. \tag{7.101}$$

Corollary 7.38. *Let* $(\mathcal{M}, d)$ *be the direct* p*-sum of spaces* $\mathbb{H}^{n_i}_{\omega_i}$, $1 \le i \le N$*. Assume that every* ω_i *satisfies* (7.101)*. Then for some* $c > 1$ *and* $\frac{1}{q} := 1 - \frac{1}{p}$,

$$\lambda(\mathcal{M}, X) \le c \left(\sum_{i=1}^{N} n_i + \left[\sum_{i=1}^{N} n_i^{2q} \right]^{1/q} \right) \cdot \max_{1 \le i \le N} \sqrt{n_i}.$$

Proof. According to Theorem 4.58 of Volume I the space $\mathbb{H}^n_\omega$ is bi-Lipschitz homeomorphic to the metric space of $(n-1)$-cubes in $\mathbb{R}^{n-1}$ denoted by $\mathcal{B}^{n-1}_\omega := (\mathcal{B}(\ell^{n-1}_\infty), d_\omega)$, see subsection 4.41 of Volume I for its definition and properties. In subsection 4.4.2 of Volume I, the family of measures $\mathcal{F}_\omega := \{\mu_Q\}_{Q \in \mathcal{B}(\ell^{n-1}_\infty)}$ is constructed, see formula (4.88) there, such that $(\mathcal{B}^{n-1}_\omega, \mathcal{F})$ turns into a 1-uniform $\mathcal{PHT}$ space and for every ball $\overline{B}_R(Q) \subset \mathcal{B}^n_\omega$ and $0 < t < 1$,

$$\mu_Q(\overline{B}_R(Q)) = 2^{n-1} R^n \quad \text{and} \quad C(\mathcal{F}_\omega; t) \le \frac{3n}{2} \cdot \frac{(1+t)^n - 1}{t}. \tag{7.102}$$

Now let $\widehat{\mathcal{M}} := \oplus^{(p)} \mathcal{B}^{n_i - 1}_{\omega_i}$ and $\widehat{\mathcal{F}} := \otimes_{i=1}^N \mathcal{F}_{\omega_i}$. By (7.102) a measure $\widehat{\mu} \in \widehat{\mathcal{F}}$ satisfies for all $\widehat{m} \in \widehat{M}$ and $R > 0$,

$$\widehat{\mu}(\overline{B}_R(\widehat{m})) = 2^{n-N} R^n$$

where $n := \sum_{i=1}^N n_i$. Hence, we may apply Corollary 7.36 to get

$$\lambda(\widehat{M}, X) \le c \left(n + C\left(\widehat{\mathcal{F}}; \frac{1}{n} \right) \right).$$

Further, by Proposition 3.92 of Volume I we have for $0 < t \le 1$,

$$C(\widehat{\mathcal{F}}; t) \le \gamma_p(t) \left\{ \sum_{i=1}^N C(\mathcal{F}_{\omega_i}; t)^q \right\}^{1/q}.$$

To estimate each term of this sum we use inequality (7.102) and the inequality

$$\frac{(1+t)^k - 1}{t} \le k(1+t)^{k-1}$$

to have

$$C\left(\mathcal{F}_{\omega_i}; \frac{1}{n} \right) \le \frac{3}{2} n_i^2 \left(1 + \frac{1}{n} \right)^{n_i - 1} < \frac{3e}{2} n_i^2.$$

Moreover, by (7.102) and the definition of γ_p,

$$\begin{aligned} \gamma_p\left(\frac{1}{n}\right) &\le \left[\left(1 + \frac{1}{n}\right)^p - \frac{1}{n^p} \right]^{-\frac{1}{p}} D\left(\widehat{\mathcal{F}}; 1 + \frac{1}{n} \right) \\ &\le D\left(\widehat{\mathcal{F}}; 1 + \frac{1}{n} \right) = \left(1 + \frac{1}{n} \right)^n < e. \end{aligned}$$

Combining these inequalities we have

$$\lambda(\widehat{\mathcal{M}}, X) \leq \frac{3}{2}e^2 c \left(n + \left[\sum_{j=1}^{N} n_i^{2q} \right]^{1/q} \right). \tag{7.103}$$

To conclude the proof we use the argument of Theorem 4.58 of Volume I asserting that $\mathbb{H}_\omega^n$ is bi-Lipschitz homeomorphic to the space of Euclidean balls $(\mathcal{B}(\mathbb{R}^{n-1}), d_\omega)$ with distortion at most 3, see formula (4.98) there. Further, the identity map of $\mathbb{R}^{n-1}$ $(= \ell_2^{n-1})$ onto ℓ_∞^n has distortion $\sqrt{n-1}$. By Theorem 4.53 (b) of Volume I $(\mathcal{B}(\mathbb{R}^{n-1}), d_\omega)$ is bi-Lipschitz homeomorphic to $\mathcal{B}_\omega^{n-1} := (\mathcal{B}(\ell_\infty^{n-1}), d_\omega)$ with distortion at most $\sqrt{n}$. Then distortion of the bijection of $\widehat{\mathcal{M}}$ onto $\mathcal{M} := \oplus_{i=1}^N \mathbb{H}_{\omega_i}^{n_i}$ is at most $3(\max_{1\leq i\leq N} \sqrt{n_i})$ (use the Hölder inequality). This implies

$$\lambda(\mathcal{M}, X) \leq 3 \left(\max_{1\leq i\leq N} \sqrt{n_i} \right) \lambda(\widehat{\mathcal{M}}, X).$$

Together with (7.103) this yields the required result. □

Now from the results established we derive estimates of $\lambda(\mathcal{M}, X)$ for $\mathcal{M}$ being one of the classical spaceforms.

Corollary 7.39. *It is true that*

$$\lambda(\mathbb{S}^n, X), \quad \lambda(\mathbb{R}^n, X) \leq c_1 n \quad \textit{and} \quad \lambda(\mathbb{H}^n, X) \leq c_2 n^{5/2}$$

where $c_1 < 311$ *and* $c_2 < 20640$.

Proof. Since the surface measure of $\mathbb{S}^n$ and the Lebesgue measure of $\mathbb{R}^n$ are n-homogeneous, we may apply Corollary 7.36 with $\sigma = n$ and $C(\mathcal{F}; \cdot) = 0$ to prove the first two inequalities.

The third inequality follows from Corollary 7.38 with $N = 1$ and $\omega(t) = \frac{1}{t}$, $t > 0$. Under this choice $\mathbb{H}_\omega^n$ coincides with the hyperbolic space $\mathbb{H}^n$ while an accurate computation of the constants involved in the proof gives the above estimate for c_2. □

Using another method, discussed at the end of the chapter, one can establish that $\lambda(\mathbb{R}^n, X) \leq cn^{1/2}$ for a numerical constant $c > 0$, [LN-2005].

Problem. *What is the sharp order of growth for* $\lambda(\mathcal{M}, X)$ *as* $\dim \mathcal{M} \to \infty$ *where* $\mathcal{M}$ *is one of the classical spaceforms?*

We guess that the sharp order is $n^{1/2}$ for all these spaces. The proof of this conjecture apparently requires an approach which is considerably more profound than those described in the present book.

Finally, we briefly discuss another construction of the extension operator (7.74) which allows us to obtain essentially better numerical constants in all results of this section for an X-valued function with X being a Banach space constrained in its dual. For example, the constant in Corollary 7.36 will be less than 24 (see [BB-2007b, Cor. 2.27]).

The new extension operator is a simple modification of that given by (7.74) where the Dugundji operator $f \mapsto \widehat{f}$ is replaced by the operator composing f and a metric projection onto the subspace $S \subset \mathcal{M}$ denoted by p_S. For card $S < \infty$ and a suitable definition of p_S we obtain an operator $E \in \operatorname{Ext}(S, \mathcal{M})$ which yields better estimates than that in (7.74). To transfer from finite subsets S to the general case we, for scalar functions, exploit the finiteness property of Theorem 7.12 and then, for X-valued functions with X being constrained in its dual, Proposition 7.3.

Let us explain why this construction cannot be used for infinite subspaces S. In the definition of a metric or $(1+\varepsilon)$-metric projection, $\varepsilon > 0$, we encounter the following measurable selection problem.

Let P_S^ε, $\varepsilon \geq 0$, be a set-valued function on $\mathcal{M}$ given by

$$P_S^\varepsilon(m) := \{m' \in S\,;\, d(m, m') \leq (1+\varepsilon)d(m, S)\}.$$

If there exists a Borel measurable selection $p_S \in P_S^\varepsilon$, the composition $f \circ p_S$ with a continuous locally bounded function $f : S \to \mathbb{R}$ is also Borel measurable and the average operator I may be applied to $f \circ p_S$.

Unfortunately, such a selection may not exist in general, see Theorem 3.71 on the corresponding P. Novikov's counterexample. A careless choice of a selection may lead to a Borel nonmeasurable metric projection even for a finite S (M. Wojcieakowski, personal communication). However, in this case we simply enumerate arbitrarily the points of S and then select in $P_S^0(m)$ the point with the minimal number to obtain a Borel measurable p_S.

We present only two results which can be achieved in this way, see [BB-2007b] for more facts of this kind.

Corollary 7.40. *Let $(\mathcal{M}, d)$ be the direct p-sum of spaces $\mathbb{H}^{n_i}_{\omega_i}$, $1 \leq i \leq N$, and X be the dual space. Then it is true that*

$$\lambda(\mathcal{M}, X) \leq c \left(\sum_{i=1}^{N} n_i + \left[\sum_{i=1}^{N} n_i^{2q} \right]^{1/q} \right) \cdot \max_{1 \leq i \leq N} \sqrt{n_i}$$

where $c < 800$.

Here the weights ω_i do not necessarily satisfy condition (7.101).

Proof. To prove the result we first approximate each ω_i uniformly on compact subsets of $H^{n_i}_+$ by weights $w_{i,k}$, $k \in \mathbb{N}$, satisfying condition (7.101). Then we apply Corollary 7.38 to the direct p-sums $(\mathcal{M}_k, d_k)$ of spaces $\mathbb{H}^{n_i}_{\omega_{i,k}}$, $1 \leq i \leq N$. In fact, to get the required bound for c, we apply this corollary to finite subsets of $(\mathcal{M}_k, d_k)$ for the extension operator described above. Finally, using the finiteness property of Theorem 7.12 we obtain the result. $\square$

The second result concerns the direct p-sum of a finite family of metric trees. We restrict our consideration to scalar functions.

Corollary 7.41. *Let $(\mathcal{T}_i, d_i)$ be a nontrivial metric tree, $1 \le i \le N$. Then for some numerical constant $\widetilde{c} > 1$,*

$$\lambda(\oplus^p\{(\mathcal{T}_i, d_i)\}_{1\le i\le N}) \le \widetilde{c}\, N$$

where $1 \le p \le \infty$.

Proof. Let $\mathcal{V}_i$ be the vertex set of $\mathcal{T}_i$. By Theorem 5.4, $\mathcal{V}_i$ admits a bi-Lipschitz embedding into $\mathbb{H}^2$ with distortion at most 257. Therefore $\oplus^p\{(\mathcal{V}_i, d_i)\}_{1\le i\le N}$ admits such an embedding into the direct p-sum of the spaces $\mathbb{H}^2$ taken N times. Due to Corollary 7.38 the simultaneous extension constant of the latter p-sum is bounded by

$$\sqrt{2}c\left(2N + \left(\sum_{i=1}^{N} 2^{2q}\right)^{1/q}\right) = \sqrt{2}c(2N + 4N^{1/q}).$$

Next, each finite $S \subset \oplus_p\{(\mathcal{T}_i, d_i)\}_{1\le i\le N}$ is a subset of $\oplus^p\{(S_i, d_i)\}_{1\le i\le N}$ where S_i are natural projections of S onto $\mathcal{T}_i$. In turn, each S_i is a subset of the vertex set $\mathcal{V}_{S_i}$ of a finite graph $(\mathcal{T}_{S_i}, d_i)$ in $\mathcal{T}_i$ with the underlying set being the union of geodesics in $\mathcal{T}_i$ joining distinct points of S_i and with the vertex set $\mathcal{V}_{S_i}$ consisting of points of S_i and those of $\mathcal{V}_i$ belonging to these geodesics.

Now, we apply the previous estimate to the direct p-sum of vertex sets $\mathcal{V}_{S_i}$, $1 \le i \le N$, to bound the extension constant $\lambda\left(S, \oplus^p\{\mathcal{T}_i\}_{1\le i\le N}\right)$. After that we use the finiteness property of Theorem 7.12 for the family of these extension constants with S running through all finite subsets of $\oplus^p\{(\mathcal{T}_i, d_i)\}_{1\le i\le N}$. Then we get

$$\lambda(\oplus^p\{(\mathcal{T}_i, d_i)\}_{1\le i\le N}) \le 257\sqrt{2}c(2N + 4N^{1/q}) \le \widetilde{c}\, N. \tag{7.104}$$

□

Remark 7.42. For $N = 1$ the result was proved by Matoušek [Mat-1990] by another method admitting the Banach-valued functions.

7.3 Locally doubling metric spaces with uniform lattices

Locally doubling metric spaces do not in general have the simultaneous extension property (briefly, they are not $\mathcal{LE}$ spaces); in particular we present in Chapter 8 an example of a space which even homeomorphic to $\mathbb{R}$ but is not $\mathcal{LE}$. We, however, show that under a mild restriction the space in question is $\mathcal{LE}$, if it contains a certain discrete subspace of this kind. To formulate the result we recall the corresponding notions (discussed in details in Chapter 3 of Volume I, see, e.g., Definitions 3.29 and 3.45 there).

A subset Γ of a metric space $\mathcal{M}$ is said to be an *R-lattice with parameter* $c_\Gamma \in (0, \frac{1}{2}]$ if the open balls $B_R(\gamma)$, $\gamma \in \Gamma$, cover $\mathcal{M}$ while the open balls $B_{c_\Gamma R}(\gamma)$, $\gamma \in \Gamma$, are mutually disjoint.

The existence of R-lattices for an arbitrary metric space easily follows from Zorn's lemma.

Now let $\mathcal{M}$ belong to the class of locally doubling metric spaces $\mathcal{D}(R, N)$, i.e., each of its open balls of radius $r \le R$ can be covered by at most N open balls of radius $r/2$.

Theorem 7.43. *Let $(\mathcal{M}, d)$ be a metric space from $\mathcal{D}(2R, N)$. Assume that the extension constant $\lambda(\Gamma)$ of an R-lattice $\Gamma \subset \mathcal{M}$ is finite and, moreover,*

$$\lambda_R := \sup\{\lambda(B_{2R}(m)) \ ; \ m \in \mathcal{M}\} < \infty. \tag{7.105}$$

Then $\lambda(\mathcal{M})$ is bounded by a constant depending only on $\lambda(\Gamma)$, λ_R, c_Γ, R and N.

Proof. Let Γ be an R-lattice and $\mathcal{B} := \{B_R(\gamma)\}_{\gamma\in\Gamma}$. By definition $\mathcal{B}$ and $2\mathcal{B} := \{B_{2R}(\gamma)\}_{\gamma\in\Gamma}$ cover $\mathcal{M}$.

Lemma 7.44. *The order of the cover $2\mathcal{B}$ is bounded by a constant μ depending only on c_Γ and N.*

Proof. Let m be covered by balls $B_{2R}(\gamma_i)$, $1 \le i \le k$. Then all these γ_i lie in the open ball $B_{2R}(m)$. Since $d(\gamma_i, \gamma_j) \ge c_\Gamma R$ for $i \ne j$, any cover of $B_{2R}(m)$ by open balls of radius $\frac{c_\Gamma R}{2}$ separates the set $\{\gamma_i\}_{1\le i\le k}$, i.e., distinct γ_i lie in distinct balls of the cover. Hence, such a cover has cardinality at least k. On the other hand, $\mathcal{M}$ belongs to $\mathcal{D}(2R, N)$ and therefore there exists a cover of $B_{2R}(m)$ by open balls of radius $\frac{c_\Gamma R}{2}$ whose cardinality is at most N^s where $s := \left\lfloor \log_2 \frac{1}{c_\Gamma} \right\rfloor + 2$ (apply s times the doubling condition). Hence,

$$\operatorname{ord}(2\mathcal{B}) \le \max k \le N^s. \qquad \square$$

Next we prove

Lemma 7.45. *There is a partition of unity $\{\rho_\gamma\}_{\gamma\in\Gamma}$ subordinate to $2\mathcal{B}$ such that*

$$K := \sup_\gamma ||\rho_\gamma||_{\mathrm{Lip}(\mathcal{M})} < \infty \tag{7.106}$$

where K depends only on c_Γ, N and R.

Proof. Set

$$B_\gamma := B_{2R}(\gamma) \quad \text{and} \quad B_\gamma^c := \mathcal{M} \setminus B_\gamma$$

and define

$$d_\gamma(m) := \operatorname{dist}(m, B_\gamma^c) \ , \quad m \in \mathcal{M}.$$

It is clear that

$$\operatorname{supp} d_\gamma \subset B_\gamma \quad \text{and} \quad ||d_\gamma||_{\mathrm{Lip}(\mathcal{M})} \le 1. \tag{7.107}$$

Let now $\phi : \mathbb{R}_+ \to [0, 1]$ be continuous, equal to one on $[0, R]$, zero on $[2R, \infty)$ and be linear on $[R, 2R]$. Introduce the function

$$s := \sum_{\gamma} \phi \circ d_\gamma. \tag{7.108}$$

By Lemma 7.44 only at most μ terms here are nonzero at every point. Therefore

$$||s||_{\mathrm{Lip}(\mathcal{M})} \leq 2\mu ||\phi||_{\mathrm{Lip}(\mathbb{R}_+)} \sup_{\gamma} ||d_\gamma||_{\mathrm{Lip}(\mathcal{M})}$$

and by (7.107) and the definition of ϕ we get

$$||s||_{\mathrm{Lip}(\mathcal{M})} \leq 2\mu/R. \tag{7.109}$$

On the other hand, every $m \in \mathcal{M}$ is contained in some ball $B_R(\gamma)$ of the cover $\mathcal{B}$. For this γ,

$$(\phi \circ d_\gamma)(m) \geq \phi(R) = 1$$

and therefore

$$s \geq 1. \tag{7.110}$$

Introduce now the required partition by

$$\rho_\gamma := \frac{\phi \circ d_\gamma}{s}, \quad \gamma \in \Gamma.$$

Then $\{\rho_\gamma\}$ is clearly a partition of unity subordinate to $2\mathcal{B}$. Moreover, we have

$$|\rho_\gamma(m) - \rho_\gamma(m')| \leq \frac{|\phi(d_\gamma(m)) - \phi(d_\gamma(m'))|}{s(m)} + \frac{\phi(d_\gamma(m'))}{s(m) \cdot s(m')} \cdot |s(m) - s(m')|$$

and application of (7.110), (7.109) and (7.107) leads to the desired inequality

$$||\rho_\gamma||_{\mathrm{Lip}(\mathcal{M})} \leq \frac{2\mu + 1}{R}. \qquad \square$$

Lemma 7.46.

$$\mathrm{Ext}(\Gamma, \mathcal{M}) \neq \emptyset.$$

Proof. By assumption (7.105) of the theorem, for every $\gamma \in \Gamma$ there is a linear operator $E_\gamma \in \mathrm{Ext}(\Gamma \cap B_\gamma, B_\gamma)$ such that

$$||E_\gamma|| \leq \lambda_R, \quad \gamma \in \Gamma. \tag{7.111}$$

Using this we introduce the required linear operator by

$$Ef := \sum_{\gamma \in \Gamma} (E_\gamma f_\gamma) \rho_\gamma, \quad f \in \mathrm{Lip}(\Gamma), \tag{7.112}$$

where $\{\rho_\gamma\}$ is the partition of unity from Lemma 7.45 and $f_\gamma := f|_{\Gamma\cap B_\gamma}$; here we assume that $E_\gamma f_\gamma$ is zero outside of B_γ. We have to show that

$$Ef|_\Gamma = f|_\Gamma \tag{7.113}$$

and estimate $||Ef||_{\mathrm{Lip}(\mathcal{M})}$.

Given $\hat\gamma \in \Gamma$ we can write

$$(Ef)(\hat\gamma) = \sum_{B_\gamma \ni \hat\gamma} (E_\gamma f_\gamma)(\hat\gamma)\rho_\gamma(\hat\gamma).$$

Since E_γ is an extension from $B_\gamma \cap \Gamma$, we get

$$(E_\gamma f_\gamma)(\hat\gamma) = f_\gamma(\hat\gamma) = f(\hat\gamma).$$

Moreover, $\sum_{B_\gamma \ni \hat\gamma} \rho_\gamma(\hat\gamma) = 1$, and (7.113) is proved.

To estimate the Lipschitz constant of Ef, we use the the McShane Theorem 1.27 of Volume I to extend $E_\gamma f_\gamma$ outside of B_γ so that the (nonlinear) extension F_γ satisfies

$$||F_\gamma||_{\mathrm{Lip}(\mathcal{M})} = ||E_\gamma f_\gamma||_{\mathrm{Lip}(B_\gamma)}. \tag{7.114}$$

Since $\rho_\gamma F_\gamma = \rho_\gamma E_\gamma f_\gamma$, we have

$$Ef = \sum_\gamma F_\gamma \rho_\gamma. \tag{7.115}$$

Given $\hat\gamma \in \Gamma$ we then introduce a function $G_{\hat\gamma}$ by

$$G_{\hat\gamma} := \sum_\gamma (F_\gamma - F_{\hat\gamma})\rho_\gamma := \sum_\gamma F_{\gamma\hat\gamma}\rho_\gamma \tag{7.116}$$

and write for every $\hat\gamma$,

$$Ef = F_{\hat\gamma} + G_{\hat\gamma}. \tag{7.117}$$

It follows from (7.111) and (7.114) that

$$||F_{\hat\gamma}||_{\mathrm{Lip}(\mathcal{M})} \le \lambda_R ||f||_{\mathrm{Lip}(\Gamma)}. \tag{7.118}$$

We show now that for $m \in B_\gamma \cap B_{\hat\gamma}$,

$$|F_{\gamma\hat\gamma}(m)| \le 8R\lambda_R ||f||_{\mathrm{Lip}(\Gamma)}. \tag{7.119}$$

In fact, for these m,

$$\begin{aligned} |F_{\gamma\hat\gamma}(m)| = |(E_\gamma f_\gamma - E_{\hat\gamma} f_{\hat\gamma})(m)| &\le |f(\gamma) - f(\hat\gamma)| + |(E_\gamma f_\gamma)(m) - (E_\gamma f_\gamma)(\gamma)| \\ &+ |(E_{\hat\gamma} f_{\hat\gamma})(m) - (E_{\hat\gamma} f_{\hat\gamma})(\hat\gamma)|. \end{aligned}$$

Estimating the right-hand side by (7.111) we then get

$$|F_{\gamma\hat{\gamma}}(m)| \leq \lambda_R ||f||_{\mathrm{Lip}(\Gamma)}(d(\gamma,\hat{\gamma}) + d(m,\gamma) + d(m,\hat{\gamma})) \leq 8R\lambda_R ||f||_{\mathrm{Lip}(\Gamma)}.$$

Applying this to estimate $\Delta G_{\hat{\gamma}} := G_{\hat{\gamma}}(m) - G_{\hat{\gamma}}(m')$ for $m, m' \in B_{\hat{\gamma}}$ we get

$$|\Delta G_{\hat{\gamma}}| \leq \sum_{B_\gamma \cap B_{\hat{\gamma}} \ni m} |\Delta \rho_\gamma| \cdot |F_{\gamma\hat{\gamma}}(m)| + \sum_{B_\gamma \cap B_{\hat{\gamma}} \ni m'} \rho_\gamma(m') \cdot |\Delta F_{\gamma\hat{\gamma}}|$$

(here the differences $\Delta\rho_\gamma$ and $\Delta F_{\gamma\hat{\gamma}}$ are defined similarly to $\Delta G_{\hat{\gamma}}$). The first sum is estimated by (7.119), (7.106) and Lemma 7.44, while the second one is at most $2\lambda_R||f||_{\mathrm{Lip}(\Gamma)} d(m,m')$ by (7.114) and (7.111). This leads to the estimate

$$|\Delta G_{\hat{\gamma}}| \leq (16RK\mu + 2) \cdot \lambda_R \cdot ||f||_{\mathrm{Lip}(\Gamma)} d(m,m'), \quad m, m' \in B_{\hat{\gamma}}.$$

Together with (7.118) this yields for these m, m':

$$|(Ef)(m) - (Ef)(m')| \leq C||f||_{\mathrm{Lip}(\Gamma)} d(m,m'). \tag{7.120}$$

Here and below C denotes a constant depending only on the basic parameters, that may change from line to line.

It remains to prove (7.120) for m, m' belonging to distinct balls B_γ. Let $m \in B_\gamma$ and m' be a point of some $B_R(\hat{\gamma})$ from the cover $\mathcal{B}$. Then $m \in B_\gamma \setminus B_{\hat{\gamma}}$ and therefore

$$d(m,m') \geq R. \tag{7.121}$$

Using now (7.117) we have

$$|(Ef)(m) - (Ef)(m')| \leq |F_\gamma(m) - F_{\hat{\gamma}}(m')| + |G_\gamma(m) - G_{\hat{\gamma}}(m')| := I_1 + I_2.$$

By the definition of F_γ, we then get

$$I_1 \leq |f(\gamma) - f(\hat{\gamma})| + |(E_\gamma f_\gamma)(m) - (E_\gamma f_\gamma)(\gamma)| + |(E_{\hat{\gamma}} f_{\hat{\gamma}})(m') - (E_{\hat{\gamma}} f_{\hat{\gamma}})(\hat{\gamma})|.$$

Together with (7.111) this leads to the estimate

$$\begin{aligned} I_1 &\leq \lambda_R ||f||_{\mathrm{Lip}(\Gamma)}(d(\gamma,\hat{\gamma}) + d(m,\gamma) + d(m',\hat{\gamma})) \\ &\leq 2\lambda_R ||f||_{\mathrm{Lip}(\Gamma)}(d(m,m') + d(m,\gamma) + d(m',\hat{\gamma})). \end{aligned}$$

Since $d(m,\gamma) + d(m',\hat{\gamma}) \leq 4R \leq 4d(m,m')$ by (7.121), we therefore have

$$I_1 \leq C||f||_{\mathrm{Lip}(\Gamma)} d(m,m'). \tag{7.122}$$

To estimate I_2, we note that for $m \in B_\gamma$,

$$G_\gamma(m) = \sum_{B_{\gamma'} \cap B_\gamma \ni m} (\rho_{\gamma'} F_{\gamma'\gamma})(m).$$

In combination with (7.119) and (7.121) this gives

$$|G_\gamma(m)| \le 8\lambda_R R||f||_{\mathrm{Lip}(\Gamma)} \le C||f||_{\mathrm{Lip}(\Gamma)} d(m,m').$$

The same argument estimates $G_{\hat\gamma}(m')$ for $m' \in B_{\hat\gamma}$. Hence

$$I_2 \le |G_\gamma(m)| + |G_{\hat\gamma}(m')| \le C||f||_{\mathrm{Lip}(\Gamma)} d(m,m').$$

Together with (7.122) and (7.120) this leads to the inequality

$$||Ef||_{\mathrm{Lip}(\mathcal{M})} \le C||f||_{\mathrm{Lip}(\Gamma)}.$$

Hence E is an operator from $\mathrm{Ext}(\Gamma, \mathcal{M})$. □

We are now in a position to complete the proof of Theorem 7.43. According to Theorem 7.12 we must show that

$$\sup_F \lambda(F) < \infty \tag{7.123}$$

where F runs through all finite subspaces of $\mathcal{M}$. To this end we consider the "Γ-envelope" of such F given by

$$\widehat{F} := \{\gamma \in \Gamma \ ; \ B_\gamma \cap F \ne \emptyset\}.$$

Then $\{B_\gamma \ ; \ \gamma \in \widehat{F}\} \subset 2\mathcal{B}$ is an open cover of F. By assumption (7.105) of the theorem for every $\gamma \in \widehat{F}$ there is an operator $E_\gamma \in \mathrm{Ext}(F \cap B_\gamma, B_\gamma)$ such that

$$||E_\gamma|| \le \lambda_R.$$

Introduce now a linear operator T given for $f \in \mathrm{Lip}(F)$, $\gamma \in \widehat{F}$ by

$$(Tf)(\gamma) := (E_\gamma f_\gamma)(\gamma) \ \ where \ \ f_\gamma := f|_{B_\gamma \cap F}.$$

We will show that

$$T : \mathrm{Lip}(F) \to \mathrm{Lip}(\widehat{F}) \quad \text{and} \quad ||T|| \le \lambda_R(2/c_\Gamma + 1). \tag{7.124}$$

Actually, let $\gamma_i \in \widehat{F}$ and $m_i \in B_{\gamma_i} \cap F$, $i = 1, 2$. Then $(E_{\gamma_i} f_{\gamma_i})(m_i) = f(m_i)$ and

$$\begin{aligned} |(Tf)(\gamma_1) - (Tf)(\gamma_2)| &\le \sum_{i=1,2} |(E_{\gamma_i} f_{\gamma_i})(\gamma_i) - (E_{\gamma_i} f_{\gamma_i})(m_i)| + |f(m_1) - f(m_2)| \\ &\le \lambda_R ||f||_{\mathrm{Lip}(F)} (d(\gamma_1, m_1) + d(\gamma_2, m_2) + d(m_1, m_2)). \end{aligned}$$

The sum in the brackets does not exceed $4R + d(m_1, m_2) \le 8R + d(\gamma_1, \gamma_2)$. Moreover, by the definition of an R-lattice, $d(\gamma_1, \gamma_2) \ge 4c_\Gamma R$. Combining these estimates, we have

$$|(Tf)(\gamma_1) - (Tf)(\gamma_2)| \le \lambda_R ||f||_{\mathrm{Lip}(F)} (2/c_\Gamma + 1) d(\gamma_1, \gamma_2).$$

This establishes (7.124).

Now assumption (7.105) of the theorem implies that there is an operator L from $\mathrm{Ext}(\widehat{F},\Gamma)$ whose norm is bounded by λ_Γ. Composing T and L with the operator $E \in \mathrm{Ext}(\Gamma,\mathcal{M})$ of Lemma 7.46 we obtain the operator

$$\widetilde{E} := ELT : \mathrm{Lip}(F) \to \mathrm{Lip}(\mathcal{M}) \tag{7.125}$$

whose norm is bounded by a constant depending only on $\lambda(\Gamma)$, λ_R, c_Γ, R and N. This definition also implies that

$$(\widetilde{E}f)(\gamma) := (E_\gamma f_\gamma)(\gamma), \quad \gamma \in \widehat{F}. \tag{7.126}$$

The constructed operator $\widetilde{E}$ is not an extension from F, hence we need to modify it to obtain the required extension operator. To accomplish this we, first introduce an operator $\widehat{T}$ given for $f \in \mathrm{Lip}(F)$ by

$$(\widehat{T}f)(m) := \begin{cases} (\widetilde{E}f)(m), & \text{if } m \in \widehat{F}, \\ f(m), & \text{if } m \in F \setminus \widehat{F}. \end{cases} \tag{7.127}$$

Lemma 7.47. $\widehat{T} : \mathrm{Lip}(F) \to \mathrm{Lip}(F \cup \widehat{F})$ *and* $||\widehat{T}|| \le C$.

Proof. It clearly suffices to estimate $I := |(\widehat{T}f)(m_1) - (\widehat{T}f)(m_2)|$ for $m_1 \in \widehat{F}$ and $m_2 \in F \setminus \widehat{F}$. First, let these points belong to a ball B_γ (hence $m_1 = \gamma$). Then (7.126) and the inclusion $E_\gamma \in \mathrm{Ext}(F \cap B_\gamma, B_\gamma)$ imply that

$$\begin{aligned} I &= |(E_\gamma f_\gamma)(\gamma) - f(m_2)| = |(E_\gamma f_\gamma)(\gamma) - (E_\gamma f_\gamma)(m_2)| \\ &\le \lambda_R ||f||_{\mathrm{Lip}(F)} d(\gamma, m_2) := \lambda_R ||f||_{\mathrm{Lip}(F)} d(m_1, m_2). \end{aligned}$$

In the remaining case of $m_2 \in B_{\hat{\gamma}} \setminus B_\gamma$ for some $\hat{\gamma} \in \widehat{F}$,

$$d(m_1, m_2) = d(\gamma, m_2) \ge 2R. \tag{7.128}$$

Similarly to the previous estimate we obtain for this m_2,

$$\begin{aligned} I &\le |(\widetilde{E}f)(\gamma) - (\widetilde{E}f)(\hat{\gamma})| + |(E_{\hat{\gamma}} f_{\hat{\gamma}})(\hat{\gamma}) - (E_{\hat{\gamma}} f_{\hat{\gamma}})(m_2)| \\ &\le (||\widetilde{E}|| d(\gamma, \hat{\gamma}) + \lambda_R d(\hat{\gamma}, m_2)) ||f||_{\mathrm{Lip}(F)}. \end{aligned}$$

Moreover, (7.128) implies that

$$\begin{aligned} d(\gamma, \hat{\gamma}) + d(\hat{\gamma}, m_2) &\le d(\gamma, m_2) + 2d(\hat{\gamma}, m_2) \le d(\gamma, m_2) + 4R \\ &\le 3d(\gamma, m_2) = 3d(m_1, m_2). \end{aligned}$$

Combining with the previous inequalities we estimate I and $\|\widehat{T}\|$, as required. $\square$

An operator that will be further used in our construction is defined for $f \in \mathrm{Lip}(F)$ by

$$(\widehat{S}f)(m) := (\widehat{T}f)(m) - (\widetilde{E}f)(m), \quad m \in F \cup \widehat{F}. \tag{7.129}$$

Lemma 7.48. $||\widehat{S}f||_{\ell_\infty(F\cup\widehat{F})} \le C||f||_{\mathrm{Lip}(F)}$ *and, moreover,*

$$\widehat{S} : \mathrm{Lip}(F) \to \mathrm{Lip}(F \cup \widehat{F}) \quad \text{and} \quad ||\widehat{S}|| \le C.$$

Proof. The second statement follows directly from definition (7.129). If, now, $m \in B_\gamma \cap (F \cup \widehat{F})$, then by the same definition

$$(\widehat{S}f)(m) = [(\widehat{T}f)(m) - (\widehat{T}f)(\gamma)] + [(\widetilde{E}f)(\gamma) - (\widetilde{E}f)(m)]$$

which implies that

$$|(\widehat{S}f)(m)| \le C||f||_{\mathrm{Lip}(F)} d(m,\gamma) \le CR||f||_{\mathrm{Lip}(F)}. \qquad \square$$

Finally we introduce an operator $\widehat{K}$ given for $g \in (\mathrm{Lip} \cap \ell_\infty)(F \cup \widehat{F})$ by

$$\widehat{K}g := \sum_\gamma (E_\gamma g_\gamma)\rho_\gamma; \tag{7.130}$$

here γ runs through the set $\{\gamma \in \Gamma\ ;\ (F \cup \widehat{F}) \cap B_\gamma \ne \emptyset\}$ and $\{\rho_\gamma\}$ is the partition of unity of Lemma 7.45, and $g_\gamma := g|_{(F\cup\widehat{F})\cap B_\gamma}$. Moreover, E_γ is an operator from $\mathrm{Ext}((F \cup \widehat{F}) \cap B_\gamma, B_\gamma)$ with

$$||E_\gamma|| \le \lambda_R \tag{7.131}$$

whose existence is provided by the assumption of Theorem 7.43.

Lemma 7.49.

$$||\widehat{K}g||_{\mathrm{Lip}(\mathcal{M})} \le C||g||_{(\mathrm{Lip}\cap\ell_\infty)(F\cup\widehat{F})}.$$

Proof. As in the proof of Lemma 7.46 it is convenient to extend every $E_\gamma g_\gamma$ outside B_γ so that the extension F_γ satisfies

$$||F_\gamma||_{\mathrm{Lip}(\mathcal{M})} = ||E_\gamma g_\gamma||_{\mathrm{Lip}(B_\gamma)}. \tag{7.132}$$

Then we clearly have

$$\widehat{K}g = \sum_\gamma F_\gamma \rho_\gamma. \tag{7.133}$$

Now, according to (7.130) we get for $m \in F \cup \widehat{F}$,

$$(\widehat{K}g)(m) = \sum \rho_\gamma(m) g(m) = g(m).$$

Hence, it remains to estimate the right-hand side of the inequality

$$\begin{aligned} I &:= |(\widehat{K}g)(m_1) - (\widehat{K}g)(m_2)| \\ &\le \sum_\gamma |\rho_\gamma(m_1) - \rho_\gamma(m_2)| \cdot |F_\gamma(m_1)| + \sum_\gamma \rho_\gamma(m_2)|F_\gamma(m_1) - F_\gamma(m_2)|. \end{aligned}$$

By (7.132) the second sum is at most $(\sup_\gamma ||E_\gamma||)||g||_{\mathrm{Lip}(F\cup\widehat{F})}d(m_1, m_2)$ and together with (7.131) this leads to the appropriate bound. In turn, the first sum is at most

$$2\mu \cdot K \cdot \max_\gamma |(E_\gamma g_\gamma)(m_1)| \cdot d(m_1, m_2),$$

where μ is the order of the cover $2\mathcal{B}$ and K is defined by (7.106). To estimate the maximum, one notes that $(E_\gamma g_\gamma)(\gamma) = g(\gamma)$ and therefore

$$\begin{aligned} &|(E_\gamma g_\gamma)(m_1)| \le |(E_\gamma g_\gamma)(m_1) - (E_\gamma g_\gamma)(\gamma)| + |g(\gamma)| \\ &\le \lambda_R d(m_1, \gamma)||g||_{\mathrm{Lip}(F\cup\widehat{F})} + ||g||_{\ell_\infty(F\cup\widehat{F})} \le (2R\lambda_R + 1)||g||_{(\mathrm{Lip}\cap\ell_\infty)(F\cup\widehat{F})}. \end{aligned}$$

Combining with the estimate of the second sum we prove the lemma. □

Now all is ready to define the required operator $\widehat{E}$ from $\mathrm{Ext}(F, \mathcal{M})$. Using the above introduced operators $\widetilde{E}$, $\widehat{K}$ and $\widehat{S}$ we set

$$\widehat{E}f := \widetilde{E}f + \widehat{K}(\widehat{S}f|_{F\cup\widehat{F}}).$$

Due to the properties of the operators involved we have for $m \in F$,

$$(\widehat{E}f)(m) = (\widetilde{E}f)(m) + (\widehat{S}f)(m) = (\widetilde{E}f)(m) + (\widehat{T}f)(m) - (\widetilde{E}f)(m) = f(m),$$

i.e., $\widehat{E}$ is an extension from F.

To obtain the required estimate of $||\widehat{E}f||_{\mathrm{Lip}(\mathcal{M})}$ it suffices by (7.125) to estimate $||\widehat{K}(\widehat{S}f|_{F\cup\widehat{F}})||_{\mathrm{Lip}(\mathcal{M})}$. The latter by Lemmas 7.49, 7.48, 7.47 and definition (7.125) is bounded by

$$C||\widehat{S}f||_{(\mathrm{Lip}\cap\ell_\infty)(F\cup\widehat{F})} \le C||f||_{\mathrm{Lip}(F)}.$$

Hence $\widehat{E} \in \mathrm{Ext}(F, \mathcal{M})$ and its norm is bounded, as required.

The proof of Theorem 7.43 is complete. □

Let us recall that a metric space $\mathcal{M}$ is of bounded geometry with parameters $n \in \mathbb{N}$, R, $D > 0$ (written $\mathcal{M} \in \mathcal{G}_n(R, D)$), if each of its open balls of radius R admits a bi-Lipschitz embedding into $\mathbb{R}^n$ with distortion D, see Volume I, subsection 3.3.2.

Note that if $B_R(m)$ is bi-Lipschitz homeomorphic to a subset S of $\mathbb{R}^n$ with distortion D, then

$$\frac{1}{D} \cdot \lambda(S) \le \lambda(B_R(m)) \le D \cdot \lambda(S),$$

and by the classical Whitney-Glaeser extension Theorem 2.19 of Volume I, $\lambda(S) \le \lambda(\mathbb{R}^n) < \infty$. Therefore the previous theorem leads to

Corollary 7.50. *Let $\mathcal{M} \in \mathcal{G}_n(2R, D)$. Then $\lambda(\mathcal{M})$ is finite if and only if for some R-lattice Γ in $\mathcal{M}$ we have*

$$\lambda(\Gamma) < \infty.$$

In the next corollary, we deal with finitely generated groups with word metrics, see the part of subsection 3.3.7 of Volume I devoted to these groups. Specifically, we consider groups of isometries acting *properly*, *freely* and *cocompactly* on a length metric space, see Volume I, Definition 3.132 for details. By the Efremovich-Švarc-Milnor Theorem 3.133 of Volume I such a group is finitely generated. If G is a group of this kind and A is its generating set, then d_A denotes the word metric of G. It is known that for distinct generating sets A and A',

$$c^{-1} d_A \leq d_{A'} \leq c d_A$$

for some constant $c > 1$.

Corollary 7.51. *Let $\mathcal{M}$ be a length space framed by a group G acting on $\mathcal{M}$ by isometries. Assume that*

(a) *$\mathcal{M}$ is a metric space of bounded geometry;*

(b) *G acts on $\mathcal{M}$ properly, freely and cocompactly.*

Then $\lambda(\mathcal{M})$ is finite if and only if G equipped with a word metric is an $\mathcal{LE}$ space.

Proof. Let A be a generating set of G and d_A be the associated word metric. If (G, d_A) is an $\mathcal{LE}$ space, then $(G, d_{A'})$ also is for any generating set A'. So we will work with a fixed A.

We first prove that the condition

$$\lambda(G, d_A) < \infty \tag{7.134}$$

is necessary for finiteness of $\lambda(\mathcal{M})$. Indeed, suppose that $\lambda(\mathcal{M})$ is finite. Then for a G-orbit $G(m) := \{g(m) \ ; \ g \in G\}$ we have

$$\lambda(G(m)) \leq \lambda(\mathcal{M}) < \infty. \tag{7.135}$$

Theorem 3.133 of Volume I asserts that under the hypothesis (b) of the corollary there exists a constant $C \geq 1$ independent of m so that

$$C^{-1} d_A(g, h) \leq d(g(m), h(m)) \leq C d_A(g, h) \tag{7.136}$$

for all $g, h \in G$. This, in particular, means that the metric subspace $G(m)$ is bi-Lipschitz homeomorphic to the metric space (G, d_A). Hence (7.135) implies the required inequality (7.134).

To prove sufficiency of condition (7.134) for finiteness of $\lambda(M)$, we choose a point m_0 of the generating compact set K_0, see Volume I, Definition 3.132, and show that the G-orbit $G(m_0)$ is an R-lattice for some $R > 0$. Let $B_{R_0}(m_0)$ be a ball containing K_0. Then due to the definition of the generating compact we have for $\Gamma := G(m_0)$

$$\bigcup_{m\in\Gamma} B_{R_0}(m) = G(B_{R_0}(m_0)) \supset G(K_0) = \mathcal{M}.$$

Hence the family of balls $\{B_{R_0}(m)\ ;\ m \in \Gamma\}$ covers $\mathcal{M}$. Moreover, (7.136) implies that for $m := g(m_0)$, $m' := h(m_0)$ with $g \neq h$,

$$d(m, m') \geq C^{-1} d_A(g, h) \geq C^{-1},$$

i.e., the family $\{B_{cR_0}(m)\ ;\ m \in \Gamma\}$ with $c = c_\Gamma := \frac{1}{2CR_0}$ consists of pairwise disjoint balls. Hence Γ is an R-lattice, $R := R_0$, satisfying, by (7.134) and (7.136), the condition

$$\lambda(\Gamma) < \infty.$$

We now apply Theorem 7.43 with that R-lattice Γ to derive the finiteness of $\lambda(\mathcal{M})$. To this end we should establish the validity of the assumptions of the theorem with this R.

First we prove that $\mathcal{M}$ belongs to the class of locally doubling metric spaces $\mathcal{D}(2R, N)$ for some $N = N(R, \mathcal{M})$. In other words, we show that every ball $B_r(m)$ with $r \leq 2R$ can be covered by at most N balls of radius $r/2$. Indeed, by hypothesis (a) of the corollary, $\mathcal{M} \in \mathcal{G}_n(\widetilde{R}, \widetilde{D})$ for certain $\widetilde{R}, \widetilde{D}$ and n. This implies that $\mathcal{M} \in \mathcal{D}(\widetilde{R}/2, N)$ for some $N = N(\widetilde{D}, n)$ and shows that the required statement is true for $2R \leq \widetilde{R}/2$. Suppose now that

$$\widetilde{R}/2 \leq r \leq 2R. \tag{7.137}$$

Note that it suffices to consider balls with $m \in K_0$. In fact, $G(K_0) = \mathcal{M}$ and therefore $g_0(m) \in K_0$ for some isometry $g_0 \in G$. Hence $g_0(B_r(m)) = B_r(g_0(m))$ and we can work with $B_r(m)$ for $m \in K_0$. Let us fix a point $m_0 \in K_0$ and set $R' := 2R + \operatorname{diam} K_0$. Then

$$B_r(m) \subset B_{R'}(m_0), \quad m \in K_0, \tag{7.138}$$

and it remains to show that $B_{R'}(m_0)$ can be covered by a finite number, say N, of (open) balls of radius $r/2$ with N independent of r. We use the following

Lemma 7.52. *Suppose that G acts properly, freely and cocompactly on a length space $\mathcal{M}$ by isometries. Then every bounded closed set $S \subset \mathcal{M}$ is compact.*

Proof. For every $m \in S$ there is a finite number of isometries $g_{im} \in G$, $i = 1, \dots, k_m$, such that $g_{im}(m) \in K_0$. Here K_0 is a generating compact set with respect to the action of G. Let $H := \{g_{im}^{-1} \in G\ ;\ 1 \leq i \leq k_m,\ m \in S\}$. Then $S \subset H(K_0)$, and, by definition, $\operatorname{diam} H(K_0) < \infty$. For a fixed $m_0 \in K_0$ let us consider the orbit $H(m_0)$. We claim that $H(m_0)$ consists of a finite number of points. Otherwise there is a sequence of points $m_i = h_i(m_0) \in H(m_0)$ such that

$d_A(h_i, 1) \to \infty$ as $i \to \infty$. This and inequality (7.136) imply $d(m_i, m_0) \to \infty$ in $\mathcal{M}$ as $i \to \infty$ and this contradicts the condition diam $H(K_0) < \infty$. From the finiteness of $H(m_0)$ we also obtain that H is finite. Thus S is covered by a finite number of compact sets, and, since S is closed, it is compact. □

According to this lemma the closed ball $\overline{B}_{R'}(m_0)$ is compact. Thus $B_{R'}(m_0)$ can be covered by a finite number N of open balls of radius $\widetilde{R}/4$. This, (7.137) and (7.138) show that $B_r(m)$ can be covered by N open balls of radius $r/2$ as is required.

To establish the second condition of Theorem 7.43, i.e., the finiteness of

$$\lambda_R := \sup\{\lambda(B_{2R}(m))\ ;\ m \in \mathcal{M}\},$$

we first use the previous argument and (7.138) which immediately yield

$$\lambda_R \le \lambda(B_{R'}(m_0)).$$

We now show that the right-hand side is bounded. Since $\mathcal{M} \in \mathcal{G}_n(\widetilde{R}, \widetilde{D})$, for every $m \in \mathcal{M}$,

$$\lambda(B_{\widetilde{R}}(m)) < \infty.$$

This, compactness of $\overline{B}_{R'}(m_0)$ and the argument used in the proof of Lemma 7.9 lead to the required inequality

$$\lambda(B_{R'}(m_0)) < \infty.$$

The proof of Corollary 7.51 is complete. □

Corollary 7.50 tells us that the desired extension property for a metric space may be reduced to that for some of its lattices regarded as a metric space. We assume that for the class of the, so-called, *uniform lattices* the answer is positive, i.e., each such a lattice belongs to $\mathcal{SLE}$.

Uniformity of a lattice Γ means that for some increasing function $\phi_\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$ and constant $0 < c \le 1$, the number of points of $\Gamma \cap B_R(m)$ for every $R > 0$ and $m \in \Gamma$ satisfies

$$c\phi_\Gamma(R) \le |\Gamma \cap B_R(m)| \le \phi_\Gamma(R). \tag{7.139}$$

According to Theorem 3.134 of Volume I uniform lattices bi-Lipschitz homeomorphic to finitely generated groups of polynomial growth (in this case $\phi_\Gamma(R) = aR^n$ for some $a, n \ge 0$) equipped with word metrics are of homogeneous type. Hence, they are $\mathcal{SLE}$ spaces, see Corollary 7.37. Also, it follows from the results of Lang and Schlichenmaier [LSchl-2005] and of the authors of the present book [BB-2007b] that the result holds for uniform lattices bi-Lipschitz homeomorphic to Gromov hyperbolic groups, see Volume I, Definition 3.137 (in this case the function ϕ is generally of exponential growth). However, one can construct a uniform lattice bi-Lipschitz homeomorphic to a countable group with infinite number of generators equipped with the word metric which does not belong to $\mathcal{SLE}$, see subsection 8.3.3.

These facts lead to the following

Conjecture 7.53. *A uniform lattice bi-Lipschitz homeomorphic to a finitely generated group equipped with the word metric belongs to $\mathcal{SLE}$.*

Returning to Corollary 7.51 we will single out the case of a group acting on a metric space for which we can give rather sharp estimates of the characteristics λ. To formulate the results we introduce the functional

$$\lambda_{\mathrm{conv}}(\mathcal{M}) := \sup_C \lambda(C, \mathcal{M})$$

where C runs through all convex subsets of a normed linear space $\mathcal{M}$. As before we set

$$\lambda(S, \mathcal{M}) := \inf\{||E|| \ ; \ E \in \mathrm{Ext}(S, \mathcal{M})\}.$$

Theorem 7.54. *There exists a numerical constant $c_0 > 1/4$ such that for all n and $1 \leq p \leq \infty$,*

$$c_0 \leq n^{-\left|\frac{1}{p}-\frac{1}{2}\right|} \cdot \lambda_{\mathrm{conv}}(\ell_p^n) \leq 1.$$

Proof. We first prove that

$$\lambda_{\mathrm{conv}}(\ell_2^n) = 1. \tag{7.140}$$

Since the lower bound 1 is clear, we have to prove that

$$\lambda_{\mathrm{conv}}(\ell_2^n) \leq 1. \tag{7.141}$$

Let $C \subset (\ell_2^n, 0)$ be a closed convex set containing 0, and $p_C(x)$ be the (unique) closest to x point from C. Then, see, e.g., [BL-2000, Sect. 3.2], the metric projection p_C is Lipschitz and

$$||p_C(x) - p_C(y)||_2 \leq ||x - y||_2. \tag{7.142}$$

Using this we introduce a linear operator E given on $\mathrm{Lip}_0(C)$ by

$$(Ef)(x) := (f \circ p_C)(x), \quad x \in \ell_2^n. \tag{7.143}$$

Since p_C is the identity on C and $p_C(0) = 0$ as $0 \in C$, this operator belongs to $\mathrm{Ext}(C, \ell_2^n)$. Moreover, by (7.142),

$$||(Ef)(x) - (Ef)(y)||_2 \leq ||f||_{\mathrm{Lip}_0(C)} ||x - y||_2,$$

i.e., $||E|| \leq 1$, and (7.141) is established.

Using now the inequalities

$$||x||_p \leq ||x||_2 \leq n^{\frac{1}{2}-\frac{1}{p}} \ ||x||_p \quad \text{for} \quad 2 \leq p \leq \infty, \tag{7.144}$$

and

$$n^{\frac{1}{2}-\frac{1}{p}} \ ||x||_p \leq ||x||_2 \leq ||x||_p \quad \text{for} \quad 1 \leq p \leq 2 \tag{7.145}$$

we derive from (7.141) the required upper bound:

$$\lambda_{\mathrm{conv}}(\ell_p^n) \leq n^{\left|\frac{1}{2}-\frac{1}{p}\right|}. \tag{7.146}$$

In order to prove the lower estimate we need the following result where Banach spaces are regarded as pointed metric spaces with $m^* = 0$.

Proposition 7.55. *Let Y be a linear subspace of a finite-dimensional Banach space X, and the operator E belong to* $\mathrm{Ext}(Y,X)$. *Then there is a linear projection P from X onto Y such that*

$$||P|| \leq ||E||. \tag{7.147}$$

Proof. We use an argument similar to that of the proof of Theorem 1.49 of Volume I. First, we introduce an operator $S : \mathrm{Lip}_0(Y) \to \mathrm{Lip}_0(X)$ given at $z \in X$ by

$$(Sf)(z) := \int_X \left\{ \int_Y [(Ef)(x+y+z) - (Ef)(x+y)]dy \right\} dx. \tag{7.148}$$

Here $\int_A \dots da$ is a translation invariant mean on the space $\ell_\infty(A)$ of all bounded functions on the abelian group A. Since the function within [ ] is bounded for every fixed z (recall that $Ef \in \mathrm{Lip}(X)$), this operator is well defined. Moreover, as $\int_A da = 1$ we get

$$||Sf||_{\mathrm{Lip}_0(X)} \leq ||E|| \cdot ||f||_{\mathrm{Lip}_0(Y)}. \tag{7.149}$$

By the translation invariance of dx we then derive from (7.148) that

$$(Sf)(z_1 + z_2) = (Sf)(z_1) + (Sf)(z_2), \quad z_1, z_2 \in X.$$

Together with (7.149) and the equality

$$||f||_{\mathrm{Lip}_0(X)} = ||f||_{X^*}, \quad f \in X^*, \tag{7.150}$$

this shows that Sf belongs to X^* and therefore S maps $\mathrm{Lip}_0(Y)$ linearly and continuously into X^*. Further, Y^* is a linear subset of $\mathrm{Lip}_0(Y)$ whose norm coincides with that induced from $\mathrm{Lip}_0(Y)$. Therefore the restriction

$$T := S|_{Y^*}$$

is a bounded linear operator from Y^* to X^*. As in the proof of Theorem 1.49 of Volume I we obtain that

$$(Tf)(z) = f(z), \quad z \in Y, \tag{7.151}$$

i.e., T is an extension from Y^*.

Consider now the conjugate to T operator T^* acting from $X^{**} = X$ to $Y^{**} = Y$. Since T is a linear extension operator from Y^*, its conjugate is a projection onto Y. Finally, (7.149) and (7.150) give for the norm of this projection the required estimate

$$||T^*|| = ||T|| \leq ||E||.$$

The proposition is proved. □

Proposition 7.56. *Let X be either ℓ_1^n or ℓ_∞^n. Then there is a subspace $Y \subset X$ such that* $\dim Y = \lfloor n/2 \rfloor$ *and its projection constant* $\pi(Y,X) := \inf ||P||$ *where P runs through all linear projections from X onto Y and satisfies*

$$\pi(X,Y) \geq c_0\sqrt{n} \tag{7.152}$$

with c_0 independent of n.

Proof. The inequality follows from Theorem 1.2 of the paper [So-1941] by Sobczyk with the optimal c_0 greater than 1/4. □

We now complete the proof of Theorem 7.54. Applying Propositions 7.55 and 7.56 we get for an arbitrary $E \in \mathrm{Ext}(Y, \ell_1^n)$ the inequality

$$||E|| \geq c_0\sqrt{n} \tag{7.153}$$

with $c_0 > 0$ independent of n. A similar estimate is valid for $E \in \mathrm{Ext}(Y^\perp, \ell_\infty^n)$ as well. Hence for $p = 1, \infty$,

$$\lambda_{\mathrm{conv}}(\ell_p^n) \geq c_0\sqrt{n}. \tag{7.154}$$

Using this estimate for $p = 1$ and applying an inequality similar to (7.145) comparing $||x||_1$ and $||x||_p$, we get for $1 \leq p \leq 2$ the estimate

$$\lambda_{\mathrm{conv}}(\ell_p^n) \geq c_0 n^{\frac{1}{p}-\frac{1}{2}}.$$

Then using (7.154) for $p = \infty$ and an inequality similar to (7.144) comparing $||x||_p$ and $||x||_\infty$, we get for $2 \leq p \leq \infty$ the inequality

$$\lambda_{\mathrm{conv}}(\ell_p^n) \geq c_0 n^{\frac{1}{2}-\frac{1}{p}}.$$

The proof of the theorem is complete. □

As a corollary of Theorem 7.54 we get

Corollary 7.57. *For a Banach space X,*

$$c_0\sqrt{n} \leq \lambda(\ell_p^n, X) \leq cn, \quad p = 1, \infty,$$

for some numerical constants $c \leq 311$ and $c_0 > \frac{1}{4}$.

Proof. The right-hand inequality follows directly from Corollary 7.36. And, the left-hand inequality follows from Theorem 7.54 and the inequalities (see the proof of Proposition 7.3)

$$\lambda(\ell_p^n, X) \geq \lambda(\ell_p^n) \geq \lambda_{\mathrm{conv}}(\ell_p^n) \geq c_0\sqrt{n}.$$ □

Now we derive from here a result for the abelian group $\mathbb{Z}^n$ with the word metric d_A where A is the standard orthonormal basis of $\mathbb{R}^n$. The reader may easily check that d_A coincides with the restriction to $\mathbb{Z}^n$ of ℓ_1^n-metric. This fact and the result of Corollary 7.19 with $(\mathcal{M}, d) = (\mathbb{Z}^n, d_A)$ and the dilation $\phi : x \mapsto \frac{1}{2}x$ yield

$$\lambda(\mathbb{Z}^n, d_A) = \lambda(\ell_1^n).$$

This and Corollary 7.57 immediately imply the following result.

Corollary 7.58.

$$c_0\sqrt{n} \leq \lambda((\mathbb{Z}^n, d_A), X) \leq cn.$$

In a similar way we may consider other finitely generated groups introduced in subsection 3.3.7 of Volume I. We present several illustrating examples of groups having the simultaneous extension property referring to the paper [BB-2007b] for more results and missing proofs.

Example 7.59. (a) **Carnot groups.** Let G be an m-step Carnot group whose Lie algebra is stratified into the direct sum of nontrivial vector spaces V_i, $1 \le i \le m$. The number $Q := \sum_{i=1}^{m} i \dim V_i$ is recalled to be the *fractal* (or homogeneous) *dimension* of G see Volume I, formula (3.152).

Let d_C be the Carnot-Carathéodory metric on G. Then the Haar measure of G is Q-homogeneous, see Volume I, formula (3.151). Applying Corollary 7.36 we then have

$$\lambda((G, d_C), X) \le cQ$$

for a dual space X with the constant $c < 24$, see [BB-2007b, Cor. 2.27].

For instance, $\mathbb{R}^n$ equipped with any Banach norm $\|\cdot\|$ is a Carnot group with $Q = n$ and d_C generated by this norm. The above inequality, in particular, gives the Johnson-Lindenstrauss-Schechtman theorem [JLS-1986] for $\lambda((\mathbb{R}^n, \|\cdot\|), X)$ with a better constant but for a dual X.

Another special case is the Heisenberg group H_n, see Volume I, Example 3.141 (b), in which case $Q = 2n + 2$ and therefore

$$\lambda(H_n, d_C) \le 48(n + 1).$$

Finally, let us consider the discrete subgroup $H_n(\mathbb{Z})$ of H_n consisting of elements from the set $\mathbb{Z}^n \times \mathbb{Z}^n \times \mathbb{Z}$, see Volume I, Example 3.135. It is readily seen that the map $\phi : (x, y, t) \mapsto \frac{1}{2}(x, y, t)$ is a dilation in the sense of Corollary 7.19. Since $\phi(H_n(\mathbb{Z})) \supset H_n(\mathbb{Z})$ and $\cup_{j=0}^{\infty} \phi^j(H_n(\mathbb{Z}))$ is dense in (H_n, d_C) the aforementioned corollary yields

$$\lambda(H_n(\mathbb{Z})) = \lambda(H_n) \le 48(n + 1).$$

(b) **Groups of polynomial growth.** Let now G be a finitely generated group with a word metric associated with a generating set A. Assume that G is of polynomial growth, i.e., for all $R > 0$ the cardinality of a ball of radius R is bounded by CR^ρ for some $C > 0$ and $\rho \ge 0$. Then the counting measure of G is equivalent to a Q-homogeneous measure where the constants of equivalence depend on G and A, and Q is the degree of the maximal torsion free nilpotent subgroup of G, see Volume I, Theorem 3.134. Therefore due to Corollary 7.36 for some $c = c(G, A)$,

$$\lambda(\mathcal{M}, X) \le c.$$

If, in addition, G is torsion free, the constants of equivalence in the previous statement depend only on the degree Q and the constant c does as well.

(c) **Hyperbolic groups.** Let G be a (finitely generated) hyperbolic group and d_A be some of its word metrics. Then (G, d_A) is a metric subspace in a δ-hyperbolic space with $\delta \geq 0$ (its Cayley graph), see Definition 3.137 and Theorem 3.138 of Volume I. If G is not virtually cyclic (i.e., its quotient by a finite subgroup is cyclic), the ambient δ-hyperbolic space is of bounded geometry. Due to Corollary 6.38 $\lambda((G, d_A), X) < \infty$ for such G.

7.4 Spaces with the universal linear Lipschitz extension property

The basic characteristic studied in this section is described by

Definition 7.60. A metric space $\mathcal{M}$ is said to be universal with respect to simultaneous Lipschitz extensions if, for an arbitrary metric space $\widetilde{\mathcal{M}}$ and every subspace S of $\widetilde{\mathcal{M}}$ isometric to a subspace of $\mathcal{M}$,

$$\lambda(S, \widetilde{\mathcal{M}}, X) \leq c$$

where c depends only on $\mathcal{M}$.

The optimal constant in this inequality will be denoted by $\lambda_u(\mathcal{M})$.

Remark 7.61. In fact, in all our results related to universality we will establish a much stronger property: if S is C-isometric ($C \geq 1$) to a subspace of $\mathcal{M}$, then

$$\lambda(S, \widetilde{\mathcal{M}}, X) \leq C^2 c$$

with c depending only on $\mathcal{M}$. This clearly implies the universality of $\mathcal{M}$.

We will show that the direct sum of a finite combination of Gromov hyperbolic spaces of bounded geometry and homogeneous metric spaces is universal.

It was shown in Section 6.4, see Corollary 6.41 and Remark 6.42, that all these results with unspecified estimates of the extension constants follow from the Lang-Schlichenmaier theory [LSchl-2005]. Universality of doubling metric spaces with an almost optimal extension constant was firstly proved by Lee and Naor, see [LN-2005, Thm. 1.6], in a nonconstructive way. Universality of Gromov hyperbolic spaces of bounded geometry was proved by the authors of this book [BB-2007c].

In this section, we prove all of the aforementioned universality results by the method of this paper. This method is constructive and allows us to obtain relatively good estimates of the extension constants. We first apply this approach to the proof of the result equivalent to the Lee-Naor theorem, cf. Corollary 7.37. Since the basic ideas had already presented in the proof of Theorem 7.22, we refer to that proof to shorten our derivation.

Theorem 7.62. *Let $\mathcal{M}_0$ be a metric subspace of an arbitrary metric space $\mathcal{M}$. Assume that $\mathcal{M}_0$ is of homogeneous type with respect to a doubling measure μ_0.*

Then for any Banach space X there exists a simultaneous extension operator E : $\mathrm{Lip}(\mathcal{M}_0, X) \to \mathrm{Lip}(\mathcal{M}, X)$ satisfying

$$||E|| \le c(\log_2 D(\mu_0) + 1)$$

with some numerical constant $c \ge 1$.

Proof. We begin with the following remark reducing the required result to a special case. Let $\mathcal{M}$ and $\mathcal{M}_0$ be isometric to subspaces of a new metric space $\widehat{\mathcal{M}}$ and its subspace $\widehat{\mathcal{M}}_0$, respectively. Assume that there exists a simultaneous extension operator $\widehat{E} : \mathrm{Lip}(\widehat{\mathcal{M}}_0, X) \to \mathrm{Lip}(\widehat{\mathcal{M}}, X)$. Then, after the corresponding identification, the operator $\widehat{E}$ gives rise to a simultaneous extension operator $E : \mathrm{Lip}(\mathcal{M}_0, X) \to \mathrm{Lip}(\mathcal{M}, X)$ satisfying

$$||E|| \le ||\widehat{E}||.$$

If, in addition, $||\widehat{E}||$ is bounded by $c(\log_2 D(\mu_0) + 1)$ with a numerical constant $c \ge 1$, then the desired result immediately follows.

We choose as the above pair $\widehat{\mathcal{M}}_0 \subset \widehat{\mathcal{M}}$ metric spaces denoted by $\mathcal{M}_{0N}$ and $\mathcal{M}_N$ defined as follows.

The underlying sets of these spaces are

$$\mathcal{M}_N := \mathcal{M} \times \mathbb{R}^N, \quad \mathcal{M}_{0N} := \mathcal{M}_0 \times \mathbb{R}^N; \tag{7.155}$$

a metric d_N on $\mathcal{M}_N$ is given by

$$d_N((m, x), (m', x')) := d(m, m') + \|x - x'\|_1 \tag{7.156}$$

where $m, m' \in \mathcal{M}$ and $x, x' \in \mathbb{R}^N$, and $\|x\|_1 := \sum_{i=1}^N |x_i|$ is the ℓ_1^N-metric of $x \in \mathbb{R}^N$. Further, d_{0N} denotes the metric on $\mathcal{M}_{0N}$ induced by d_N.

Finally, we define a Borel measure μ_{0N} on $\mathcal{M}_{0N}$ by

$$\mu_{0N} := \mu_0 \otimes \lambda_N. \tag{7.157}$$

We extend this measure to the σ-algebra consisting of subsets $S \subset \mathcal{M}_N$ such that $S \cap \mathcal{M}_{0N}$ is a Borel subset of $\mathcal{M}_{0N}$ and for these S,

$$\overline{\mu}_N(S) := \mu_{0N}(S \cap \mathcal{M}_{0N}).$$

It is important for the subsequent part of the proof that each open ball $B_R((m, x)) \subset \mathcal{M}_N$ belongs to this σ-algebra. In fact, its intersection with $\mathcal{M}_{0N}$ is a Borel subset of this space, since the function $(m', x') \mapsto d_N((m, x), (m', x'))$ is continuous on $\mathcal{M}_{N0}$. Hence,

$$\overline{\mu}_N(B_R((m, x))) = \mu_{0N}(B_R((m, x)) \cap \mathcal{M}_{0N}). \tag{7.158}$$

Hereafter we denote by $\widehat{m}$ the pair (m, x) with $m \in \mathcal{M}$ and $x \in \mathbb{R}^N$, and by $B_R^o(\widehat{m})$ the open ball in $\mathcal{M}_{0N}$ centered at $\widehat{m} \in \mathcal{M}_{0N}$ and of radius R. The open ball $B_R(\widehat{m})$ of $\mathcal{M}_N$ relates to that by

$$B_R^o(\widehat{m}) = B_R(\widehat{m}) \cap \mathcal{M}_{0N}$$

provided that $\widehat{m} \in \mathcal{M}_{0N}$.

Since the measure μ_{0N} is clearly doubling, its dilation function given for $l \geq 1$ by

$$D_{0N}(l) := \sup \left\{ \frac{\mu_{0N}(B_{lR}^o(\widehat{m}))}{\mu_{0N}(B_R^o(\widehat{m}))} \ ; \ \widehat{m} \in \mathcal{M}_{0N} \text{ and } R > 0 \right\}$$

is finite.

We also define a (modified) dilation function D_N for the extended measure $\overline{\mu}_N$. This is given for $l \geq 1$ by

$$D_N(l) := \sup \left\{ \frac{\overline{\mu}_N(B_{lR}(\widehat{m}))}{\overline{\mu}_N(B_R(\widehat{m}))} \right\} \tag{7.159}$$

where the supremum is taken over all R satisfying

$$R > 4d(\widehat{m}, \mathcal{M}_{0N}) := 4 \inf\{d_N(\widehat{m}, \widehat{m}') \ ; \ \widehat{m}' \in \mathcal{M}_{0N}\} \tag{7.160}$$

and then over all $\widehat{m} \in \mathcal{M}_N$.

Due to (7.158) and (7.160) the denominator in (7.159) is not zero and $D_N(l)$ is well defined.

Comparison with the dilation function for M_{0N} shows that $D_{0N}(l) \leq D_N(l)$. We will see that the converse is also true for l close to 1.

Lemma 7.63. *Assume that N and the doubling constant $D := D(\mu_0)$ are related by*

$$N \geq \lfloor 3 \log_2 D \rfloor + 5. \tag{7.161}$$

Then the following is true:

$$D_N(1 + 1/N) \leq \frac{6}{5} e^4.$$

Proof. The proof repeats the proof of Lemma 3.94 of Volume I. □

Next, we estimate $\overline{\mu}_N$-measure of the spherical layer $B_{R_2}(\widehat{m}) - B_{R_1}(\widehat{m})$ with $R_2 \geq R_1$ by a kind of a surface measure. For its formulation we set

$$A_N := \frac{12}{5} e^4 N. \tag{7.162}$$

Lemma 7.64. *Assume that*

$$N \geq \lfloor 3\log_2 D \rfloor + 6.$$

Then for all $\widehat{m} \in \mathcal{M}_N$ *and* $R_1, R_2 > 0$ *satisfying*

$$R_2 \geq \max\{R_1, 8d_N(\widehat{m}, \mathcal{M}_{0N})\}$$

the following is true:

$$\overline{\mu}_N(B_{R_2}(\widehat{m}) \setminus B_{R_1}(\widehat{m})) \leq A_N \frac{\overline{\mu}_N(B_{R_2}(\widehat{m}))}{R_2}(R_2 - R_1).$$

Proof. By definition $\mathcal{M}_N = \mathcal{M}_{N-1} \times \mathbb{R}$ and $\overline{\mu}_N = \overline{\mu}_{N-1} \otimes \lambda_1$. Then by the Fubini theorem we have for $R_1 \leq R_2$ and $\widehat{m} = (\widetilde{m}, t)$,

$$\begin{aligned}\overline{\mu}_N(B_{R_2}(\widehat{m})) - \overline{\mu}_N(B_{R_1}(\widehat{m})) &= 2\int_{R_1}^{R_2} \overline{\mu}_{N-1}(B_s(\widetilde{m}))ds \\ &\leq \frac{2R_2\overline{\mu}_{N-1}(B_{R_2}(\widetilde{m}))}{R_2}(R_2 - R_1).\end{aligned}$$

We claim that for arbitrary $l > 1$ and $R \geq 8d_N(\widehat{m}, \mathcal{M}_{0N}) := 8d_{N-1}(\widetilde{m}, \mathcal{M}_{0\,N-1})$

$$R\overline{\mu}_{N-1}(B_R(\widetilde{m})) \leq \frac{lD_{N-1}(l)}{l-1}\overline{\mu}_N(B_R(\widehat{m})). \tag{7.163}$$

Together with the previous inequality this would yield

$$\overline{\mu}_N(B_{R_2}(\widehat{m})) - \overline{\mu}_N(B_{R_1}(\widehat{m})) \leq \frac{2lD_{N-1}(l)}{l-1} \cdot \frac{\overline{\mu}_N(B_{R_2}(\widehat{m}))}{R_2}(R_2 - R_1).$$

Finally choose here $l = 1 + \frac{1}{N-1}$ and use Lemma 7.64 to obtain the required inequality.

Hence, it remains to establish (7.81). By the definition of $D_{N-1}(l)$ and the previous lemma we have for $l > 1$,

$$\begin{aligned}\overline{\mu}_N(B_{lR}(\widehat{m})) = 2l\int_0^R \overline{\mu}_{N-1}(B_{ls}(\widetilde{m}))ds &\leq 4l\int_{R/2}^R \overline{\mu}_{N-1}(B_{ls}(\widetilde{m}))ds \\ \leq 4lD_{N-1}(l)\int_{R/2}^R \overline{\mu}_{N-1}(B_s(\widetilde{m}))ds &\leq 2lD_{N-1}(l)\overline{\mu}_N(B_R(\widehat{m})).\end{aligned}$$

On the other hand, replacing $[0, R]$ by $[l^{-1}R, R]$ we also have

$$\overline{\mu}_N(B_{lR}(\widehat{m})) \geq 2l\overline{\mu}_{N-1}(B_R(\widetilde{m}))(R - l^{-1}R) = 2(l-1)R\overline{\mu}_{N-1}(B_R(\widetilde{m})).$$

Combining the last two inequalities we get (7.81). □

Extension operator

We define the required simultaneous extension operator $E : \mathrm{Lip}(\mathcal{M}_{0N}, X) \to \mathrm{Lip}(\mathcal{M}_N, X)$ using the standard average operator Ave defined on continuous and locally bounded functions $g : \mathcal{M}_{0N} \to X$ by

$$Ave(g; \widehat{m}, R) := \frac{1}{\overline{\mu}_N(B_R(\widehat{m}))} \int_{B_R(\widehat{m})} g \, d\overline{\mu}_N.$$

Then we define the E on functions $f \in \mathrm{Lip}(\mathcal{M}_{0N}, X)$ by

$$(Ef)(\widehat{m}) := \begin{cases} f(\widehat{m}), & \text{if } \widehat{m} \in \mathcal{M}_{0N}, \\ Ave(f; m, R(\widehat{m})), & \text{if } \widehat{m} \notin \mathcal{M}_{0N}, \end{cases} \tag{7.164}$$

where we set $R(\widehat{m}) := 8d_N(\widehat{m}, \mathcal{M}_{0N})$. Since $\overline{\mu}_N(B_{R(\widehat{m})}(\widehat{m})) > 0$, this definition is correct.

To give the required estimate of $||E||$ we set

$$K_N(l) := A_N D_N(l)(4l + 1) \tag{7.165}$$

where the first of two factors are defined by (7.162) and (7.159).

Proposition 7.65. *The following inequality*

$$||E|| \le 20A_N + \max\left(\frac{4l+1}{2(l-1)}, K_N(l)\right)$$

is true provided that $l := 1 + 1/N$.

Proof. The proof repeats the steps of the proof of Proposition 7.29 where we use A_N instead of A_n and do not use any results related to $\widehat{C}_n$, see also [BB-2006] for details. □

Choosing here

$$N := \lfloor 3 \log_2 D \rfloor + 6$$

and using Lemma 7.63 and (7.162) to estimate $D_N(1 + 1/N)$ and A_N we get

$$||E|| \le C(\log_2 D + 2)$$

with some numerical constant C. This clearly gives the required result. □

Our next result, proved in [BB-2007b, Thm. 1.5], establishes the universality of the direct sum of Gromov hyperbolic spaces of bounded geometry, see subsection 3.3.4 of Volume I for the corresponding definitions.

Theorem 7.66. *Let* $\mathcal{M} := \oplus^p\{(\mathcal{M}_i, d_i)\}_{1 \le i \le N}$ *where every* $(\mathcal{M}_i, d_i)$ *is a (Gromov) hyperbolic metric space of bounded geometry. Then* $\mathcal{M}$ *is universal.*

Here $\oplus^{(p)}\{(\mathcal{M}_i, d_i)\}_{1\le i\le N}$ is the metric space with underlying set $\prod_{i=1}^N \mathcal{M}_i$ and metric $d_p := \left\{\sum_{i=1}^N d_i^p\right\}^{1/p}$.

Proof. Clearly it suffices to proof the result for $p = \infty$ only.

We need several auxiliary results the first of which is Theorem 5.14 of Volume I and the second result used is a variant of Corollary 7.39 which requires trivial changes in its proof.

Lemma 7.67. *Let $(\mathcal{M}, d)$ be the direct sum $\oplus^{(\infty)}\{(\mathcal{M}_i, d_i)\}_{0\le i\le N}$ where $(\mathcal{M}_0, d_0)$ is an n_0-dimensional Banach space and $\mathcal{M}_i = \mathbb{H}^{n_i}$, $1 \le i \le N$. Then for some $c > 1$ we have*

$$\lambda(S, \mathcal{M}, X) \le c\left(n_0 + \sum_{i=0}^{N}[n_i^2 + n_i]\right)\cdot \max_{1\le i\le N}\sqrt{n_i}.$$

We are now ready to prove Theorem 7.66. So, let S be a subspace of an arbitrary metric space $(\widehat{\mathcal{M}}, \widehat{d})$ and let $\phi : S \to \oplus^{(\infty)}\{(\mathcal{M}_i, d_i)\}_{1\le i\le N}$ be a C-isometric embedding, i.e., for all $m, m' \in S$,

$$C^{-1}\widehat{d}(m, m') \le d_\infty(\phi(m), \phi(m')) \le C\widehat{d}(m, m').$$

We must find a linear extension operator $E : \mathrm{Lip}(S, X) \to \mathrm{Lip}(\widehat{\mathcal{M}}, X)$ whose norm is bounded by a constant depending only on the geometric characteristics of the spaces $\mathcal{M}_i$ and the embedding constant C.

For this aim we first use Theorem 5.14 of Volume I to find a C_1-isometric embedding ψ of $\oplus^{(\infty)}\{(\mathcal{M}_i, d_i)\}_{1\le i\le N}$ into the space $(\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1\le i\le N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}$. Note that C_1 depends only on the characteristics of the spaces $\mathcal{M}_i$. Then $\psi \circ \phi$ is a CC_1-isometric embedding of S into $(\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1\le i\le N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}$.

Set

$$\widehat{S} := \mathrm{Image}(\psi \circ \phi) \subset (\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1\le i\le N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}$$

and define the linear operator E_1 on $\mathrm{Lip}(S, X)$ by the formula

$$E_1 f := f \circ \phi^{-1} \circ \psi^{-1}. \tag{7.166}$$

Then $E_1 : \mathrm{Lip}(S, X) \to \mathrm{Lip}(\widehat{S}, X)$ and

$$||E_1|| \le CC_1. \tag{7.167}$$

Further, we use Lemma 7.67 to find a linear bounded operator

$$E_2 : \mathrm{Lip}(\widehat{S}, X) \to \mathrm{Lip}((\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1\le i\le N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}, X)$$

such that

$$E_2 g|_{\widehat{S}} = g \quad \text{for} \quad g \in \mathrm{Lip}(\widehat{S}, X) \quad \text{and} \quad ||E_2|| \le c(\overline{n}) \tag{7.168}$$

where

$$c(\overline{n}) := c(n_0, n_1, \dots, n_N) \le c\left(n_0 + \sum_{i=0}^{N} [n_i^2 + n_i]\right) \cdot \max_{1\le i\le N} \sqrt{n_i}.$$

Finally, the coordinatewise application of the Lang-Pavlović-Schroeder Theorem 6.38 allows us to extend the map $\psi \circ \phi : S \to (\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1\le i\le N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}$ to a Lipschitz map $\Phi : \widehat{\mathcal{M}} \to (\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1\le i\le N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}$ such that

$$\Phi|_S = \psi \circ \phi \quad \text{and} \quad L(\Phi) \le c_1(\overline{n})CC_1 \tag{7.169}$$

where

$$c_1(\overline{n}) \le 2\sqrt{2}\left(\max_{1\le i\le N} \sqrt{n_i}\right).$$

Next, define a linear operator E_3 on $\mathrm{Lip}((\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{1\le i\le N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}, X)$ given by

$$E_3 h := h \circ \Phi.$$

Then $\mathrm{Lip}(\widehat{\mathcal{M}}, X)$ is the target space of E_3 and

$$||E_3|| \le L(\Phi) \le c_1(\overline{n})CC_1. \tag{7.170}$$

Moreover, by (7.169),

$$(E_3 h)|_S = h(\Phi|_S) = h \circ \psi \circ \phi. \tag{7.171}$$

Finally, define the desired linear extension operator E by

$$E = E_3 E_2 E_1.$$

According to (7.166), (7.168) and (7.171), E acts from $\mathrm{Lip}(S, X)$ into $\mathrm{Lip}(\widehat{\mathcal{M}}, X)$ and

$$Ef|_S = f.$$

In addition, (7.167), (7.168) and (7.170) imply that

$$||E|| \le C^2 C_1^2 c(\overline{n}) c_1(\overline{n}).$$

Hence, the simultaneous extension constant $\lambda(S, \widehat{\mathcal{M}}, X)$ is bounded by a constant which depends only on the characteristics of the spaces $\mathcal{M}_i$ and the embedding constant C of the map ϕ. □

Theorem 7.68. *Let $\mathcal{M} := \oplus^{(p)}\{(\mathcal{M}_i, d_i)\}_{1\le i\le N}$ where every $(\mathcal{M}_i, d_i)$ is either of homogeneous type or a Gromov hyperbolic space of bounded geometry. Then $\mathcal{M}$ is universal.*

Proof. Without loss of generality we assume that $p = \infty$ and $(\mathcal{M}_i, d_i)$ is homogeneous for $i = 1$ and Gromov hyperbolic of bounded geometry for $i \geq 2$.

Let S be a subspace of an arbitrary metric space $(\widetilde{\mathcal{M}}, \widetilde{d})$ and $\phi : S \to \mathcal{M}$ be a C-isometric embedding. Set $\mathcal{M}^1 := \oplus^{(\infty)}\{(\mathcal{M}_i, d_i)\}_{2\leq i\leq N}$ so that $\mathcal{M} = \mathcal{M}_1 \oplus^{(\infty)} \mathcal{M}^1$. By Theorem 5.14 of Volume I we embed $\mathcal{M}^1$ bi-Lipschitz homeomorphically into $H := (\oplus^{(\infty)}\{\mathbb{H}^{n_i}\}_{2\leq i\leq N}) \oplus^{(\infty)} \{\mathbb{R}^{n_0}\}$. Further, using Lemma 4.92 of Volume I we embed $\mathcal{M}_1$ isometrically into the predual space $\mathcal{F}(\mathcal{M}_1)$ of the space $\mathrm{Lip}_0(\mathcal{M}_1)$. In turn, we embed $\mathcal{F}(\mathcal{M}_1)$ isometrically into the Banach space $\ell_\infty(B)$ where B is the unit ball of $\mathcal{F}(\mathcal{M}_1)$. This allows us to identify the set $\mathcal{M}$ with its image in $\ell_\infty(B) \oplus^{(\infty)} H$ and the map $\phi : S \to \mathcal{M}$ with a bi-Lipschitz embedding into this image, here $\phi = \phi_1 \oplus \phi_2$ where $\phi_1 : S \to \ell_\infty(B)$ and $\phi_2 : S \to H$.

Next, by the McShane extension Theorem 1.27 of Volume I, ϕ_1 admits a Lipschitz extension to all of $\widetilde{\mathcal{M}}$ preserving its Lipschitz constant while ϕ_2 can be extended to all of $\widetilde{\mathcal{M}}$ with Lipschitz constant bounded by $c(\sum_{i=2}^N n_i, n_0)L(\phi_2)$, by Theorem 6.38. Hence there is a Lipschitz map $\widetilde{\phi} : \widetilde{\mathcal{M}} \to \ell_\infty(B) \oplus H$ such that $\widetilde{\phi}|_S = \phi$ and $L(\widetilde{\phi})$ is bounded by a constant $c(\mathcal{M})C$.

Following the arguments of the proof of Theorem 7.66, we now determine continuous linear extension operators $E_1 : \mathrm{Lip}(\phi(S), X) \to \mathrm{Lip}(\mathcal{M}_1 \oplus^{(\infty)} H, X)$ and $E_2 : \mathrm{Lip}(\mathcal{M}_1 \oplus^{(\infty)} H, X) \to \mathrm{Lip}(\ell_\infty(B) \oplus^{(\infty)} H, X)$ with bounds of their norms depending only on the basic parameters of $\mathcal{M}$. Setting then

$$E(f)(x) := (E_2E_1)(f \circ \phi^{-1})(\widetilde{\phi}(x)), \quad x \in \widetilde{\mathcal{M}}, \quad f \in \mathrm{Lip}(S, X),$$

we obtain a linear extension operator $\mathrm{Lip}(S, X) \to \mathrm{Lip}(\widetilde{\mathcal{M}}, X)$ whose norm is bounded by the basic parameters of $\mathcal{M}$ and C. This would complete the proof of the corollary.

The operator E_1 has already been defined by Theorem 7.22 with $\mathcal{M}_1$ being a metric space of homogeneous type.

To define E_2 we first use Theorem 7.62 to find a bounded linear extension operator $\widetilde{E} : \mathrm{Lip}(\mathcal{M}_1, X) \to \mathrm{Lip}(\ell_\infty(B), X)$ whose norm is controlled by some constant $D(\mathcal{M}_1)$. Moreover, since $\widetilde{E}$ is an average operator,

$$\widetilde{E}f \subset \overline{\mathrm{conv}\, f(\mathcal{M}_1)} \quad \text{(closure in } X).$$

Now for every $h \in H$ we define a linear operator $\pi_h : \mathrm{Lip}(\mathcal{M}_1 \oplus^{(\infty)} H, X) \to \mathrm{Lip}(\mathcal{M}_1, X)$ by setting $\pi_h f := f(\cdot, h)$, and then introduce the required operator E_2 given for $f \in \mathrm{Lip}(\mathcal{M}_1 \oplus^{(\infty)} H, X)$ by

$$(E_2 f)(m, h) := (\widetilde{E}\pi_h f)(m), \quad (m, h) \in \ell_\infty(B) \oplus^{(\infty)} H.$$

Using this definition we get

$$\begin{aligned}&||(E_2 f)(m_1, h_1) - (E_2 f)(m_2, h_2)||_X \\ &\leq ||\widetilde{E}(\pi_{h_1} f)(m_1) - \widetilde{E}(\pi_{h_1} f)(m_2)||_X + ||\widetilde{E}[(\pi_{h_1} - \pi_{h_2})f](m_2)||_X.\end{aligned}$$

The first summand is at most $||\widetilde{E}||L(f)d_1(m_1, m_2)$ while the second one is bounded by

$$\begin{aligned}&\sup\{||x||_X \ ; \ x \in \mathrm{conv}[f(\cdot, h_1) - f(\cdot, h_2)]\}\\&= \sup\{||\sum \alpha_i[f(m_i, h_1) - f(m_i, h_2)]||_X \ ; \ \alpha_i \geq 0, \ \sum \alpha_i = 1 \text{ and } \{m_i\} \subset \mathcal{M}_1\}\end{aligned}$$

which is clearly bounded by $L(f)d_H(h_1, h_2)$.

Together with the previous estimates this bounds the Lipschitz constant of $E_2 f$ in $\mathrm{Lip}(\ell_\infty(B) \oplus^{(\infty)} H, X)$ by that of f, as required. □

It seems to be highly plausible that Theorem 7.68 is true for more general hyperbolic metric spaces. In particular, one can ask the following question.

Problem. *Is it true that a complete simply connected length space $\mathcal{M}$ of non-positive curvature in the Alexandrov sense is universal?*

Finally we present three main results of the Lee and Naor paper [LN-2005] and briefly discuss the extension method of these authors. To their formulation we need to add the notion of a *graph minor*.

Let $G = (V, E)$ be a combinatorial graph. Its minor is a graph obtained from G by a finite sequence of the following two operations:

- Removal an edge.
- Contracting an edge to a (new) vertex which replaces the endpoints of the edge.

A finite graph G is said to be *n-incomplete*, if the complete graph K_n on n vertices differs from any minor of G.

Theorem 7.69. (a) *Let $\mathcal{M}$ be doubling with the doubling constant $\delta_{\mathcal{M}}$. Then its universal extension constant satisfies*

$$\lambda_u(\mathcal{M}) \leq c_1 \log_2 \delta_M.$$

(b) *Let $G = (V, E)$ be an n-incomplete finite graph equipped by a weight $w : E \to \mathbb{R}_+ \cup \{+\infty\}$. Then the universal extension constant of the associated metric space $(\mathcal{M}_G, d_w)$ satisfies*

$$\lambda_u((\mathcal{M}_G, d_w)) \leq c_2 n^2.$$

(c) *For the Euclidean space $\mathbb{R}^n$,*

$$\lambda_u(\mathbb{R}^n) \leq c_3 \sqrt{n}.$$

Here $c_1, c_2, c_3 > 0$ are numerical constants.

The first assertion of the theorem is equivalent to that of Theorem 7.62, cf. the proof of Corollary 7.37.

At the present time constructive proofs of the remaining assertions of Theorem 7.69 are unknown.

Using the limiting procedure of the proof of Theorem 7.12 we easily derive from part (b) of the theorem the following result:

Let $G = (V, E)$ be a (not necessarily finite) graph and $w : E \to \mathbb{R}_+ \cup \{+\infty\}$ be a weight. Assume that for some $n \in \mathbb{N}$ each finite subgraph of G is n-incomplete. Then

$$\lambda_u^*((\mathcal{M}_G, d_w)) \leq c_2 n^2.$$

Here λ_u^* is defined similarly to λ_u but with target spaces being dual Banach spaces.

This result implies an extension of the Matoušek theorem [Mat-1990] on the simultaneous Lipschitz extension property of metric trees to the class of planar graphs. Let us recall that a metric graph is *planar* if each of its finite subgraphs is homeomorphic to a subset of $\mathbb{R}^2$ (in particular, a metric tree is a planar graph).

To derive the above assertion we use the Kuratowski planarity theorem stating that every finite planar graph is 5-incomplete. This implies that any planar graph $G = (V, E)$ with a weight $w : E \to \mathbb{R}_+ \cup \{+\infty\}$ satisfies $\lambda_u^*((\mathcal{M}_G, d_w)) \leq 25c_2$.

Now we describe the extension method of these authors.

Let S be a closed subspace of a metric space $(\mathcal{M}, d)$. As in other proofs the main point for constructing an extension operator is existence of a Lipschitz partition of unity subordinate to a suitable cover of $\mathcal{M} \setminus S$. In the Lee-Naor proof, these objects are obtained by averaging a family of partitions of unity $\{P_i^\omega\}_{i \in I(\omega)}$ depending on stochastic variable ω. In turn, this family is constructed using padding stochastic decompositions, see, e.g., Lemma 4.2 and Remark 4.4 of Section 4.1 of Volume I where such an object is described. Existence of these decompositions is a deep fact established by Gupta, Krauthgamer and Lee [GKL-2003] for doubling metric spaces, and followed from the result of Klein, Plotkin and Rao [KPR-1993] for n-incomplete graphs.

Now the construction of the required extension operator is as follows.

Using a padded stochastic decomposition one defines, for a metric space $(\mathcal{M}, d)$ and some measure space (Ω, μ), a function $\Psi : \Omega \times \mathcal{M} \to [0, +\infty]$ with the following properties:

(i) For every $m \in \mathcal{M} \setminus S$ the function $\omega \mapsto \Psi(\omega, m)$ is μ-measurable and

$$\int_\Omega \Psi(\omega, m) d\mu(\omega) = 1;$$

(ii) For all $\omega \in \Omega$

$$\Psi(\omega, \cdot)|_S = 0;$$

(iii) There exists a function $\gamma : \Omega \to S$ such that for some constant $K > 0$ and all $m, m' \in \mathcal{M}$,

$$\int_\Omega d(\gamma(\omega), m)|\Psi(\omega, m) - \Psi(\omega, m')|d\mu(\omega) \leq Kd(m, m').$$

Clearly, Ψ may be seen as the aforementioned stochastic partition of unity. Using Ψ we define the required extension operator $E : \mathrm{Lip}(S, X) \to \mathrm{Lip}(\mathcal{M}, X)$ by setting

$$Ef(m) := \int_\Omega f(\gamma(\omega))\Psi(\omega, m)d\mu(\omega)$$

for $m \in \mathcal{M} \setminus S$ and

$$Ef(m) := f(m)$$

for $m \in S$.

An easy computation exploiting (i)-(iii) gives for the Lipschitz constant of Ef the inequality

$$L(Ef; X) \leq 3K.$$

It is worth also noting that this promising method is applied in [LN-2005] to several other interesting settings. In particular, it established an estimate of $\lambda_u(\mathcal{M})$ for an n-point metric space $\mathcal{M}$ which improves the estimate $O(\log n)$ given in the paper [JLS-1986]. It was also proved that $\lambda_u(S)$ where S is a subset of a Riemann surface of genus g is majorated by $O(g + 1)$.

Comments

Unlike the continuous extension case where linear operators play a role right from the beginning (see Chapter 1 of Volume I) the small number of results on Lipschitz simultaneous extensions arise only as by-products in the papers devoted to the nonlinear case. It was noted by Pelczyński [Pe-1968] that "… our knowledge of existence of linear extension operators for uniformly continuous or Lipschitz functions is rather unsatisfactory." However, only several isolated results have been obtained since then, the most important of which are those of the seminal papers by Marcus and Pisier [MP-1984], Johnson and Lindenstrauss [JL-1984], and Johnson, Lindenstrauss and Schechtman [JLS-1986].

The methods and results presented in the last two chapters were developed independently and at about the same time by Lang and Schlichenmaier, Lee and Naor, and the authors of this book, and presented in the papers [LSchl-2005], [LN-2005] and [BB-2007b], [BB-2007c], respectively.

The simultaneous extension results of the first two papers are consequences of more general theories devoted, respectively, to a Lipschitz generalization of the Hurewicz theory of continuous extensions and to a probabilistic approach to Computer Science combinatorial algorithms.

The third approach due to the authors of this book stems from the work of Yu. Brudnyi and Shvartsman on Whitney's linear extension problem for functions of Zygmund class announced in [BSh-1985] and published in detail in [BSh-1999]. To construct the corresponding extension operators these authors introduced generalized hyperbolic spaces $\mathbb{H}^n_\omega$ and associated spaces of balls and established the simultaneous Lipschitz extension property for them. The corresponding operator is given there as the composition of the average operator and the metric $(1+\varepsilon)$-projection (difficulties of this construction are discussed in Section 7.2 and the way to overcome them in Section 7.4).

The account of most of the results of Chapter 7 follows, with some improvements, that of aforementioned papers of A. and Yu. Brudnyis.

Our attempt to present in detail the proofs of the Lee-Naor results requires the understanding of a great deal of material that for now is mostly available only in the proceedings of Computer Science conferences in the form of extended abstracts. In fact, a detailed description of the methods and results of this area would probably form a volume comparable in size with this book.

Chapter 8

Linearity and Nonlinearity

The chapter contains several results comparing the linear and nonlinear Lipschitz extension properties and the corresponding Lipschitz constants.

In the first section, one studies the influence of general snowflake transforms of metric spaces on the properties in question. Although such a transform worsens considerably the geometry of a metric space, in both cases, linear and nonlinear, it preserves the Lipschitz extension property of the space.

In the second section, it is proved that the linear Lipschitz constant is the supremum of the corresponding nonlinear extension constants. In particular, unlike the nonlinear case, the linear Lipschitz extension constant is unbounded for every infinite-dimensional convex subset of a Banach space.

The final section contains three examples of metric spaces which do not possess the simultaneous Lipschitz extension property. The first two show that an analog of the McShane theorem is not true in the linear case even for low-dimensional proper metric spaces of bounded geometry (of dimensions one and two in these examples). The third example shows that uniform lattices (introduced in Section 7.3) formed by abelian groups with word metrics and an infinite number of generators may not possess the simultaneous Lipschitz extension property.

8.1 Snowflake stability of Lipschitz extension properties

We show that snowflake transforms preserve both the nonlinear and simultaneous Lipschitz extension properties and estimate the corresponding nonlinear and linear Lipschitz constants; the result is due to Yu. Brudnyi and Shvartsman [BSh-2002].

Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a concave nondecreasing function equals 0 at 0. The operation called a *snowflake transform* associates to a metric space $(\mathcal{M}, d)$ a new metric space $(\mathcal{M}, d_\omega)$ where $d_\omega := \omega \circ d$. This space is denoted by $\omega(\mathcal{M})$.

This operation takes its name from the familiar constructions of the snowflake curves in the plane. In particular, the space $(\mathbb{R}, |x-y|^s)$ with $0 < s < 1$ is

A. Brudnyi and Y. Brudnyi, *Methods of Geometric Analysis in Extension and Trace Problems: Volume 2*, Monographs in Mathematics 103, DOI 10.1007/978-3-0348-0212-3_8,

bi-Lipschitz homeomorphic to the classical von Koch snowflake curve, see, e.g., [Sem-1999, p. 406].

For a Banach space X we have

Theorem 8.1. (a) $\Lambda(\omega(\mathcal{M}), X) \leq C[\Lambda(\mathcal{M}, X)]^2$;

(b) $\lambda(\omega(\mathcal{M}), X) \leq C[\lambda(\mathcal{M}, X)]^2$.
Hereafter C stands for a numerical constant.

Proof. The crucial steps of the proof are some basic results of Interpolation Space Theory presented in Chapters 2 and 4 of Volume I (see Remark 2.78 (a) and Theorem 4.79 there). For convenience of the reader we formulate the corresponding results and definitions (for other facts see Volume I, Section 4.6.2).

Let $\vec{X} := (X_0, X_1)$ be a Banach couple and let $K(\cdot, x; \vec{X})$ be the K-functional of an element $x \in \Sigma(\vec{X}) := X_0 + X_1$.

Theorem 8.2 (On K-divisibility of $\vec{X}$, [BK-1991]). *Assume that ω is a sum of concave functions $\omega_i : \mathbb{R}_+ \to \mathbb{R}_+$, $i \in \mathbb{N}$,*

$$\omega = \sum_{i=1}^{\infty} \omega_i \quad (\textit{pointwise convergence}) \tag{8.1}$$

and for every $t > 0$ the inequality

$$K(t, x; \vec{X}) \leq \omega(t) \tag{8.2}$$

is true. Then there exists a decomposition

$$x = \sum_{i=1}^{\infty} x_i \qquad (\textit{convergence in } \Sigma(\vec{X}))$$

such that

$$K(t, x_i; \vec{X}) \leq \gamma \omega_i(t)$$

for all $t > 0$ and $i \in \mathbb{N}$. Here $\gamma > 0$ is a numerical constant.

For the formulation of a "linear" version of this result recall the following:

Definition 8.3. A Banach couple $\vec{X}$ is said to be K-linearizable if there exist a positive constant λ and two families of linear bounded operators $\{U_t : \Sigma(\vec{X}) \to X_0\ ;\ t > 0\}$ and $\{V_t : \Sigma(\vec{X}) \to X_1\ ;\ t > 0\}$ such that $U_t + V_t = Id_{\Sigma(\vec{X})}$ for all $t > 0$ and for every $x \in \Sigma(\vec{X})$ and $t > 0$,

$$||U_t x||_{X_0} + t||V_t x||_{X_1} \leq \lambda K(t, x; \vec{X}).$$

We set $\lambda(\vec{X}) := \inf \lambda$.

Theorem 8.4. *Suppose $\vec{X}$ is K-linearizable and condition* (8.1) *holds. Then there exists a family of linear operators $\{T_i : \Sigma(\vec{X}) \to \Sigma(\vec{X})\ ;\ i \in \mathbb{N}\}$ such that*

$$Id_{\Sigma(\vec{X})} = \sum_{i=1}^{\infty} T_i \quad (\textit{pointwise convergence})$$

and for every element $x \in \Sigma(\vec{X})$ satisfying (8.2) *and every $t > 0$, $i \in \mathbb{N}$,*

$$K(t, T_i x; \vec{X}) \le \gamma\lambda(\vec{X})\omega_i(t).$$

The proof of this result literally follows the proof of Theorem 3.2.7 in [BK-1991]. The only change is required in the definition of elements $x_0(t)$, $x_1(t)$ of that proof, see (3.2.30), p. 327 of [BK-1991]. Instead of that we now use operators U_t, V_t of Definition 8.3 to set $x_0(t) := U_t x$ and $x_1(t) := V_t x$.

We apply these results to the Banach couple $\vec{L} := (\ell_\infty(\mathcal{M}, X), \mathrm{Lip}(\mathcal{M}, X))$. Here $\ell_\infty(\mathcal{M}, X)$ is the space of bounded mappings $f : \mathcal{M} \to X$ equipped with the norm $||f||_{\ell_\infty(\mathcal{M},X)} := \sup\{||f(m)||\ ;\ m \in \mathcal{M}\}$. The K-functional of this couple is related to the *modulus of continuity* of a mapping $f : \mathcal{M} \to X$, i.e.,

$$\omega(t, f) := \sup\{||f(m) - f(m')||\ ;\ d(m, m') \le t\}.$$

The least concave majorant of ω will be denoted by $\widehat{\omega}$.

Proposition 8.5. *For every $f \in \Sigma(\vec{L})$ and $t > 0$*

$$\frac{1}{2}\widehat{\omega}(t, f) \le K(t, f; \vec{L}) \le 3\Lambda(\mathcal{M}, X)\,\widehat{\omega}(t, f). \tag{8.3}$$

Proof. Given $\varepsilon > 0$ and $t > 0$ choose a decomposition $f = f_0 + f_1$ such that

$$||f_0||_{\ell_\infty} + t||f_1||_{\mathrm{Lip}} \le K(t, f; \vec{L}) + \varepsilon.$$

Here and below we omit $\mathcal{M}$ and X from notation, writing, e.g., $||f||_{\ell_\infty}$ instead of $||f||_{\ell_\infty(\mathcal{M},X)}$. Then we have

$$\omega(t, f) \le \omega(t, f_0) + \omega(t, f_1) \le 2||f_0||_{\ell_\infty} + t||f_1||_{\mathrm{Lip}} \le 2K(t, f; \vec{L}) + 2\varepsilon.$$

Since the K-functional is concave and $\varepsilon > 0$ is arbitrary, this implies the left inequality in (8.3).

Fix now $t > 0$ and choose a t-net N_t in $\mathcal{M}$; hence

(i) $d(x, y) \ge t$ for distinct $x, y \in N_t$;

(ii) for every $x \in \mathcal{M}$ there is $\hat{x} \in N_t$ such that $d(x, \hat{x}) < t$.

Estimating the norm of $f|_{N_t}$ in $\mathrm{Lip}(N_t, X)$ we have for $x, y \in N_t$, $x \neq y$,

$$||f|_{N_t}(x) - f|_{N_t}(y)|| = ||f(x) - f(y)|| \leq \widehat{\omega}(d(x,y), f).$$

Since $\widehat{\omega}$ is concave, the function $\widehat{\omega}(t)/t$ is nonincreasing so that by (i),

$$||f|_{N_t}||_{\mathrm{Lip}(N_t,X)} \leq \sup\left\{\frac{\widehat{\omega}(d(x,y),f)}{d(x,y)}\ ;\ x \neq y,\, x, y \in N_t\right\} \leq \frac{\widehat{\omega}(t,f)}{t}.$$

Now let $\Lambda(\mathcal{M}, X) < \infty$. Given $\epsilon > 0$ there exists an extension $f_1 : \mathcal{M} \to X$ of $f|_{N_t}$ such that

$$||f_1||_{\mathrm{Lip}} \leq (\Lambda(\mathcal{M}, X) + \varepsilon)\frac{\widehat{\omega}(t,f)}{t}. \tag{8.4}$$

Set $f_0 := f - f_1$ and estimate $||f_0||_{\ell_\infty}$. To this end given $x \in \mathcal{M}$ we choose $\hat{x} \in N_t$ satisfying $d(x, \hat{x}) < t$. Since

$$||f_0(x)|| \leq ||f(x) - f(\hat{x})|| + ||f(\hat{x}) - f_1(x)||$$

and $f(\hat{x}) = f_1(\hat{x})$, one gets for the left-hand side the bound

$$\omega(d(x,\hat{x}), f) + ||f_1||_{\mathrm{Lip}}\, d(x, \hat{x}).$$

Together with (8.4) this implies

$$\begin{aligned} ||f_0||_{\ell_\infty} &\leq \widehat{\omega}(t,f) + (\Lambda(\mathcal{M}, X) + \varepsilon)\widehat{\omega}(t,f) \\ &\leq (2\Lambda(\mathcal{M}, X) + \varepsilon)\widehat{\omega}(t,f). \end{aligned} \tag{8.5}$$

From this inequality and (8.4) it follows that

$$\begin{aligned} K(t, f; \overrightarrow{L}) &\leq ||f_0||_{\ell_\infty} + t||f_1||_{\mathrm{Lip}} \\ &\leq (2\Lambda(\mathcal{M}, X) + \varepsilon)\widehat{\omega}(t,f) + (L_{\mathrm{ext}}(\mathcal{M}, X) + \varepsilon)\widehat{\omega}(t,f). \end{aligned} \qquad \square$$

In the "linear" case, the following is true.

Proposition 8.6. *If* $\lambda(\mathcal{M}, X) < \infty$*, then the couple* $\overrightarrow{L}$ *is* K*-linearizable and the optimal constant* $\lambda(\overrightarrow{L})$*, see Definition* 8.3*, is at most* $6\lambda(\mathcal{M}, X)$*.*

Proof. Let N_t be as in Proposition 8.5. Consider a linear extension operator E_t from $\mathrm{Lip}(N_t, X)$ into $\mathrm{Lip}(\mathcal{M}, X)$ with the norm $||E_t|| \leq \lambda(\mathcal{M}, X) + \varepsilon$. Now define the required operator V_t of Definition 8.3 by $V_t f := E_t(f|_{N_t})$. Similarly to (8.4), (8.5) we then obtain the inequalities

$$\begin{aligned} ||V_t f||_{\mathrm{Lip}} &\leq (\lambda(\mathcal{M}, X) + \varepsilon)\frac{\widehat{\omega}(t,f)}{t}, \\ ||f - V_t f||_{\ell_\infty} &\leq (2\lambda(\mathcal{M}, X) + \varepsilon)\widehat{\omega}(t,f). \end{aligned}$$

Using these and the left inequality in (8.3) we get

$$\begin{aligned}||f - V_t f||_{\ell_\infty} + t||V_t f||_{\text{Lip}} &\le (3\lambda(\mathcal{M}, X) + 2\varepsilon)\widehat{\omega}(t, f)\\ &\le 2(3\lambda(\mathcal{M}, X) + 2\varepsilon)K(t, f; \overrightarrow{L}).\end{aligned}$$

Hence $\lambda(\overrightarrow{L}) \le 6\lambda(\mathcal{M}, X)$. □

As a consequence of Theorem 8.2 and Proposition 8.5 one has

Corollary 8.7. *Let $\Lambda(\mathcal{M}, X) < \infty$ and ω, ω_i be as in Theorem 8.2. If $f \in \text{Lip}(\omega(\mathcal{M}), X)$, then there exists a sequence of maps $f_i \in \text{Lip}(\omega(\mathcal{M}), X)$, $i \in \mathbb{N}$, such that $f = \sum_{i=1}^\infty f_i$ (pointwise convergence) and, in addition,*

$$||f_i||_{\text{Lip}(\omega_i(\mathcal{M}),X)} \le C\Lambda(\mathcal{M}, X)||f||_{\text{Lip}(\omega(\mathcal{M}),X)}. \tag{8.6}$$

Proof. Since by (8.3),

$$K(t, f; \overrightarrow{L}) \le 3\Lambda(\mathcal{M}, X)\widehat{\omega}(t, f) \le 3\Lambda(\mathcal{M}, X)||f||_{\text{Lip}(\omega(\mathcal{M}),X)}\omega(t),$$

one can apply Theorem 8.2 to $\overrightarrow{X} := \overrightarrow{L}$ and ω replaced by the function $3\Lambda(\mathcal{M}, X)||f||_{\text{Lip}(\omega(\mathcal{M}),X)}\omega$. By this theorem there exists a decomposition $f = \sum_{i=1}^\infty f_i$ with convergence in $\Sigma(\overrightarrow{L})$ such that for every $t > 0$ and $i \in \mathbb{N}$,

$$K(t, f_i; \overrightarrow{L}) \le C\Lambda(\mathcal{M}, X)||f||_{\text{Lip}(\omega(\mathcal{M}),X)}\omega_i(t).$$

From here it easily follows that $\sum_{i=1}^\infty f_i$ pointwise converges to $f + c$ where c is a suitable constant. Thus we can redefine f to retain the equality $f = \sum_{i=1}^\infty f_i$ with the pointwise convergence. According to the left inequality in (8.3) this implies (8.6). □

Now applying Theorem 8.4 and Proposition 8.6 we obtain in the very same fashion the following

Corollary 8.8. *Let $\lambda(\mathcal{M}, X) < \infty$, and ω, ω_i be as in Theorem 8.2. Then there exists a family $\{T_i : \text{Lip}(\omega(\mathcal{M}), X) \to \text{Lip}(\omega_i(\mathcal{M}), X)\ ;\ i \in \mathbb{N}\}$ such that*

$$Id_{\text{Lip}(\omega(\mathcal{M}),X)} = \sum_{i=1}^\infty T_i \quad (\textit{pointwise convergence})$$

and, in addition, $\sup_i ||T_i|| \le C\lambda(\mathcal{M}, X)$.

After these preliminary results let us prove Theorem 8.1.

According to Lemma 3.2.8 in [BK-1991],

$$\omega(t) \approx \sum_{i=1}^\infty \min\{\lambda_i, \mu_i t\}, \quad t \in \mathbb{R}_+ \tag{8.7}$$

with suitable $\lambda_i, \mu_i > 0$ and numerical constants of equivalence. Thus it is natural to prove first the required result for "atoms" $\min\{\lambda_i, \mu_i t\}$. In turn, the latter follows from the corresponding result for $\alpha(t) := \min\{1, t\}$, $t \in \mathbb{R}_+$.

Lemma 8.9. (a) $\Lambda(\alpha(\mathcal{M}), X) \le 3\Lambda(\mathcal{M}, X)$;

(b) $\lambda(\alpha(\mathcal{M}), X) \le 3\lambda(\mathcal{M}, X)$.

Proof. Let $S \subset \mathcal{M}$ and let $f \in \mathrm{Lip}(\alpha(S), X)$ be such that $||f||_{\mathrm{Lip}(\alpha(S),X)} \le 1$. Letting $A := \{m \in \mathcal{M}\ ;\ d(m, S) \ge 1\}$ and then fixing $m_0 \in S$ we define a map $g : S \cup A \to X$ by setting $g(m) := f(m)$, $m \in S$, and $g(m) := f(m_0)$, $m \in A$.

Clearly, $||g||_{\mathrm{Lip}(S\cup A,X)} \le 1$ so that given $\varepsilon > 0$ we can extend g to a Lipschitz map $\tilde{g} : \mathcal{M} \to X$ with constant $K := \Lambda(\mathcal{M}, X) + \varepsilon$. In particular, $\tilde{g}$ extends f. If $m_1, m_2 \in \mathcal{M} \setminus A$, then we can find $m_1', m_2' \in S$ with $d(m_1, m_1') \le 1$ and $d(m_2, m_2') \le 1$. Hence

$$\begin{aligned} ||\tilde{g}(m_1) - \tilde{g}(m_2)|| &\le ||\tilde{g}(m_1) - \tilde{g}(m_1')|| + ||f(m_1') - f(m_2')|| + ||\tilde{g}(m_2') - \tilde{g}(m_2)|| \\ &\le Kd(m_1, m_1') + \alpha(d(m_1', m_2')) + Kd(m_2, m_2') \le 2K + 1. \end{aligned}$$

Similarly, if $m_1 \in A$ and $m_2 \in \mathcal{M} \setminus A$ find $m_2' \in S$ with $d(m_2, m_2') \le 1$. Hence

$$||\tilde{g}(m_1) - \tilde{g}(m_2)|| \le ||f(m_0) - f(m_2')|| + ||\tilde{g}(m_2') - \tilde{g}(m_2)|| \le 1 + K.$$

Thus for all $m_1, m_2 \in \mathcal{M}$,

$$\begin{aligned} ||\tilde{g}(m_1) - \tilde{g}(m_2)|| &\le \min\{Kd(m_1, m_2), 2K + 1\} \le (2K + 1)\alpha(d(m_1, m_2)) \\ &\le (3\Lambda(\mathcal{M}, X) + 2\varepsilon)\alpha(d(m_1, m_2)) \end{aligned}$$

and the statement (a) follows. The corresponding modification of the proof for the "linear" case is obvious. □

Let $S \subset \mathcal{M}$ and $f \in \mathrm{Lip}(\omega(S), X)$. We have to find an extension $\tilde{f} \in \mathrm{Lip}(\omega(\mathcal{M}), X)$ of f satisfying

$$\|\tilde{f}\|_{\mathrm{Lip}(\omega(\mathcal{M}),X)} \le C[\Lambda(\mathcal{M}, X)]^2 \|f\|_{\mathrm{Lip}(\omega(S),X)}. \tag{8.8}$$

In the second part of the theorem we also should define $\tilde{f}$ linearly depending on f. Using decomposition (8.7) and Corollary 8.7 with $\omega_i(t) := \min\{\lambda_i, \mu_i t\}$, $i \in \mathbb{N}$, we represent f as a sum $\sum_{i=1}^{\infty} f_i$ (pointwise convergence) with $f_i : S \to X$ satisfying

$$\|f_i\|_{\mathrm{Lip}(\omega_i(S),X)} \le C\Lambda(\mathcal{M}, X)\|f\|_{\mathrm{Lip}(\omega(S),X)}, \quad i \in \mathbb{N}.$$

Using this and Lemma 8.9 (a) we now obtain an extension $\tilde{f}_i : \mathcal{M} \to X$ satisfying

$$\|\tilde{f}_i\|_{\mathrm{Lip}(\omega_i(\mathcal{M}),X)} \le 3C[\Lambda(\mathcal{M}, X)]^2 \|f\|_{\mathrm{Lip}(\omega(S),X)}, \quad i \in \mathbb{N}. \tag{8.9}$$

Now, set $\tilde{f} := \sum_{i=1}^{\infty} \tilde{f}_i$ and show that $\tilde{f}$ is well defined and satisfies (8.8).

Given $m \in \mathcal{M}$ we choose a point $m_0 \in S$ and write

$$\left\| \sum_{i=M}^{N} \tilde{f}_i(m) \right\| \le \sum_{i=M}^{N} \|\tilde{f}_i(m) - \tilde{f}_i(m_0)\| + \left\| \sum_{i=M}^{N} \tilde{f}_i(m_0) \right\|.$$

Since $\tilde{f}_i(m_0) = f_i(m_0)$, this and (8.9) imply

$$\left\| \sum_{i=M}^{N} \tilde{f}_i(m) \right\| \leq 3C[\Lambda(\mathcal{M}, X)]^2 ||f||_{\mathrm{Lip}(\omega(S),X)} \sum_{i=M}^{N} \omega_i(d(m, m_0)) + \left\| \sum_{i=M}^{N} \tilde{f}_i(m_0) \right\| .$$

But $\sum_{i=1}^{\infty} f_i(m_0)$ and $\sum_{i=1}^{\infty} \omega_i(d(m, m_0))$ are convergent, therefore the right-hand side of this inequality tends to 0 as $M, N \to \infty$. Thus $\tilde{f}$ is well defined. It remains to note that by (8.9) and (8.7) we have for $m_1, m_2 \in \mathcal{M}$,

$$\begin{aligned} \|\tilde{f}(m_1) - \tilde{f}(m_2)\| &\leq \sum_{i=1}^{\infty} \|\tilde{f}_i(m_1) - \tilde{f}_i(m_2)\| \\ &\leq 3C[\Lambda(\mathcal{M}, X)]^2 \|f\|_{\mathrm{Lip}(\omega(S),X)} \left(\sum_{i=1}^{\infty} \omega_i(d(m_1, m_2)) \right) \\ &\leq C_1[\Lambda(\mathcal{M}, X)]^2 \|f\|_{\mathrm{Lip}(\omega(S),X)} \omega(d(m_1, m_2)) \end{aligned}$$

with a suitable numerical constant C_1.

Thus part (a) of Theorem 8.1 is proved.

As for part (b) one should repeat word-for-word the proof of part (a) replacing Corollary 8.7 by Corollary 8.8 and statement (a) of Lemma 8.9 by statement (b). If in this case E_i denotes the corresponding linear extension operator from $\mathrm{Lip}(\omega_i(S), X)$ into $\mathrm{Lip}(\omega(\mathcal{M}), X)$ for which $||E_i|| \leq 3\lambda(\mathcal{M}, X)$, then the required extension operator is defined by $E = \sum_{i=1}^{\infty} E_i T_i$ where T_i are linear operators of Corollary 8.8.

The proof of the theorem is complete. □

8.2 Relation between linear and nonlinear extension constants

According to the classical results presented in Chapter 1 there exist pairs of infinite-dimensional Banach spaces B, X such that the Lipschitz extension constant $\Lambda(B, X)$ is finite. In particular, the McShane Theorem 1.27 of Volume I gives

$$\Lambda(\mathcal{M}, \ell_\infty) = 1$$

even for an arbitrary metric space $\mathcal{M}$ and the Kirszbraun theorem yields the same result for the pair of Hilbert spaces (see also some recent results of this kind presented in Section 6.5).

Our goal is to show that for simultaneous Lipschitz extensions the corresponding constant is infinity for any infinite-dimensional Banach spaces B even in the case of scalar Lipschitz functions; this result was proved by the authors of the book, see [BB-2008]. It will be derived from a relation between simultaneous and nonlinear Lipschitz extension constants.

To formulate the result on the relation of linear and nonlinear extension constants, we let $\mathcal{B}_{fin}$ denote the category of all finite-dimensional Banach spaces. We also set

$$\Lambda_{fin}(\mathcal{M}) := \sup\{\Lambda(\mathcal{M}, X)\,; X \in \mathcal{B}_{fin}\}.$$

Theorem 8.10. *The following equality holds:*

$$\lambda(\mathcal{M}) = \Lambda_{fin}(\mathcal{M}).$$

Proof. In the proof of the theorem, we use results of Sections 4.6.3 (of Volume I) and 7.1.2. It was proved in the former:

Lemma 8.11. *Let* $E \in \mathrm{Ext}(S, \mathcal{M})$ *and* $\operatorname{card} S < \infty$. *There exists a linear bounded operator* $E_0 : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(S)$ *such that*

(1) E_0 *is predual to* E*; in particular,* $||E_0|| = ||E||$.

(2) E_0 *is a projector onto* $\mathcal{F}(S)$.

Here $\mathcal{F}(\mathcal{M})$ and $\mathcal{F}(S)$ are the free-Lipschitz spaces predual to $\mathrm{Lip}(\mathcal{M})$ and $\mathrm{Lip}(S)$ constructed in Section 4.6.3 of Volume I.

We first prove

Claim I.

$$\Lambda_{fin}(\mathcal{M}) \le \lambda(\mathcal{M}).$$

The proof is based on two inequalities, the first of which asserts that

$$\Lambda(F, \mathcal{M}, X) \le \lambda(F, \mathcal{M}) \tag{8.10}$$

provided that $\operatorname{card} F < \infty$.

To formulate the second one we define a "finite-dimensional" analog of the nonlinear extension constant $\Lambda(S, \mathcal{M}, X)$ by

$$\widehat{\Lambda}(S, \mathcal{M}, X) := \sup\{\Lambda(F, \mathcal{M}, X)\ ;\ F \subset S\ ,\ \operatorname{card} F < \infty\}. \tag{8.11}$$

Then, for $\dim X < \infty$, the following inequality holds:

$$\Lambda(S, \mathcal{M}, X) \le \widehat{\Lambda}(S, \mathcal{M}, X). \tag{8.12}$$

We will prove these inequalities later, but now derive from them Claim I.

Choose in (8.10) a Banach space X from $\mathcal{B}_{fin}$ and take the supremum over all $F \subset S$ with $\operatorname{card} F < \infty$. Using (8.11) we then get

$$\widehat{\Lambda}(S, \mathcal{M}, X) \le \sup\{\lambda(F, \mathcal{M})\ ;\ F \subset S\ ,\ \operatorname{card} F < \infty\}.$$

By the variant of the finiteness property given by (7.56) the right-hand side equals $\lambda(S, \mathcal{M})$. Together with (8.12) this leads to the inequality

$$\Lambda(S, \mathcal{M}, X) \le \lambda(S, \mathcal{M}).$$

Taking here the supremum over all $X \in \mathcal{B}_{fin}$ and $S \subset \mathcal{M}$, we prove Claim I.

Now we establish inequalities (8.10) and (8.12). In their proofs, $\mathcal{M}$ is a pointed metric space with a distinguished point m_0 and all subsets of $\mathcal{M}$ are assumed to be pointed subspaces of $\mathcal{M}$ (with the same distinguished point).

To prove (8.10), we pick $F \subset \mathcal{M}$ with $\operatorname{card} F < \infty$. For $f \in \mathrm{Lip}_0(F, X)$, let T_f be the linear operator of Theorem 4.91 of Volume I. Hence, $T_f : \mathcal{F}(F) \to X$ and

$$||T_f|| = L(f) := ||f||_{\mathrm{Lip}(F,X)}.$$

Given an extension operator $E \in \mathrm{Ext}(F, \mathcal{M})$, we take the predual operator $E_0 : \mathcal{F}(\mathcal{M}) \to \mathcal{F}(F)$ of Lemma 8.11 and set

$$E_f := T_f E_0.$$

Then $E_f : \mathcal{F}(\mathcal{M}) \to X$ and its norm is at most $||E_0|| \cdot ||T_f|| = L(f)||E||$. By Theorem 4.91 of Volume I, $E_f|_{\mathcal{M}} : \mathcal{M} \to X$ is a Lipschitz extension of f from F to the whole of $\mathcal{M}$ and, moreover,

$$L(E_f|_{\mathcal{M}}) \leq ||E_f|| \leq L(f)||E||.$$

According to the definition of the nonlinear Lipschitz extension constant this implies

$$\Lambda(F, \mathcal{M}, X) \leq ||E||.$$

Taking here the infimum over all $E \in \mathrm{Ext}(F, \mathcal{M})$ we get (8.10).

To prove (8.12), given $\epsilon > 0$, we choose a function $f \in \mathrm{Lip}_0(S, X)$ with $L(f) = 1$ admitting an *ϵ-optimal* extension to the whole of $\mathcal{M}$; this means that there exists $\widetilde{f} \in \mathrm{Lip}_0(\mathcal{M}, X)$ such that

$$-2\epsilon + \Lambda(S, \mathcal{M}, X) \leq -\epsilon + L(\widetilde{f}) \leq \inf\{L(g)\ ;\ g \in \mathrm{Lip}_0(\mathcal{M}, X)\ ,\ g|_S = f\}. \quad (8.13)$$

Now let $\mathcal{F}$ be the family of all finite subsets of S partially ordered by inclusion. For every $F \in \mathcal{F}$, define g_F to be an ϵ-optimal extension of the trace $f|_F$ to $\mathcal{M}$, that is,

$$-\epsilon + L(g_F) \leq \inf\{L(g)\ ;\ g \in \mathrm{Lip}_0(\mathcal{M}, X)\ ,\ g|_F = f|_F\}. \quad (8.14)$$

By the definition of $\widetilde{f}$ we get for all $F \in \mathcal{F}$,

$$L(g_F) \leq L(\widetilde{f}) + \epsilon.$$

Hence, the family $\{g_F\}_{F \in \mathcal{F}}$ is equicontinuous. It is also uniformly bounded on every ball in $\mathcal{M}$, since $g_F(m_0) = f(m_0) = 0$ for all F. Finally, every g_F maps $\mathcal{M}$ into the finite-dimensional Banach space X. Therefore a standard compactness argument using the Tychonov theorem, see, e.g., the proof of Proposition 7.16 allows us to find a subnet of the *net* $\{g_F\}_{F \in \mathcal{F}}$ which pointwise converges to a Lipschitz function $g : \mathcal{M} \to X$. By the definition of the net convergence

$$g|_S = f. \quad (8.15)$$

Further, passing to the limit in (8.14) and using definition (8.11) we also have

$$\begin{aligned}-\epsilon + L(g) &\le -\epsilon + \sup_{F\in\mathcal{F}} L(g_F)\\ &\le \sup_{F\in\mathcal{F}} \inf\{L(g)\ ;\ g\in \mathrm{Lip}_0(\mathcal{M},X)\ ,\ g|_F = f|_F\} \le \widehat{\Lambda}(S,\mathcal{M},X).\end{aligned}$$

Since $\widetilde{f}$ is an ϵ-optimal extension of f, see (8.13), we derive from (8.15) that

$$L(\widetilde{f}) \le L(g) + \epsilon.$$

Together with the previous inequality and (8.13) this yields

$$\Lambda(S,\mathcal{M},X) \le L(\widetilde{f}) + \epsilon \le \widehat{\Lambda}(S,\mathcal{M},X) + 3\epsilon.$$

This completes the proofs of (8.12) and Claim I.

Claim II.

$$\lambda(\mathcal{M}) \le \Lambda_{fin}(\mathcal{M}).$$

We derive this fact from the inequality

$$\lambda(F,\mathcal{M}) \le \Lambda(F,\mathcal{M}) := \sup\{\Lambda(F,\mathcal{M},X)\ ;\ X\in\mathcal{B}_{fin}\} \tag{8.16}$$

valid for $\operatorname{card} F < \infty$.

Taking here the supremum over all such F and applying the finiteness result Theorem 7.12 with $S = \mathcal{M}$ we get

$$\lambda(\mathcal{M}) = \sup_F \lambda(F,\mathcal{M}) \le \sup_F \Lambda(F,\mathcal{M}) \le \Lambda_{fin}(\mathcal{M}),$$

as required.

It remains to establish (8.16).

Let $\delta_F : F \to \mathcal{F}(F)$ be the isometric embedding of Theorem 4.89 of Volume I. Then $\delta_F \in \mathrm{Lip}(F,\mathcal{F}(F))$ and by the definition of $\Lambda(F,\mathcal{M})$, given $\epsilon > 0$, there exists an extension $\widetilde{\delta}_F \in \mathrm{Lip}(\mathcal{M},\mathcal{F}(F))$ of δ_F such that

$$-\epsilon + L(\widetilde{\delta}_F) \le \Lambda(F,\mathcal{M})L(\delta_F)\ (= \Lambda(F,\mathcal{M})).$$

Now let $f \in \mathrm{Lip}_0(F)$. Since $\mathrm{Lip}_0(F) = \mathcal{F}(F)^*$, the function f can be regarded as a linear bounded functional on $\mathcal{F}(F)$. Define a linear operator $E : \mathrm{Lip}_0(F) \to \mathrm{Lip}_0(\mathcal{M})$ by

$$(Ef)(m) := \langle f, \widetilde{\delta}_F(m)\rangle_{\mathcal{F}(F)}.$$

Then the norm $||Ef||_{\mathrm{Lip}_0(\mathcal{M})}$ is bounded by

$$||f||_{\mathrm{Lip}_0(F)} \sup_{m\ne m'} \frac{||\widetilde{\delta}_F(m) - \widetilde{\delta}_F(m')||_{\mathcal{F}(F)}}{d(m,m')} = L(f)L(\widetilde{\delta}_F) \le (\Lambda(F,\mathcal{M}) + \epsilon)L(f).$$

Hence, $\|E\| \le \Lambda(F,\mathcal{M}) + \epsilon$ and, moreover, $(Ef)|_F = f$, i.e., $E \in \operatorname{Ext}(F,\mathcal{M})$. By the definition of $\lambda(F,\mathcal{M})$ this implies inequality (8.16):

$$\lambda(F,\mathcal{M}) \le \Lambda(F,\mathcal{M}).$$

Finally, from the inequalities of Claims I and II we get the required statement of the theorem. □

We will apply the result of Theorem 8.10 to obtain some bounds for the simultaneous Lipschitz constants of the spaces $\lambda(\ell_p^n)$.

Corollary 8.12. *For some positive numerical constants c_1, c_2 and all $p \in [1,\infty]$ and $n \in \mathbb{N}$,*

$$c_1 n^{\ell(p)} \le \lambda(\ell_p^n) \le c_2 n^{\frac{1}{2}+\left|\frac{1}{2}-\frac{1}{p}\right|}$$

where

$$\ell(p) := \begin{cases} \frac{1}{p} - \frac{1}{2}, & \text{if } 1 \le p \le \frac{8}{5}, \\ \frac{1}{8}, & \text{if } \frac{8}{5} \le p \le \frac{8}{3}, \\ \frac{1}{2} - \frac{1}{p}, & \text{if } \frac{8}{3} \le p \le \infty. \end{cases}$$

Proof. We will first prove these inequalities for $p = \frac{1}{2}$.

The following fact was established in the proof of Lemma 1.13 of the paper [LN-2005], see Remark 5.3 there.

Lemma 8.13. *Let $2 < p < \infty$ and $m \in \mathbb{N}$, $m \ge \max\{6, \lfloor 4^{\frac{p^2}{p-2}} \rfloor + 1\}$. There exists a finite subset $F \subset \ell_2^{2m}$ of $\operatorname{card} F \le m^{2m}$ such that*

$$\Lambda(F, \ell_2^{2m}, \ell_p^{2m}) \ge \frac{1}{6}\left(m^{\frac{p-2}{p^2}} - 4\right).$$

Proof. Set

$$\varepsilon := \frac{1}{m^{1/2-1/p}}$$

and let N be an ε-net in the unit ball of ℓ_2^{2m}, denoted B. By standard volume estimates

$$|N| \le \left(\frac{1+\varepsilon/2}{\varepsilon/2}\right)^{2m} \le (2m^{1/2}+1)^{2m} \le m^{2m}$$

for $m \ge 6$. Consider a map $f : \ell_2^{2m} \to \ell_p^{2m}$ given in coordinates by

$$f(x)_i = |x_i|^{2/p} \cdot \operatorname{sign}(x_i), \quad 1 \le i \le 2m.$$

From the numerical inequality

$$|a^{2/p} \cdot \operatorname{sign}(a) - b^{2/p} \cdot \operatorname{sign}(b)| \le 2^{1-2/p} \cdot |a-b|^{2/p}$$

it follows that for every $x, y \in \ell_2^{2m}$,

$$||f(x) - f(y)||_p \le 2||x-y||_2^{2/p}.$$

Since the elements of N are ε separated, the restriction of f to N is then Lipschitz with constant $L(f|_N) \leq \frac{2}{\varepsilon^{1-2/p}}$.

Assume that it is possible to extend $f|_N$ to a function $g : \ell_2^{2m} \to \ell_p^{2m}$ which is K-Lipschitz. Since N is ε-dense in B, for every $x \in B$ and $x' \in N$ such that $\|x - x'\| \leq \varepsilon$,

$$||f(x) - g(x)||_p \leq ||f(x) - f(x')||_p + ||g(x) - g(x')||_p \leq 2\varepsilon^{2/p} + K\varepsilon.$$

Let Σ_{2m} be the finite group of linear transformations of ℓ_2^{2m} generated by all permutations and sign changes of coordinates of its points. We introduce a function $h : \ell_2^{2m} \to \ell_p^{2m}$ given by

$$h(x) := \frac{1}{|\Sigma_{2m}|} \left(\sum_{\sigma \in \Sigma_m} \sigma^{-1}[g(\sigma(x))] \right).$$

Let $\mathbf{1}_A \in \ell_2^{2m}$ be a point whose ith coordinate is 1 if $i \in A$ and 0 otherwise. Then h satisfies $h(a\mathbf{1}_A) = b\mathbf{1}_A$ for all scalars a and $A \subset \{1, 2, \ldots, 2m\}$, where b depends only on the a and the cardinality of A.

Additionally, h is K-Lipschitz, and since f is equivariant under permutations and sign changes of coordinates (i.e., $f(\sigma(x)) = \sigma(f(x))$ for $\sigma \in \Sigma_{2m}$), we get for every $x \in B$,

$$||h(x) - f(x)||_p \leq 2\varepsilon^{2/p} + K\varepsilon.$$

Setting $x_k := \frac{1}{\sqrt{2m}} \mathbf{1}_{\{k,\ldots,k+m-1\}}$ we derive from the properties of h that

$$||h(x_{m+1}) - h(x_1)||_p^p = \sum_{k=1}^{m} ||h(x_{k+1}) - h(x_k)||_p^p,$$

and since h is K-Lipschitz we get the estimate

$$||h(x_{m+1}) - h(x_1)||_p \leq \frac{K}{m^{1/2-1/p}} = K\varepsilon.$$

On the other hand,

$$\begin{aligned} 2K\varepsilon + 4\varepsilon^{2/p} &\geq ||h(x_{m+1}) - f(x_{m+1})||_p + ||h(x_1) - f(x_1)||_p \\ &\geq ||f(x_{m+1}) - f(x_1)||_p - ||h(x_{m+1}) - h(x_1)||_p \geq 1 - K\varepsilon. \end{aligned}$$

This implies that the ratio between K and the Lipschitz constant $L(f|_N)$ satisfies

$$\frac{K}{L(f|_N)} \geq \frac{(1 - 4\varepsilon^{2/p}) \cdot \varepsilon^{1-2/p}}{6\varepsilon} \geq \frac{1}{6} \left(m^{\frac{p-2}{p^2}} - 4 \right).$$

The proof of the lemma is complete. $\square$

Choosing in Lemma 8.13, $p = 4$ and $m \geq 4^8 + 1$ we get

$$\Lambda(F, \ell_2^{2m}, \ell_4^{2m}) \geq \frac{m^{1/8} - 4}{6}.$$

By the definition

$$\sup\{\Lambda(\ell_2^{2m}, X)\ ;\ X \in \mathcal{B}_{fin}\} \geq \Lambda(\ell_2^{2m}, \ell_4^{2m}) \geq \sup \Lambda(\Gamma, \ell_2^{2m}, \ell_4^{2m})$$

where Γ runs over all finite point metric subspaces of ℓ_2^{2m}. This and the previous inequality imply that for each $m \in \mathbb{N}$,

$$\sup\{\Lambda(\ell_2^{2m}, X)\ ;\ X \in \mathcal{B}_{fin}\} \geq cm^{1/8}$$

for a numerical constant $c > 0$.

Finally, by Theorem 8.10 the left-hand side equals $\lambda(\ell_2^{2m})$ and this yields the required lower bound

$$\lambda(\ell_2^n) \geq c_1 n^{1/8}, \quad n \in \mathbb{N},$$

with a numerical constant $c_1 > 0$.

The upper bound

$$\lambda(\ell_2^n) \leq c_2 n^{1/2}, \quad n \in \mathbb{N},$$

follows from [LN-2005, Thm. 1.11]. These authors construct a linear extension operator described in Section 7.4 to prove that

$$\Lambda(\ell_2^n, X) \leq cn^{1/2}, \quad n \in \mathbb{N},$$

for a Banach space X.

Now let $p \neq 2$. We consider, for instance, $1 \leq p \leq \frac{8}{5}$ leaving the other two cases to be derived in the same fashion by the reader.

To obtain the required upper bound for $\lambda(\ell_p^n)$ we choose a linear operator $E \in \mathrm{Ext}(S, \ell_2^n)$ and apply the Hölder and Jensen inequalities to write for $f \in \mathrm{Lip}(S, \|\cdot\|_2)$,

$$\begin{aligned} \|Ef\|_{\mathrm{Lip}(\ell_p^n)} &\leq n^{1/p-1/2}\|Ef\|_{\mathrm{Lip}(\ell_2^n)} \leq n^{1/p-1/2}\|E\|_2 \cdot \|f\|_{\mathrm{Lip}(S,\|\cdot\|_2)} \\ &\leq n^{1/p-1/2}\|E\|_2 \cdot \|f\|_{\mathrm{Lip}(S,\|\cdot\|_p)}. \end{aligned}$$

Here $\|E\|_q$ stands for the norm of E regarded as an operator from $\mathrm{Lip}(S, \|\cdot\|_q)$ into $\mathrm{Lip}(\ell_q^n)$.

This and the definition of the simultaneous extension constant imply

$$\lambda(\ell_p^n) := \inf\{\|E\|_p\ ;\ S \subset \mathbb{R}^n\} \leq n^{1/p-1/2} \inf\{\|E\|_2\ ;\ S \subset \mathbb{R}^n\} =: n^{1/p-1/2}\lambda(\ell_2^n).$$

It remains to apply the inequality $\lambda(\ell_2^n) \leq c_2 n^{1/2}$, see Section 7.4, to obtain the required upper estimate of $\lambda(\ell_p^n)$.

The same argument gives first for $E \in \mathrm{Ext}(S, \ell_p^n)$,

$$\|Ef\|_{\mathrm{Lip}(\ell_p^n)} \geq n^{1/p-1} \|Ef\|_{\mathrm{Lip}(\ell_1^n)}.$$

Choosing here f from $\mathrm{Lip}(S, \|\cdot\|_1)$ with norm 1 and then taking the supremum over all these f we have

$$\begin{aligned}\|E\|_p = \|E\|_p \cdot \|f\|_{\mathrm{Lip}(S,\|\cdot\|_1)} &\geq \|E\|_p \cdot \|f\|_{\mathrm{Lip}(S,\|\cdot\|_p)} \geq \|Ef\|_{\mathrm{Lip}(\ell_p^n)} \\ &\geq n^{1/p-1} \sup_f \|Ef\|_{\mathrm{Lip}(\ell_1^n)} =: n^{1/p-1}\|E\|_1 \geq n^{1/p-1}\lambda(\ell_1^n).\end{aligned}$$

Using Corollary 7.57 to estimate the right-hand side and then taking the infimum over E we obtain that

$$\lambda(\ell_p^n) := \inf\{\|E\|_p \ ; \ S \subset \mathbb{R}^n\} \geq n^{1/p-1} \cdot c_1 n^{1/2} = c_1 n^{1/p-1/2}$$

with $c_1 > \frac{1}{4}$.

This completes the proof of Corollary 8.12 for this case. □

Problem 8.14. *Find asymptotics for $\lambda(\ell_p^n)$ as $n \to \infty$.*

Corollary 8.15. *Let $\mathcal{M}$ be an n-dimensional Riemannian manifold equipped with the geodesic metric. Then*

$$\lambda(\mathcal{M}) \geq cn^{1/8}$$

for a numerical constant $c > 0$.

Proof. By the definition of $(\mathcal{M}, d)$ for each point $m \in \mathcal{M}$ there is a neighborhood U of m such that (U, d) is 2-isometric to an open ball in ℓ_2^n. Thus by Corollary 8.12,

$$\lambda(\mathcal{M}) \geq \lambda((U, d)) = \lambda(\ell_2^n) \geq cn^{1/8}$$

for a numerical constant $c > 0$. □

Now we present a construction of a metric space and a Lipschitz map from one of its subspaces to $\mathbb{R}^n$, which will be used below to estimate the simultaneous Lipschitz extension constants for finite metric spaces. The construction is due to Lang [L-1999] who used it to prove the inequality $\Lambda(\mathbb{R}^n) > \sqrt[4]{n}$.

Theorem 8.16. *There exist a metric space $\mathcal{M}$ of cardinality 2^{n+1}, its subspace S and a 1-Lipschitz map $f : S \to \mathbb{R}^n$ such that for any Lipschitz extension $\tilde{f} : \mathcal{M} \to \mathbb{R}^n$ of f,*

$$L(\tilde{f}) \geq \sqrt[4]{n}.$$

Proof. The derivation is based on the properties of a certain finite graph $G = (V_G, E_G)$ whose vertex set is the union of the sets

$$I_G := \{-1, 1\}^n \quad \text{and} \quad T_G := \{-2, 2\}^n.$$

Hence, I_G consists of all vertices of the closed ℓ_∞^n-ball (cube) $\overline{B}_1(0)$ while T_G consists of all those of $\overline{B}_2(0)$.

Further, we define the edge set E_G as follows. Two vertices $x, y \in V_G := I_G \cup T_G$ determine an edge denoted by $[x, y]$ if either $||x - y|| = 2$ or $y = 2x$; here $\| \cdot \|$ stands for the canonical norm of $\mathbb{R}^n$.

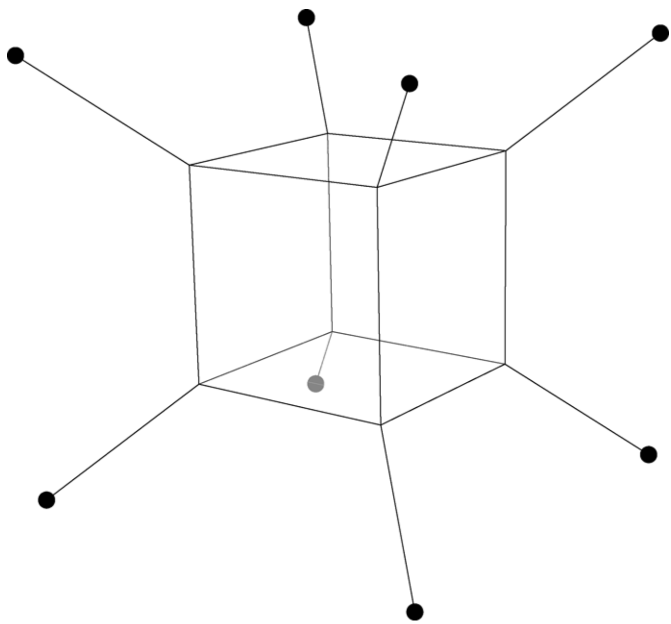

Figure 8.1: Graph $G = (V_G, E_G)$.

Hence, an edge joins either adjacent vertices of the cube $\overline{B}_1(0)$ or a vertex of $\overline{B}_1(0)$ with the closest vertex of $\overline{B}_2(0)$. Due to this definition, each vertex from T_G belongs to exactly one edge. Therefore vertices of T_G are called *terminals*.

The required result is immediately derived from the following two lemmas.

Lemma 8.17. *For every path* $\{x^0, \ldots, x^{l+2}\}$ *in* G *(i.e.,* $x_i \in V_G$ *and* $[x_i, x_{i+1}] \in E_G$ *for all* i*) with* $x^0, x^{l+2} \in T_G$ *and* $x^0 \neq x^{l+2}$,

$$\sum_{i=0}^{l+1} ||x^i - x^{i+1}|| \geq \sqrt[4]{n} \cdot ||x^0 - x^{l+2}||. \tag{8.17}$$

Proof. It suffices to consider a path $\{x^0, \dots, x^{l+2}\}$ with $x^i \in I_G$ for $1 \le i \le l+1$. Then

$$\sum_{i=0}^{l+1} ||x^i - x^{i+1}|| = ||x^0 - x^1|| + ||x^{l+1} - x^{l+2}|| + \sum_{i=1}^{l} ||x^i - x^{i+1}|| = 2\sqrt{n} + 2l.$$

Moreover, $||x^0 - x^{l+2}|| \le 4\sqrt{l}$ and

$$\frac{\sqrt{n}+l}{2\sqrt{l}} \ge \sqrt{\sqrt{\frac{n}{l}} \cdot \sqrt{l}} = \sqrt[4]{n}.$$

The result is established. □

The harder part of the proof exploits a property of the natural embedding $\mathcal{E} : V_G \to \mathbb{R}^n$ presented by

Lemma 8.18. *Let a map $F : V_G \to \mathbb{R}^n$ satisfy the conditions*

(a) *$F(x) = x$ for all $x \in T_G$;*

(b) *$||F(x) - F(y)|| \le ||x - y||$ for all $[x, y] \in E_G$.*

Then $F = \mathcal{E}$.

Proof. Let $\mathcal{F}$ be the class of maps F satisfying (a) and (b). Define a map $L : \mathcal{F} \to \mathbb{R}$ by

$$L(F) := \sum_{[x,y]\in E_G} ||F(x) - F(y)||.$$

By condition (b), for all $F \in \mathcal{F}$,

$$L(F) \le L(\mathcal{E}). \tag{8.18}$$

Let F_0 be an element of $\mathcal{F}$ minimizing $L(F)$:

$$L(F_0) = \min_{F\in\mathcal{F}} L(F).$$

We will show that $F_0 = \mathcal{E}$ but first obtain from here the required result. In fact, we then have $L(\mathcal{E}) \ge L(F) \ge L(F_0) = L(\mathcal{E})$ for every $F \in \mathcal{F}$ and therefore $||F(x) - F(y)|| = ||x - y||$ for all $[x, y] \in E_G$. Together with (a) this implies $F = \mathcal{E}$, i.e., $\mathcal{F}$ contains the only element $\mathcal{E}$.

To prove (8.18) we first show that $F_0(x)$ for every $x \in I_G$ belongs to the line $\mathbb{R}x$. Fix $x^0 \in I_G$ and pick an isometry $\rho : \mathbb{R}^n \to \mathbb{R}^n$ (rotation about $\mathbb{R}x^0$) such that $\rho(V_G) = V_G$. Then $\rho(\lambda x^0) = \lambda x^0$ for all $\lambda \in \mathbb{R}$ and $\rho(y) \ne y$ for $y \ne \mathbb{R}x^0$. It can be readily derived from here that along with F_0 both maps $\widetilde{F}_0 := (\rho \circ F_0 \circ \rho^{-1})|_V$ and $\widehat{F}_0 := \frac{1}{2}[F_0 + \widetilde{F}_0]$ are elements of $\mathcal{F}$.

Now suppose that

$$F_0(x^0) \notin \mathbb{R}x^0.$$

Then, by the choice of ρ, $(\rho \circ F_0 \circ \rho^{-1})(x^0) = (\rho \circ F_0)(x^0) \neq F_0(x^0)$. Since, moreover,

$$||(\rho \circ F_0 \circ \rho^{-1})(x^0) - 2x^0|| = ||F_0(x^0) - 2x^0||,$$

it follows that

$$||\widehat{F}(x^0) - 2x^0|| < ||F_0(x^0) - 2x^0||.$$

This and condition (a), in turn, imply that $L(\widehat{F_0}) < L(F_0)$ in contradiction with the choice of F_0.

Hence, $F_0(x^0) \in \mathbb{R}x$ for all $x \in I_G$. Moreover, $F_0(x) \in [x, 3x]$ for these x, since

$$||F_0(x) - 2x|| \leq ||x - 2x|| = ||x||.$$

Suppose now that

$$F_0(x^0) \neq x^0 \text{ for some } x^0 \in I_G.$$

Then $F_0(x^0) = \lambda x^0$ for some $1 < \lambda \leq 3$. This would imply for every $y \in I_G$ with $[x^0, y] \in E_G$ that

$$||F_0(x^0) - F_0(y)|| > ||x^0 - y||.$$

But $F_0 \in \mathcal{F}$ and cannot satisfy this inequality, see (b).

Hence, $F_0(x) = x$ for all $x \in I_G$ and this also holds for $x \in T_G$ by (a). In other words, $F_0 = \mathcal{E}$, as required. □

In order to include the extension constant $\Lambda(\mathcal{M}, \mathbb{R}^n)$, see (6.47), in the proof we construct a metric space $(\mathcal{M}, d)$ with the following two properties.

(i) $\mathbb{R}^n$ sits in $\mathcal{M}$ and the identity map from $\mathbb{R}^n$ as a subset of $\mathcal{M}$ into $(\mathbb{R}^n, ||\cdot||)$ is 1-Lipschitz.

To formulate the second property, we first note that by Proposition 1.48 of Volume I for every $\epsilon > 0$ there exists a retraction R of $\mathcal{M}$ onto $\mathbb{R}^n$ such that

$$L(R) \leq (1 + \epsilon)\Lambda(\mathcal{M}, \mathbb{R}^n). \tag{8.19}$$

Further, identifying V_G with the subset $\mathcal{E}(V_G)$ we denote by d^* the inner metric on $\mathcal{E}(V_G)$ induced by $||\cdot||$. In other words, for $v, w \in V_G$

$$d^*(v, w) := \inf \sum_i ||\mathcal{E}(v^i) - \mathcal{E}(v^{i+1})||,$$

where the infimum is taken over all paths $\{v^i\}$ in G joining v and w.

Then we set

$$\lambda := \min \left\{ \frac{d^*(v, w)}{||\mathcal{E}(v) - \mathcal{E}(w)||} \ ; \ v, w \in T_G, \ v \neq w \right\}. \tag{8.20}$$

(ii) There exists a map $C : V_G \to \mathcal{M}$ such that

$$(R \circ C)(v) = \mathcal{E}(v) \text{ for all } v \in T_G$$

and, moreover, for all $[v, w] \in E_G$,

$$d(C(v), C(w)) = \lambda^{-1} ||\mathcal{E}(v) - \mathcal{E}(w)||. \tag{8.21}$$

If such $\mathcal{M}$ is constructed, then the derivation of the desired result is as follows. The map $R \circ C : V_G \to \mathbb{R}^n$ satisfies

$$(R \circ C)(v) = v \text{ for } v \in T_G.$$

If $R \circ C$ belongs to the class $\mathcal{F}$ of Lemma 8.18 then $R \circ C = \mathcal{E}$ and therefore

$$||(R \circ C)(v) - (R \circ C)(w)|| = ||\mathcal{E}(v) - \mathcal{E}(w)||$$

for all $[v, w] \in E_G$.

Otherwise, for some $[v, w] \in E_G$,

$$||(R \circ C)(v) - (R \circ C)(w)|| \geq ||\mathcal{E}(v) - \mathcal{E}(w)||.$$

In both cases, for some $[v, w] \in E_G$ we get

$$||\mathcal{E}(v) - \mathcal{E}(w)|| \leq L(R) \cdot d(C(v), C(w)).$$

Combining with (8.19) and (8.21) we obtain

$$||\mathcal{E}(v) - \mathcal{E}(w)|| \leq (1 + \epsilon)\Lambda(\mathcal{M}, \mathbb{R}^n)\lambda^{-1} ||\mathcal{E}(v) - \mathcal{E}(w)||,$$

i.e., $\lambda \leq (1 + \epsilon)\Lambda(\mathbb{R}^n)$.

By the definition of λ, see (8.21), this implies for some path $\{v^i\}_{i=0}^{l+2}$ in G joining two distinct points v and w from T_G,

$$\sum_{i=0}^{l+1} ||\mathcal{E}(v^i) - \mathcal{E}(v^{i+1})|| \leq (1 + \epsilon)\Lambda(\mathcal{M}, \mathbb{R}^n) \cdot ||\mathcal{E}(v) - \mathcal{E}(w)||.$$

Applying now (8.17) and sending $\epsilon \to 0$ we obtain the desired inequality $\Lambda(\mathcal{M}, \mathbb{R}^n) \geq \sqrt[4]{n}$.

It remains to construct the metric space $(\mathcal{M}, d)$. Let first d' be a maximal pseudonorm on $V_G \sqcup \mathbb{R}^n$ satisfying

$$d'(x, y) \leq ||x - y|| \text{ for } x, y \in \mathbb{R}^n,$$
$$d'(v, w) \leq \lambda^{-1} d^*(\mathcal{E}(v), \mathcal{E}(w)) \text{ for } v, w \in V_G \text{ and}$$
$$d'(\mathcal{E}(v), v) = 0 \text{ for } v \in T_G.$$

Since $||\mathcal{E}(v) - \mathcal{E}(w)|| \leq \lambda^{-1} d^*(\mathcal{E}(v), \mathcal{E}(w))$ for $v, w \in T_G$ by (8.21), we obtain that d' satisfies

$$d'(x, y) = ||x - y|| \text{ for } x, y \in \mathbb{R}^n.$$

Now we define the desired metric space $(\mathcal{M}, d)$ by identification points in $V_G \sqcup \mathbb{R}^n$ with d'-distance zero. Further, C is defined to be the canonical inclusion of V_G into $\mathcal{M}$.

We leave it to the reader to check that $(\mathcal{M}, d)$ satisfies the above formulated conditions (i), (ii).

The estimate $\Lambda(\mathcal{M}, \mathbb{R}^n) \geq \sqrt[4]{n}$ implies the existence of a subspace $\widetilde{S} \subset \mathcal{M}$ and a 1-Lipschitz map $\tilde{f} : \widetilde{S} \to \mathbb{R}^n$ which does not admit an extension to $\mathcal{M}$ with the Lipschitz constant smaller than $\sqrt[4]{n}$.

Since $\mathcal{M}$ is a factorization of the disjoint union $V_G \bigsqcup \mathbb{R}^n$, the subspace $\widetilde{S}$ and the map $\tilde{f}$ give rise to a subspace $S \subset (V_G, d)$ and a 1-Lipschitz map $f : S \to \mathbb{R}^n$ which does not admit an extension to (V_G, d) with the Lipschitz constant smaller than $\sqrt[4]{n}$. Since $\operatorname{card} V_G = 2^{n+1}$, the required result is proved. □

Using this result and Theorem 8.10 we obtain an estimate for the constant λ_n defined as follows.

Let $\mathbb{F}_n$ be the family of finite metric spaces with the number of points $\leq n$. Set

$$\lambda_n := \sup_{\mathcal{M} \in \mathbb{F}_n} \lambda(\mathcal{M}). \tag{8.22}$$

Corollary 8.19. *There are numerical constants $c_1, c_2 > 0$ such that*

$$c_1 \sqrt[4]{\log_2(n+1)} \leq \lambda_n \leq c_2 \log_2(n+1).$$

Proof. Assume that $(\mathcal{M}, d) \in \mathbb{F}_n$. Then $\mathcal{M}$ is of homogeneous type with respect to a measure μ defined by

$$\mu(S) := \#S, \quad S \subset \mathcal{M}.$$

In particular, the doubling constant $D(\mu)$ of μ is $\leq n$. Thus from Corollary 7.37 we obtain that

$$\lambda(\mathcal{M}) \leq c_2 \log_2(n+1)$$

for a numerical constant $c_2 > 0$.

Further, according to Theorem 8.16, for each $n \in \mathbb{N}$, there exist a metric space $\mathcal{M}$ consisting of 2^{d+1} points where $d := \lfloor \log_2(n+1) \rfloor$, a subspace $S \subset \mathcal{M}$ and a Lipschitz map $f : S \to \mathbb{R}^d$ such that $L(f) = 1$ and every Lipschitz extension $\bar{f}$ of f to $\mathcal{M}$ has Lipschitz constant $L(\bar{f}) \geq \sqrt[4]{d}$. By Theorem 8.10 we obtain from here

$$\lambda(\mathcal{M}) = \Lambda_{fin}(\mathcal{M}) \geq \sqrt[4]{d}.$$

This implies that

$$\lambda_n \geq c_1 \sqrt[4]{\log_2(n+1)}$$

for a numerical constant $c_1 > 0$. □

To formulate the next corollary of Theorem 8.10 we first introduce the following definition.

A convex subset K of a Banach space X containing $0 \in X$ is said to be *generating* if the union of all its dilations λK, $\lambda \geq 1$, is dense in X.

Corollary 8.20. $\lambda(K) < \infty$ *for a generating set* $K \subset X$ *if and only if* $\dim X < \infty$.

In particular, $\lambda(X) = \infty$ if $\dim X = \infty$. This naturally leads to the following

Proof. We use Corollary 7.19 with the dilation $\delta(x) := 2x$, $x \in X$, and S being the generating set K of the corollary. Check that the conditions of this corollary hold for this setting. Since K is convex and contains 0,

$$K \subset \delta(K) := 2K.$$

By the same reason $\cup_{i=1}^{\infty} \delta^i(K) = \cup_{\lambda \geq 1} \lambda K$ and the latter is dense in X by the definition of K. Finally, the operator $\Delta : \mathrm{Lip}(X) \to \mathrm{Lip}(X)$ is now defined by $(\Delta f)(x) := f(2x)$, $x \in X$, and therefore $||\Delta|| = 2$ and $||\Delta^{-1}|| = 1/2$. Hence, Corollary 7.19 yields

$$\lambda(K) = \lambda(X). \tag{8.23}$$

Let, in particular, U_X be the closed unit ball of the Banach space X. This, clearly, is a generating set and therefore (8.23) implies

$$\lambda(U_X) = \lambda(X). \tag{8.24}$$

Applying this to the Banach space ℓ_2^n and using the lower bound of Corollary 8.12 we get

$$\lambda(U_{\ell_2^n}) \geq c_1 n^{1/8}, \quad n \in \mathbb{N}. \tag{8.25}$$

Now let X be an infinite-dimensional Banach space. By the Dvoretzki theorem [Dv-1961] for every sufficiently large n, say $n \geq n_0$, there is an n-dimensional linear subspace $X_n \subset X$ such that the ball $U_X \cap X_n$ is 2-isometric to $U_{\ell_2^n}$. In other words, there is a bi-Lipschitz map ϕ of $U_{\ell_2^n}$ onto $U_X \cap X_n$ so that $\max\{L(\phi), L(\phi^{-1})\} \leq 2$. This immediately implies the inequality

$$\lambda(U_X) \geq \lambda(U_X \cap X_n) \geq \frac{1}{4}\lambda(U_{\ell_2^n}).$$

Together with (8.25) and (8.24) this yields for $n \geq n_0$,

$$\lambda(X) = \lambda(U_X) \geq \frac{1}{4} c_1 n^{1/8} \to \infty \text{ as } n \to \infty.$$

Hence,

$$\lambda(X) = \infty, \text{ if } \dim X = \infty\ .$$

In the case $n := \dim X < \infty$, the Whitney-Glaeser Theorem 2.19 of Volume I implies that $\lambda(X) < \infty$.

The proof of the corollary is complete. □

8.3 Metric spaces without simultaneous Lipschitz extension property

We first present (in subsections 8.3.1 and 8.3.2) two examples showing that the linear analog of the McShane extension result (see Volume I, Theorem 1.27) is not true even for proper metric spaces of low dimension (one and two in these examples). The first one is a direct construction based on the properties of $\mathbb{Z}^n$ regarded as a subspace of ℓ_1^n, while the second is a kind of a locally Lipschitz Peano curve placed in an infinite-dimensional Banach space.

The third example concerns Conjecture 7.53 on the simultaneous Lipschitz extension properties of uniform lattices. It explains why the conjecture is not true for lattices generated by groups with an infinite number of generators equipped with the word metric.

It is worth remarking that since the metric spaces of these examples do not belong to $\mathcal{LE}$, they are of infinite Nagata dimension.

8.3.1 Two dimensional space of bounded geometry

In view of Corollary 7.50 it would be natural to conjecture that $\lambda(\mathcal{M})$ is finite for every $\mathcal{M}$ of bounded geometry. The following counterexample disproves this claim.

Theorem 8.21. *There exists a connected two-dimensional metric space $\mathcal{M}_0$ of bounded geometry such that*

$$\operatorname{Ext}(S, \mathcal{M}_0) = \emptyset$$

for some subset $S \subset \mathcal{M}_0$.

Proof. First we will construct a metric graph that does not belong to $\mathcal{LE}$.

Let us recall (see Volume I, subsection 3.3.6 for details) that a metric graph $(\mathcal{M}_\Gamma, d_w)$ is generated by a weighted combinatorial graph $\Gamma = (\mathcal{V}, \mathcal{E}, w)$ where $\mathcal{V}$ and $\mathcal{E}$ are, respectively, the vertex and edge sets of Γ and $w : \mathcal{E} \to [0, +\infty]$ is a weight. Every edge e is identified with the segment of the real line of length $w(e)$, and the one-dimensional CW-complex obtained in this way is the underlying set $\mathcal{M}_\Gamma$ of the metric graph whose (length) pseudometric d_w is defined by formula (3.137) of Volume I. In particular, the vertex set $\mathcal{V}$ is a metric subspace of $(\mathcal{M}_\Gamma, d_w)$.

Below we will deal with a weighted graph whose weight $w = 1$ (hence, each of its edges is isometric to $[0, 1]$) and such that $\sup_{v \in \mathcal{V}} \deg v < \infty$. Here $\deg v$, the degree of $v \in \mathcal{V}$, is the number of edges incident to v. According to Proposition 3.126 of Volume I the corresponding metric graph is a complete geodesic space of *bounded geometry.*

Proposition 8.22. *There exists a weighted graph $\Gamma = (\mathcal{V}, \mathcal{E}, w)$ with $w = 1$ and a subspace $S \subset \mathcal{V}$ such that*

(a) *the degree* $\deg v$ *of every vertex* $v \in \mathcal{V}$ *is at most* 3*;*

(b) *it is true that*

$$\operatorname{Ext}(S, \mathcal{V}) = \emptyset. \tag{8.26}$$

Remark 8.23. Since $\lambda(S, \mathcal{M}_\Gamma) \geq \lambda(S, \mathcal{V})$, (8.26) implies that $\operatorname{Ext}(S, \mathcal{M}_\Gamma) = \emptyset$.

Proof. Our argument is based on the following result.

Let $\mathbb{Z}_1^n$ denote $\mathbb{Z}^n$ regarded as a metric subspace of ℓ_1^n. We reformulate Corollary 7.58 as

Lemma 8.24. *There is a numerical constant* $c_0 > 0$ *such that for every* n*,*

$$\lambda(\mathbb{Z}_1^n) \geq c_0\sqrt{n}. \tag{8.27}$$

Now let $\mathbb{Z}_1^n(l)$ denote the discrete cube of side length $l \in \mathbb{N}$, i.e.,

$$\mathbb{Z}_1^n(l) := \mathbb{Z}_1^n \cap [-l, l]^n.$$

Lemma 8.25. *For every* $n \in \mathbb{N}$ *there are an integer* $l = l(n)$ *and a subset* $S_n \subset \mathbb{Z}_1^n(l)$ *such that*

$$\lambda(S_n, \mathbb{Z}_1^n(l)) \geq c_1\sqrt{n} \tag{8.28}$$

with $c_1 > 0$ *independent of* n*.*

Proof. By Corollary 7.13,

$$\lambda(\mathbb{Z}_1^n) = \sup_F \lambda(F)$$

where F runs through all finite subsets $F \subset \mathbb{Z}^n$. On the other hand

$$\lambda(\mathbb{Z}_1^n) \geq \sup_{l \in \mathbb{N}} \lambda(\mathbb{Z}_1^n(l)).$$

Finally, Corollary 7.13 gives

$$\lambda(\mathbb{Z}_1^n(l)) = \sup_{F \subset \mathbb{Z}_1^n(l)} \lambda(F). \tag{8.29}$$

These three relations imply that $\lambda(\mathbb{Z}_1^n) = \sup_{l \in \mathbb{N}} \lambda(\mathbb{Z}_1^n(l))$. Together with (8.27) this yields for some $l = l(n)$,

$$\lambda(\mathbb{Z}_1^n(l(n))) > \frac{c_0}{2}\sqrt{n}.$$

Applying now (8.29) with $l := l(n)$ we then find $S_n \subset \mathbb{Z}_1^n(l(n))$ such that for $l = l(n)$,

$$\lambda(S_n, \mathbb{Z}_1^n(l)) := \inf\{||E|| \,;\; E \in \operatorname{Ext}(S_n, \mathbb{Z}_1^n(l))\} \geq \frac{c_0}{2}\sqrt{n}.$$

The result has been established. □

Now let $G_n := (\mathbb{Z}^n, \mathcal{E}^n)$ be the graph whose edge set is given by

$$\mathcal{E}^n := \{(x, y) \; ; \; x, y \in \mathbb{Z}^n, \; ||x - y||_{\ell_1^n} = 1\}.$$

Let $\Gamma_n := (V_n, E_n)$ be a subgraph of G_n whose vertex set is

$$V_n := \mathbb{Z}^n \cap [-l(n), l(n)]^n$$

where $l(n)$ is defined in Lemma 8.25. In particular, the set S_n of this lemma is contained in V_n. The metric graph $\mathcal{M}_{\Gamma_n}$ is then a (metric) subspace of the space ℓ_1^n, but it also can and will be regarded below as a subspace of ℓ_2^n with the path metric induced by the Euclidean metric.

Lemma 8.26. *There exists a finite connected graph $\widehat{\Gamma}_n := (\widehat{V}_n, \widehat{E}_n)$ and a subset $\widehat{S}_n \subset \widehat{V}_n$ such that*

(a) *for every vertex $v \in \widehat{V}_n$,*

$$\deg v \leq 3; \tag{8.30}$$

(b) *the underlying set $\mathcal{M}_{\widehat{\Gamma}_n}$ of the associated metric graph is a subset of the n-dimensional Euclidean space and its metric coincides with the restriction to $\mathcal{M}_{\widehat{\Gamma}_n}$ of the Euclidean metric;*

(c) *there is a numerical constant $c > 0$ such that for every n,*

$$\lambda(\widehat{S}_n, \widehat{V}_n) \geq c\sqrt{n}; \tag{8.31}$$

here $\widehat{S}_n$ and $\widehat{V}_n$ are regarded as subspaces of $\mathcal{M}_{\widehat{\Gamma}_n}$.

Proof. Let $\epsilon := \frac{1}{q\sqrt{2}}$ for some natural $q \geq 2$. For a vertex $v \in V_n \subset \ell_2^n$ of the graph Γ_n, let $S(v)$ stand for the $(n-1)$-dimensional Euclidean sphere centered at v and of radius $\frac{\epsilon}{1+2\epsilon}$. Then $S(v)$ intersects $N(v)$ edges of $\mathcal{M}_{\Gamma_n}$ ($n \leq N(v) \leq 2n$) at some points denoted by $p_i(v)$, $i = 0, \dots, N(v) - 1$. The ordering of these points is chosen in such a way that any interval $\operatorname{conv}\{p_i(v), p_{i+1}(v)\}$ does not belong to $\mathcal{M}_{\Gamma_n}$ (here and below $p_{N(v)}(v)$ is identified with $p_0(v)$). Let us introduce a new graph with the set of vertices $\{p_i(v) \; ; \; i = 0, \dots, N(v) - 1, \; v \in V_n\}$ and the set of edges defined as follows.

The set contains the edges determined by all pairs $(p_i(v), p_{i+1}(v))$ with $0 \leq i \leq N(v) - 1$ and $v \in V_n$ and, moreover, all edges formed by all pairs $(p_i(v'), p_j(v''))$ where v', v'' are, respectively, the head and the tail of an edge $e \in E_n$, and $i \neq j$ satisfy the condition

$$\operatorname{conv}\{p_i(v'), p_j(v'')\} \subset e \subset \mathcal{M}_{\Gamma_n}. \tag{8.32}$$

In this way we obtain a new graph (and the associated metric space with the length metric induced by the Euclidean one) whose vertices have degree at most 3.

The lengths of edges $(p_i(v), p_{i+1}(v))$ of this metric graph equal $\frac{1}{q(1+2\epsilon)}$ while the lengths of edges $(p_i(v'), p_i(v''))$ satisfying (8.32) equal $\frac{1}{1+2\epsilon}$.

Then we add new vertices (and edges) by inserting into every edge satisfying (8.32) $(q-1)$ equally distributed new vertices. (Note that every new vertex obtained in this way has degree 2.) Finally, by dilation (with respect to $0 \in \ell_2^n$) with factor $q(1+2\epsilon)$ we obtain a new graph $\widehat{\Gamma}_n := (\widehat{E}_n, \widehat{V}_n)$ whose edges are of length 1, and such that

$$\deg v \leq 3 \text{ for all } v \in \widehat{V}_n, \text{ and there exists } v_n \in \widehat{V}_n \text{ with } \deg v_n = 2.$$

Moreover, the metric graph $\mathcal{M}_{\widehat{\Gamma}_n}$ is a (metric) subspace of ℓ_2^n.

It remains to define the required subset $\widehat{S}_n \subset \widehat{V}_n$. To this end we use a map $i : V_n \to \widehat{V}_n$ given for $v \in V_n$ by

$$i(v) := q(1+2\epsilon) \cdot p_0(v).$$

Recall that $p_0(v)$ is a point of the sphere $S(v) \subset \ell_2^n$. Since our construction depends on ϵ continuously, and Γ_n is a finite graph, we clearly have for a sufficiently small ϵ and all $v', v'' \in V_n$,

$$(q/2)(1+2\epsilon) \cdot d(v', v'') \leq \widehat{d}(i(v'), i(v'')) \leq 2q(1+2\epsilon) \cdot d(v', v''). \tag{8.33}$$

Here $d, \widehat{d}$ stand for the metrics of $\mathcal{M}_{\Gamma_n}$ and $\mathcal{M}_{\widehat{\Gamma}_n}$, respectively.

Now note that the constant $\lambda(S_n, V_n)$ does not change if we replace the metric d by $q(1+2\epsilon) \cdot d$. Therefore (8.33) and (8.28) imply the estimate

$$\lambda(i(S_n), i(V_n)) \geq \frac{1}{4}\lambda(S_n, V_n) \geq \frac{c_1}{4}\sqrt{n}.$$

Finally, we set $\widehat{S}_n := i(S_n) \cup \{v_n\}$ where $v_n \in \widehat{V}_n$ satisfies $\deg v_n = 2$ and use the next two facts:

(a) $i(V_n) \subset \widehat{V}_n$;

(b) there exists a linear operator L extending Lipschitz functions from $i(S_n)$ to the single point v_n with norm $\|L\| \leq 2$.

In fact, $\deg v_n = 2$, i.e., v_n is the middle point of the interval with endpoints in $i(\widehat{S}_n)$. Therefore, L may be taken to be the linear polynomial interpolating at the endpoints.

Hence, we have

$$\lambda(\widehat{S}_n, \widehat{V}_n) \geq \frac{1}{\|L\|}\lambda(i(S_n), i(V_n)) \geq \frac{c_1}{8}\sqrt{n}.$$

This proves (8.31) and the lemma. □

Now we identify ℓ_2^n with its isometric copy P_n, an n-dimensional plane of the Hilbert space $\ell_2(\mathbb{N})$ orthogonal to the line $\{x \in \ell_2(\mathbb{N})\ ;\ x_i = 0 \text{ for } i > 1\}$ and intersecting this line at the point $(n, 0, \dots)$. Then $\mathcal{M}_{\widehat{\Gamma}_n}$ becomes a subspace of $P_n \subset \ell_2(\mathbb{N})$ equipped with the trace metric of that from $\ell_2(\mathbb{N})$. Using an appropriate translation we also may and will assume that the point v_n with $\deg v_n = 2$ from $\widehat{S}_n$, see Lemma 8.26, coincides with $(n, 0, \dots)$.

Note that $\operatorname{dist}(P_n, P_{n+1}) = \|v_n - v_{n+1}\| = 1$ and therefore the sets $\mathcal{M}_{\widehat{\Gamma}_n}$ are pairwise disjoint.

Further, we will introduce now the sets of vertices and edges of the required graph $\Gamma = (\mathcal{V}, \mathcal{E})$ by setting

$$\mathcal{V} := \bigcup_{n\in\mathbb{N}} \widehat{V}_n \quad \text{and} \quad \mathcal{E} := \bigcup_{n\in\mathbb{N}} (\widehat{E}_n \cup e_n)$$

where e_n denotes the new edge joining v_n with v_{n+1}.

This definition and Lemma 8.26 imply that for $v \in \mathcal{V}$

$$\deg v \le 3,$$

and so assertion (a) of Proposition 8.22 holds.

Set now

$$S := \bigcup_{n\in\mathbb{N}} \widehat{S}_n.$$

We claim that this S satisfies (8.26) and assertion (b) of the proposition. If, on the contrary, there is an operator $E \in \operatorname{Ext}(S, \mathcal{V})$, we choose n so that

$$c\sqrt{n} > ||E|| \tag{8.34}$$

with the constant c from (8.31). For this n we introduce an operator T_n given on $f \in \operatorname{Lip}(\widehat{S}_n)$ by

$$(T_n f)(v) := \begin{cases} f(v), & \text{if } v \in \widehat{S}_n, \\ f(v_n), & \text{if } v \in S \setminus \widehat{S}_n. \end{cases}$$

We show that T_n maps $\operatorname{Lip}(\widehat{S}_n)$ into $\operatorname{Lip}(S)$ and its norm is 1. To accomplish this we must show that for $v' \in \widehat{S}_n$ and $v'' \in S \setminus \widehat{S}_n$,

$$|(T_n f)(v') - (T_n f)(v'')| \le \|f\|_{\operatorname{Lip}(\widehat{S}_n)} d(v', v'').$$

But the left-hand side here equals $|f(v') - f(v_n)| \le \|f\|_{\operatorname{Lip}(\widehat{S}_n)} d(v', v_n)$ and also $d(v', v_n) \le d(v', v'')$ by the definition of S and the metric d of $\mathcal{M}_\Gamma$. This clearly implies the required statement for T_n.

Finally, introduce the restriction operator $R_n : \operatorname{Lip}(\mathcal{V}) \to \operatorname{Lip}(\widehat{V}_n)$ by

$$R_n f = f|_{\widehat{V}_n}$$

and set $E_n := R_n E T_n$. Then $E_n \in \mathrm{Ext}(\widehat{S}_n, \widehat{V}_n)$ and its norm is bounded by $\|E\|$. This immediately implies that

$$\lambda(\widehat{S}_n, \widehat{V}_n) \le \|E\|$$

in contradiction with (8.31) and our choice of n, see (8.34).

So we establish (8.26) and complete the proof of the proposition. □

In order to complete the proof of Theorem 8.21 we use Proposition 8.22 to construct a connected two-dimensional metric space $\mathcal{M}$ of bounded geometry so that

$$\mathrm{Ext}(S, \mathcal{M}) = \emptyset$$

for some subspace S. (In fact, $\mathcal{M}$ will be a Riemannian manifold with the geodesic (inner) metric.) This will be done by sewing surfaces of three types along the metric graph $\mathcal{M}_\Gamma$ of the proposition.

To begin with, we introduce an open cover of $\mathcal{M}_\Gamma$ by balls and the related coordinate system and partition of unity. To simplify evaluation we replace the metric of $\mathcal{M}_\Gamma$ by $\widetilde{d}_\Gamma := 4d_\Gamma$. So every edge $e \subset \mathcal{M}_\Gamma$ is a closed interval of length 4. (Note that the abstract graph $\Gamma = (\mathcal{V}, \mathcal{E})$ remains unchanged.) The required cover $\{B(v)\}_{v\in\mathcal{V}}$ is given by

$$B(v) := \{m \in \mathcal{M}_\Gamma \ ; \ \widetilde{d}(m, v) < 3\}. \tag{8.35}$$

So $B(v)$ is the union of at most three intervals of length 3 each of which has the form $e \cap B(v)$ where every e belongs to the set of edges $\mathcal{E}(v)$ incident to v. We enumerate these intervals by numbers from the set $\omega \subset \{1, 2, 3\}$ where $\omega = \{1\}$, $\{1, 2\}$ or $\{1, 2, 3\}$, if $\deg v = 1$, 2 or 3, respectively. In particular, $e \cap B(v)$ obtains number 1 if $\deg v = 1$ and 1 or 2 if $\deg v = 2$. The set of indices will be denoted by $\omega(v)$, while $i(e, v)$ (briefly, $i(e)$) will stand for the number of $e \cap B(v)$ in this enumeration.

We then introduce a *coordinate system* $\{\psi_v : B(v) \to \mathbb{R}^3 \,;\, v \in \mathcal{V}\}$ for $\mathcal{M}_\Gamma$ as follows. Let $\{b_1, b_2, b_3\}$ be the standard basis in $\mathbb{R}^3 (:= \ell_2^3)$. We define ψ_v as the isometry sending v to 0 and each interval $e \cap B(v)$, $e \in \mathcal{E}(v)$, to the interval $\{tb_i \ ; \ 0 \le t < 3\}$ of the x_i-axis with $i := i(e)$.

Now we introduce the desired *partition* of unity $\{\rho_v\}_{v\in\mathcal{V}}$ subordinate to the cover $\{B(v)\}_{v\in\mathcal{V}}$. To this end one first considers a function $\widetilde{\rho}_v : \psi_v(B(v)) \to [0, 1]$ with support strictly inside its domain such that $\widetilde{\rho}_v = 1$ in the neighborhood $\cup_{e\in\mathcal{E}(v)}\{tb_{i(e)} \ ; \ 0 \le t \le 1\}$ of 0 and is C^∞-smooth outside 0. This function gives rise to the function $\widehat{\rho}_v : \mathcal{M}_\Gamma \to [0, 1]$ equal to $\widetilde{\rho}_v \circ \psi_v$ on $B(v)$ and 0 outside. It is important to note that there exist only three types of the functions $\widetilde{\rho}_v$ corresponding to the types of the balls $B(v)$. Finally we define the required partition of unity by setting

$$\rho_v := \widehat{\rho}_v / \sum_v \widehat{\rho}_v, \quad v \in \mathcal{V}. \tag{8.36}$$

Secondly, we introduce the building blocks of our construction, C^∞-smooth surfaces $\Sigma_{\{1\}}$, $\Sigma_{\{1,2\}}$ and $\Sigma_{\{1,2,3\}}$ embedded in $\mathbb{R}^3$. We begin with a C^∞-function $f : [-1, 3) \to [0, 1]$ given by

$$f(t) := \begin{cases} \sqrt{1-t^2}, & \text{if } -1 \le t \le 1-\epsilon := \frac{3}{4}, \\ \sqrt{1-10\epsilon^2}, & \text{if } 1 \le t < 3. \end{cases}$$

In the remaining interval $[1-\epsilon, 1](:= [3/4, 1])$ f is an arbitrary decreasing function smoothly joining the given endpoint values. Then we introduce $\Sigma_{\{1\}}$ as the surface of revolution

$$\Sigma_{\{1\}} := \{(t, f(t)\cos\theta, f(t)\sin\theta) \in \mathbb{R}^3 \ ; \ -1 \le t < 3 \ , \ 0 \le \theta < 2\pi\}, \tag{8.37}$$

the result of rotating the graph of f about the x_1-axis. By the definition of f, this surface is the union of the unit sphere $S^2 \subset \mathbb{R}^3$ with the spherical hole $S(b_1)$ centered at b_1 and of the curvilinear (near the bottom) cylinder $T(b_1)$ attached to the circle $\partial S(b_1)$ (of radius $\sqrt{1-(1-\epsilon)^2}$). In turn, $T(b_1)$ is the union of a curvilinear cylinder and that of a circular one. The latter, denoted by $\widehat{T}(b_1)$, is of height 2.

Similarly $\Sigma_{\{1,2\}}$ and $\Sigma_{\{1,2,3\}}$ are the unions of the unit sphere S^2 with the holes $S(b_i)$ and of the cylinders $T(b_i)$ attached to $\partial S(b_i)$; here $i = 1, 2$ or $i = 1, 2, 3$, respectively. Note that each $T(b_i)$ with $i \ne 1$ is obtained from $T(b_1)$ by a fixed rotation around the x_j-axis, $j \ne 1, i$. This determines the isometry

$$J_i : \widehat{T}(b_i) \to \widehat{T}(b_1), \quad i = 1, 2, 3, \tag{8.38}$$

where J_1 stands for the identity map.

Using these blocks and the previous notation for $B(v)$ we now assign to every $v \in \mathcal{V}$ the smooth surface

$$\Sigma(v) := \Sigma_{\omega(v)} \subset \mathbb{R}^3. \tag{8.39}$$

We denote by $S(v) \subset \Sigma_{\omega(v)}$ the corresponding sphere S^2 with holes $\{S(b_i)\}$, $i \in \omega(v)$, and by $T(e)$, $e \in \mathcal{E}(v)$, the corresponding curvilinear cylinder $(= T(b_{i(e)}))$. The circular part of the latter is denoted by $\widehat{T}(e)$ and the corresponding isometry of $\widehat{T}(e)$ onto $\widehat{T}(b_1)$ is denoted by J_e $(= J_{i(e)})$. Since e belongs to two different sets, say, $\mathcal{E}(v)$ and $\mathcal{E}(v')$, we will also write $T(e, v)$, $\widehat{T}(e, v)$ and $J_{e,v}$ to distinguish them from the corresponding objects determined by e as an element of $\mathcal{E}(v')$. Finally, we equip $\Sigma(v)$ with the Riemannian metric induced by the canonical Riemannian structure of $\mathbb{R}^3$, and denote the corresponding geodesic metric by d_v.

According to our construction for every $\Sigma(v)$ there exists a continuous surjection $p_v : \Sigma(v) \to \psi_v(B(v))$ such that the restriction of p_v to every cylinder $\widehat{T}(e, v)$, $e \in \mathcal{E}(v)$, is the orthogonal projection onto its axis $I_e := \{tb_{i(e,v)} \ ; \ 1 < t < 3\}$. Using this and the polar coordinate θ from (8.37) we then equip each $x \in \widehat{T}(e, v)$ with the cylindrical coordinates:

$$r(x) := \psi_v^{-1}(p_v(x)), \quad \theta(x) := \theta(J_{e,v}(x)).$$

Now we define the required smooth surface $\mathcal{M}$ as the quotient of the disjoint union $\sqcup_{v\in\mathcal{V}}\Sigma(v)$ by the equivalence relation:

$$x \sim y \text{ if } x \in \widehat{T}(e,v_0),\ y \in \widehat{T}(e,v_1) \text{ for some } e \in \mathcal{E}(v_0)\cap\mathcal{E}(v_1) \text{ and } (r(x),\theta(x)) = (r(y),\theta(y)).$$

Let $\pi : \sqcup_{v\in\mathcal{V}}\Sigma(v) \to \mathcal{M}$ be the quotient projection. Then $\{\pi(\Sigma(v))\}_{v\in\mathcal{V}}$ is an open cover of $\mathcal{M}$. Using the partition of unity (8.36) we now introduce a partition of unity subordinate to this cover as follows.

Define the function $\widehat{\phi}_v : \Sigma(v) \to [0,1]$ as the pullback of the function $\rho_v : B(v) \to [0,1]$ given by

$$\widehat{\phi}_v := \rho_v(\psi_v^{-1}(p_v(x))), \quad x \in \Sigma(v). \tag{8.40}$$

By the definitions of all functions used here, the function $\widehat{\phi}_v$ is C^∞-smooth on every $\widehat{T}(e,v)$ and is equal to 1 outside $\cup_{e\in\mathcal{E}(v)}\widehat{T}(e,v)$. In particular, $\widehat{\phi}_v$ is C^∞-smooth and its support is strictly inside $\Sigma(v)$. Since $\pi|_{\Sigma(v)}$ is a smooth embedding, the function $\phi_v : \mathcal{M} \to [0,1]$ defined as $\widehat{\phi}_v \circ \pi$ on $\pi(\Sigma(v))$ and 0 outside is C^∞-smooth. By (8.40) the family $\{\phi_v\}_{v\in\mathcal{V}}$ forms the required partition of unity subordinate to the cover $\{\pi(\Sigma(v))\}_{v\in\mathcal{V}}$.

Using this partition we now define a Riemannian metric tensor R of $\mathcal{M}$ by

$$R := \sum_{v\in\mathcal{V}} \phi_v \cdot (\pi^{-1})^*(R_v)$$

where R_v stands for the metric tensor of $\Sigma(v)$. If now d is the geodesic (inner) metric of $\mathcal{M}$ determined by R, then the metric space $(\mathcal{M},d)$ is clearly of bounded geometry, because in its construction we have used objects of only three different types every of which is of bounded geometry.

It remains to find a subspace $\widetilde{S}$ of $(\mathcal{M},d)$ such that

$$\mathrm{Ext}(\mathcal{M},\widetilde{S}) = \emptyset. \tag{8.41}$$

To this end we first consider the hole spheres $\pi(S(v_i)) \subset \mathcal{M}$, $i = 1,2$, such that v_1 and v_2 are joined by an edge. Let $m_i \in \pi(S(v_i))$ be arbitrary points, $i = 1,2$. Then by the definition of the metric d and by a compactness argument

$$0 < c \le d(m_1,m_2) \le C \tag{8.42}$$

where c, C are independent of m_i and v_i. On the other hand,

$$d_\Gamma(v_1,v_2) = 1 \tag{8.43}$$

for this choice of v_i.

Now, let $m_a \in \pi(S(v_a)) \subset \mathcal{M}$, $a \in \{A, B\}$, be arbitrary points where v_A, v_B are distinct and may not necessarily be joined by an edge. Let $\{v_i\}_{i=1}^n$ be a path in the graph Γ joining v_A and v_B (here $v_1 := v_A$ and $v_n := v_B$) such that

$$d_\Gamma(v_A, v_B) = \sum_{i=1}^{n-1} d_\Gamma(v_i, v_{i+1}).$$

Together with (8.42) and (8.43) this implies that

$$c d_\Gamma(v_A, v_B) \le \sum_{i=1}^{n-1} d(m_i, m_{i+1}) \le C d_\Gamma(v_A, v_B). \tag{8.44}$$

On the other hand, the definitions of $\mathcal{M}$ and d yield

$$\widetilde{c} \cdot \sum_{i=1}^{n-1} d(m_i, m_{i+1}) \le d(m_A, m_B) \le \sum_{i=1}^{n-1} d(m_i, m_{i+1}) \tag{8.45}$$

with some $\widetilde{c} > 0$ independent of m_i's.

Finally, we introduce a map $T : \mathcal{V} \to \mathcal{M}$ sending a point $v \in \mathcal{V}$ to an arbitrary point $T(v) \in \pi(S(v))$. Because of (8.44) and (8.45) T is a bi-Lipschitz embedding of $\mathcal{V} \subset \mathcal{M}_\Gamma$ into $\mathcal{M}$. We then define the required subset $\widetilde{S}$ as the image under T of the set $S \subset \mathcal{V}$ for which $\mathrm{Ext}(S, \mathcal{V}) = \emptyset$, see (8.26). Then we have for $\widetilde{S} := T(S)$,

$$\mathrm{Ext}(\widetilde{S}, \mathcal{M}) = \emptyset.$$

The proof of Theorem 8.21 is complete. □

Remark 8.27. Using the Nash embedding theorem [Na-1966] one can realize the Riemannian manifold $\mathcal{M}$ as a C^∞-surface in an open ball of $\mathbb{R}^3$ (with the Riemannian quadratic form induced from the canonical Riemannian structure of $\mathbb{R}^3$).

8.3.2 Peano type locally Lipschitz curve

Our next result allows us to construct one-dimensional metric spaces $\mathcal{M}$ with $\lambda(\mathcal{M}) = \infty$ which "almost coincide" with the real line. In its formulation we will say that a map $f : (\mathcal{M}_0, d_0) \to (\mathcal{M}_1, d_1)$ is a *locally bi-Lipschitz embedding* if for every $m \in \mathcal{M}_0$ there is an open ball $B \subset \mathcal{M}_0$ centered at m such that for some $C = C(B) > 0$ and all $m', m'' \in B$,

$$C^{-1} d_0(m', m'') \le d_1(f(m'), f(m'')) \le C d_0(m', m'').$$

Further, a map $f : (\mathcal{M}_0, d_0) \to (\mathcal{M}_1, d_1)$ is called a *rough isometry* if for some $L \ge 0$ and all $m', m'' \in \mathcal{M}_0$,

$$d_0(m', m'') - L \le d_1(f(m'), f(m'')) \le d_0(m', m'') + L,$$

cf. Volume I, Definition 3.117 where such a map is called a rough $(1, L)$-similarity.

Theorem 8.28. *Let $(\mathcal{M}, d)$ be a complete separable length space. Then for every $\epsilon > 0$ there is a metric space $(\mathcal{M}_\epsilon, d_\epsilon)$ such that:*

(a) $\lambda(\mathcal{M}) \le \lambda(\mathcal{M}_\epsilon)$.

(b) *$\mathcal{M}_\epsilon$ is homeomorphic and locally bi-Lipschiz homeomorphic to the real line.*

(c) *There exists a closed subset $F \subset \mathbb{R}$ of measure ϵ such that $\mathbb{R} \setminus F$ admits an isometric embedding into $\mathcal{M}_\epsilon$.*

(d) *If, in addition,* $\operatorname{diam} \mathcal{M} < \infty$, *then $\mathcal{M}_\epsilon$ is roughly isometric with constant ε to the real line.*

(e) *$\mathcal{M}_\epsilon$ is a proper metric space (i.e., every closed ball in $\mathcal{M}_\epsilon$ is compact).*

Proof. Since $(\mathcal{M}, d)$ is separable, there exists an increasing sequence of finite subspaces $\{\mathcal{M}_n \subset \mathcal{M} \ ; \ n \in \mathbb{N}\}$ such that $\operatorname{card} \mathcal{M}_n = n$, $\cup_{n\in\mathbb{N}} \mathcal{M}_n$ is dense in $\mathcal{M}$ and all $\mathcal{M}_n$ contain a distinguished point $m^* \in \mathcal{M}$.

Let $m_1, \dots, m_n$ be points of $\mathcal{M}_n$ enumerated in some way such that $m_1 = m^*$. We also set $m_{n+1} := m^*$. Since $\mathcal{M}$ is a length space, one can join m_i and m_{i+1} by a curve γ_i of length $l(\gamma_i) \le 2d(m_i, m_{i+1})$, $1 \le i \le n$. By f_n we denote the concatenation (union) of all γ_i, $1 \le i \le n$. Hence f_n is a closed curve in $\mathcal{M}$ with the endpoint m^* of length $\sum_{i=1}^n l(\gamma_i)$ passing through all points of $\mathcal{M}_n$. Reparametrization of f_n by arclength divided by an appropriate positive number turns this into a Lipschitz map $g_n : [n-1, n-1+\frac{\epsilon}{2^n}] \to \{\operatorname{image} f_n\} \subset \mathcal{M}$ passing through all the points of $\mathcal{M}_n$ and such that

$$g_n(n-1) = g_n(n-1+\epsilon/2^n) = m^*. \tag{8.46}$$

Now let $(\widehat{\mathcal{M}}, \widehat{d})$ be the metric space with underlying set $\mathcal{M} \times \mathbb{R}$ and metric $\widehat{d}$ given by the formula $\widehat{d}((m,t),(m',t')) := \max\{d(m,m'), |t-t'|\}$. By $h_n := \{(g_n(t), t) \ ; \ n-1 \le t \le n-1+\frac{\epsilon}{2^n}\} \subset \widehat{\mathcal{M}}$ we denote the graph of g_n. Using the family $\{h_n\}_{n\in\mathbb{N}}$ we determine a locally bi-Lipschitz embedding $h : \mathbb{R} \to \widehat{\mathcal{M}}$ by

$$h(t) := \begin{cases} (m^*, t), & \text{if } t \le 0 \\ h_n(t), & \text{if } t \in J_n^-, \ n \in \mathbb{N}, \\ (m^*, t), & \text{if } t \in J_n^+, \ n \in \mathbb{N}. \end{cases}$$

Here $J_n^- := [n-1, n-1+\frac{\epsilon}{2^n}]$ and $J_n^+ := [n-1+\frac{\epsilon}{2^n}, n]$. Due to (8.46) this determines a homeomorphism of $\mathbb{R}$ onto the metric subspace $\mathcal{M}_\epsilon := h(\mathbb{R})$ of the space $\widehat{\mathcal{M}}$.

Clearly, $\mathcal{M}_\epsilon$ is also locally bi-Lipschitz homeomorphic to $\mathbb{R}$ equipped with the standard metric. Moreover, according to the definition, the subspace $\mathbb{R} \setminus (\cup_{n\in\mathbb{N}} J_n^-)$ of $\mathbb{R}$ is isometric under h to a subspace of $\mathcal{M}_\epsilon$. If, in addition, $\operatorname{diam} \mathcal{M} < \infty$ we, first, replace the metric d on $\mathcal{M}$ by the equivalent metric $\widetilde{d} := \epsilon d/(2 \operatorname{diam} \mathcal{M})$, and then construct the metric space $\mathcal{M}_\epsilon$ as above for the space $(\mathcal{M}, \widetilde{d})$. This space is clearly roughly isometric to $\mathbb{R}$ with constant ε.

This completes the proof of assertions (b)–(d) of the corollary.

Now we prove assertion (a). Let T_n be an n-pointed subset of the interval J_n^- such that $h(T_n) = \mathcal{M}_n \times T_n$ and $n-1 \in T_n$. Then

$$\bigcup_{n\in\mathbb{N}} \mathcal{M}_n \times T_n \subset \mathcal{M}_\epsilon. \tag{8.47}$$

Translate $\mathcal{M}_n \times T_n$ along $\mathbb{R}$ by $1-n$, and by S_n denote the set obtained. Then $S_n \subset \mathcal{M}_n \times [0, \frac{\epsilon}{2^n}] \subset \widehat{\mathcal{M}}$. Show that the sequence $\{S_n\}_{n\in\mathbb{N}}$ finitely converges to $\mathcal{M} \times \{0\} \subset \widehat{\mathcal{M}}$, see Definition 7.11 (a). Then, since S_n is isometric to $\mathcal{M}_n \times T_n$, Theorem 7.12 and (8.47) yield the desired inequality:

$$\lambda(\mathcal{M}) = \lambda(\mathcal{M} \times \{0\}) \le \overline{\lim_{n\to\infty}}\, \lambda(\mathcal{M}_n \times T_n) \le \lambda(\mathcal{M}_\epsilon).$$

To establish that $\{S_n\}_{n\in\mathbb{N}}$ finitely converges consider an arbitrary compact set $\widehat{K} \subset \mathcal{M} \times \{0\}$; then $\widehat{K} = K \times \{0\}$ for some compact set K of $\mathcal{M}$. By K_n we denote the intersection of the $\epsilon/2^n$-neighborhood of K in $\mathcal{M}$ with $\mathcal{M}_n$ and set $\widehat{K}_n := (K_n \times [0, \frac{\epsilon}{2^n}]) \cap S_n$. Since $\cup_{n\in\mathbb{N}}\mathcal{M}_n$ is dense in $\mathcal{M}$ and the sequence $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ is increasing, $\widehat{K}_n \ne \emptyset$ for sufficiently large n. Then for such n the Hausdorff distance from $\widehat{K}$ to $\widehat{K}_n$ is at most $\frac{\epsilon}{2^n}$. Hence, compact sets $\widehat{K}_n \subset S_n$ with sufficiently large n form a sequence convergent in the Hausdorff metric to $\widehat{K}$. Due to Definition 7.11 this means that $\{S_n\}_{n\in\mathbb{N}}$ finitely converges to $\mathcal{M} \times \{0\}$.

Hence assertion (a) is also proved.

Prove now that the space $\mathcal{M}_\epsilon$ is proper (assertion (e)). Let $\overline{B}_r(m) \subset \mathcal{M}_\epsilon$ be the closed ball of $\mathcal{M}_\epsilon$ centered at m and of radius r. By $p_{\mathbb{R}} : \widehat{\mathcal{M}} \to \mathbb{R}$ we denote the canonical projection onto the second component of $\widehat{\mathcal{M}} = \mathcal{M} \times \mathbb{R}$. By our definition of $\mathcal{M}_\epsilon$, this ball is contained in the image $h([p_{\mathbb{R}}(m) - r, p_{\mathbb{R}}(m) + r])$ of the compact interval $[p_{\mathbb{R}}(m) - r, p_{\mathbb{R}}(m) + r]$ of $\mathbb{R}$. Since h is continuous, this image is compact. Therefore the closed subset $\overline{B}_r(m)$ of this image is compact as well.

This proves assertion (e) and the theorem. □

Though $\dim \mathcal{M}_\epsilon = 1$ this space is metrically as large as the space $\mathcal{M}$ (which may be infinite-dimensional). In particular we get

Corollary 8.29. *Given $\varepsilon > 0$ there exists a one-dimensional metric space $\mathcal{M}_\varepsilon$ satisfying conditions* (b)–(e) *of Theorem* 8.28 *such that*

$$\lambda(\mathcal{M}_\varepsilon) = \infty.$$

Proof. It suffices to take the closed unit ball of an infinite-dimensional separable Banach space as the metric space $\mathcal{M}$ of the previous theorem. By Corollary 8.20 $\lambda(\mathcal{M}) = \infty$ and the result follows from Theorem 8.28 (a). □

Corollary 8.30. *Let $\mathcal{M}$ be the closed unit ball of a separable Banach space of infinite dimension. Then there exists a subspace S of $\mathcal{M}_\varepsilon$ such that*

$$\mathrm{Ext}(S, \mathcal{M}_\varepsilon) = \emptyset.$$

Proof. By Corollary 8.29 $\lambda(\mathcal{M}_\varepsilon) = \infty$. Also, by Theorem 8.28 (e) the metric space $\mathcal{M}_\varepsilon$ is proper. Therefore by Theorem 7.5, $\mathcal{M}_\varepsilon \notin \mathcal{SLE}$, i.e., there exists a subspace S of $\mathcal{M}_\varepsilon$ such that $\mathrm{Ext}(S, \mathcal{M}_\varepsilon) = \emptyset$. □

In fact, we can prove more for this choice of $\mathcal{M}$. It will follow from the next result showing that the Peano type curve $h : \mathbb{R} \to \mathcal{M}_\varepsilon$ constructed in Theorem 8.28 fills under some condition about the same content as the generating space does.

Corollary 8.31. *Assume that $\mathcal{M}_\epsilon$ admits a bi-Lipschitz embedding into a proper metric space $\widetilde{\mathcal{M}}$ with a cocompact action of the group of isometries. Then $\mathcal{M}$ admits a bi-Lipschitz embedding into $\widetilde{\mathcal{M}}$.*

Proof. We should construct a bi-Lipschitz embedding of $\mathcal{M}$ into $\widetilde{\mathcal{M}}$ assuming that the associated space $\mathcal{M}_\varepsilon$ admits such an embedding. To this end we will exploit the construction of Theorem 8.28 including

(a) the increasing sequence of finite subspaces $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ of $\mathcal{M}$ such that

$$\overline{\bigcup_{n\in\mathbb{N}} \mathcal{M}_n} = \mathcal{M}; \tag{8.48}$$

(b) the sequence of finite subsets $\{S_n\}_{n\in\mathbb{N}}$ of $\mathcal{M}\times\mathbb{R}$ such that

$$S_n \subset \mathcal{M}_n \times \left[0, \frac{\varepsilon}{2^n}\right] \quad \text{and} \quad \lim_{n\to\infty} S_n = \mathcal{M}\times\{0\} \text{ (finite convergence).}$$

Now let $f : \mathcal{M}_\epsilon \to \widetilde{\mathcal{M}}$ be a C-embedding. By the definition of S_n the map f gives rise to a C-bi-Lipschitz embedding $f_n : S_n \to \widetilde{\mathcal{M}}$.

Further, the group $Iso(\widetilde{\mathcal{M}})$ acts cocompactly on $\widetilde{\mathcal{M}}$, i.e., there exists a compact set $K \subset \widetilde{\mathcal{M}}$ such that

$$\widetilde{\mathcal{M}} = \bigcup \{I(K)\ ;\ I \in Iso(\mathcal{M})\}.$$

Hence, given a point $(m_n, 0) \in S_n$ there exists an isometry I_n of $\widetilde{\mathcal{M}}$ such that $(m_n, 0)$ goes to a point of K under the map

$$\widehat{f}_n := I_n \circ f_n.$$

Now let $p : \mathcal{M}\times\mathbb{R} \to \mathcal{M}$ be the canonical projection. For $k \geq n$ we set

$$S_{n,k} := p^{-1}(\mathcal{M}_n) \cap S_k.$$

Since $\mathcal{M}_n \subset \mathcal{M}_k$ for $k \geq n$, this is a nonempty finite set and $\widehat{f}_n(m_n, 0) \in K$. Therefore the image $\widehat{f}_n(S_{n,k})$ lies in the λ-neighborhood of K where $\lambda :=$

$C \max\{\operatorname{diam} \mathcal{M}_n, \frac{\epsilon}{2^n}\}$. By the properness of $\widetilde{\mathcal{M}}$ the closure of the neighborhood, denoted by K_n, is compact.

Since $p|_{S_{n,k}} : S_{n,k} \to \mathcal{M}_n$ is a bijection, the map

$$\phi_{nk} := \widehat{f}_k \circ (p|_{S_{n,k}})^{-1}$$

is correctly defined and sends $\mathcal{M}_n$ into K_n. Clearly, for all $k \geq m \geq n$,

$$\phi_{mk}|_{\mathcal{M}_n} = \phi_{nk}. \tag{8.49}$$

Further, by the definition of $S_{n,k}$ and (8.48) the sequence $\{S_{n,k}\}_{k\geq n}$ converges to $\mathcal{M}_n \times \{0\}$ in the Hausdorff metric of the metric space $(\mathcal{M} \times \mathbb{R}, \hat{d})$ where $\hat{d}((m,t),(m',t')) := \max\{d(m,m'), |t-t'|\}$. Therefore the map ϕ_{nk}, $k \geq n$, is a C_k-isometric embedding for some $C_k \geq 1$ and C_k tends to $C\,(:= L(f))$ as $k \to \infty$.

Since the target space of each ϕ_{nk} is compact, using (8.49) and the Cantor diagonal process we find a subsequence $\{\phi_{nk(n)}\}_{n\in\mathbb{N}}$ which pointwise converges on the dense subset $\cup_{n\in\mathbb{N}}\mathcal{M}_n$ of $\mathcal{M}$ and its limit, say $\tilde{f}$, is a C-isometric embedding into $\widetilde{\mathcal{M}}$. Extending $\tilde{f}$ to all of the $\mathcal{M}$ by continuity we obtain the desired bi-Lipschitz embedding of $\mathcal{M}$ into $\widetilde{\mathcal{M}}$. □

8.3.3 Uniform lattice generated by a group

Finally, we present an example of a metric abelian group (G, d_A) with the word metric d_A and a countable set of generators. We will show that this is a uniform lattice (see the text preceding Conjecture 7.53 for its definition) which does not have the simultaneous extension property. This explains why this conjecture is formulated only for finitely generated groups.

The group (G, d_A) is introduced as follows.

Given the free abelian group $\mathbb{Z}^n$ with generators $e_1, \dots, e_n$ we introduce its subgroup $(m\mathbb{Z})^n := \left\{\sum_{j=1}^n (m\alpha_j)\, e_j \in \mathbb{Z}^n;\ \alpha_j \in \mathbb{Z},\ 1 \leq j \leq n\right\}$, $m \in \mathbb{N}$.

Next, we consider the finitely generated abelian group $\widehat{G}_m := \mathbb{Z} \oplus \mathbb{Z}^2 \oplus \cdots \oplus \mathbb{Z}^m$. By definition, the set of generators of $\widehat{G}_m$ is the disjoint union of sets of generators of groups $\mathbb{Z}^k$ for $1 \leq k \leq m$. We equip $\widehat{G}_m$ with the word metric d_m corresponding to this set of generators.

The group $G_m := \mathbb{Z} \oplus (2\mathbb{Z})^2 \oplus \cdots \oplus (m\mathbb{Z})^m$ is a finitely generated subgroup of $\widehat{G}_m$. Namely, the generators of $(k\mathbb{Z})^k$ are $ke_1, \dots, ke_k$ and the set of generators of G_m is the disjoint union of those for all $(k\mathbb{Z})^k$, $1 \leq k \leq m$. We consider G_m as a metric subspace of $(\widehat{G}_m, d_m)$.

Let $\mathcal{C}_m$ be the Cayley graph associated to the group (G_m, d_m). Then every pair $g_1, g_2 \in \mathcal{C}_m$ is joined by an edge in $\mathcal{C}_m$ whenever $g_1 = g_2 \pm a$ for some generator a of G_m. Clearly, the length of this edge equals k if the a is a generator of $(k\mathbb{Z})^k$.

Further, for $m_1 < m_2$ there is the natural embedding $G_{m_1} \hookrightarrow G_{m_2}$. We identify G_{m_1} with a subgroup of G_{m_2}. Then the inductive limit of groups G_m

with respect to the system of such embeddings can be identified with the union $G := \cup_{m\in\mathbb{N}} G_m$. Clearly, G is a countable abelian group with a countable set of generators. We introduce a metric d on G by the formula

$$d(g_1, g_2) := d_m(g_1, g_2) \text{ for } g_1, g_2 \in G_m \subset G.$$

Theorem 8.32. *(G, d) is a uniform lattice which does not belong to $\mathcal{SLE}$.*

Proof. Let us prove first that (G, d) is a uniform lattice. This will imply, in particular, that it is a proper metric space. Then according to Theorem 7.5 the fact that (G, d) does not belong to $\mathcal{SLE}$ is equivalent to the equality $\lambda(G) = \infty$.

To prove that (G, d) is a uniform lattice, it suffices to prove that the number of elements of each ball centered at the unit element $1 \in G$ is finite (in fact, the metric d is invariant under left multiplication). Fix $0 < R < \infty$ and consider the ball $B_R(1) \subset G$. By the definition if $m \in B_R(1)$ then $m \in G_{\lfloor R\rfloor}$. For otherwise, m contains in the product decomposition to elements from $(k\mathbb{Z})^k$ a nontrivial element h from $(n\mathbb{Z})^n$ with $n > R$. Then by the construction of the metric d,

$$d(1, m) \geq d(1, h) = n \cdot s \text{ for some } s \in \mathbb{N}.$$

Hence, $d(1, m) > R$, a contradiction. Since $B_R(1) \subset G_{\lfloor R\rfloor}$ and the latter group is finitely generated with respect to the set of generators from $(k\mathbb{Z})^k$, $1 \leq k \leq \lfloor R\rfloor$, $|B_R(1)| < \infty$. This proves that (G, d) is a uniform lattice.

Now let us prove that $\lambda(G) = \infty$. In fact, by the definition of (G, d) for each $m \in \mathbb{N}$ it contains subgroup $(m\mathbb{Z})^m$ bi-Lipschitz homeomorphic with distortion 1 to $\mathbb{Z}^m$ equipped with the word metric. Therefore

$$\lambda(G) \geq \overline{\lim_{m\to\infty}}\, \lambda(\mathbb{Z}^m) = \infty$$

by Lemma 8.24. □

Comments

Assertion (a) of Theorem 8.1 leads to the next natural

Problem. *Under what conditions on a metric space $\mathcal{M}'$ is the inequality*

$$\Lambda(\omega(\mathcal{M}), \mathcal{M}') \leq C[\Lambda(\mathcal{M}, \mathcal{M}')]^2$$

true for every $\mathcal{M}$ satisfying $\Lambda(\mathcal{M}, \mathcal{M}') < \infty$?

One may also question the validity of this inequality for a special choice of spaces $\mathcal{M}$ and all $\mathcal{M}'$.

The proof of the above assertion exploits the Banach structure of the target space $\mathcal{M}'$ and cannot be applied directly. For the special case of $\omega(t) := t^\alpha$, $0 < \alpha \leq 1$, and a dual Banach space $\mathcal{M}'$ there exists an alternative proof of this result due to Naor [N-2001]. His approach is based on Ball's Markov type theory [B-1992]; the main result of Ball's paper may apparently be used for more general target spaces. However, this approach is restricted to power majorants, since the corresponding constant C in the Naor theorem is unbounded as α tends to 0 (it equals $\frac{50}{\alpha}$).

Sharp (preserving Lipschitz constants) extensions from subsets of $\omega(\mathcal{M})$ into a Banach space X are studied in the book [WW-1975] by Wells and Williams for the case of $\mathcal{M} := L_p$, $X := L_q$, $1 < p, q \le 2$, and $\omega(t) := t^\alpha$. Their elegant result asserts that a sharp extension exists for every S from $\omega(L_p)$ into L_q if and only if $0 < \alpha \le p\left(1 - \frac{1}{q}\right)$. Some parts of the rather involved proof of this result are simplified by Naor [N-2001].

According to Proposition 7.3 all results of Section 8.2 are true for X-valued Lipschitz functions with X being a Banach space constrained in its bidual. This restriction may be removed by using the extension method of Section 7.4 at the price of essential complication of proofs.

We present in the proof of Theorem 8.16 a special case of the general Lang result [L-1999] characterizing embeddings of finite metric graphs into metric spaces.

The examples of Section 8.3 show that no restrictions on topological or smooth structure of a metric space provide the $\mathcal{SLE}$ property of this space. The only known metric invariant which can be used for this case is the Nagata dimension. It would be assumed that finiteness of this dimension is not only sufficient but also necessary for $\mathcal{M}$ being an $\mathcal{SLE}$ space. We think that the generalized hyperbolic space $\mathbb{H}^n_\omega$ with a suitable ω may be a counterexample to the last claim.

Part IV

Smooth Extension and Trace Problems for Functions on Subsets of $\mathbb{R}^n$

Chapter 9

Traces to Closed Subsets: Criteria, Applications

Section 9.1 presents the Yu. Brudnyi and Shvartsman criteria characterizing the trace spaces for Lipschitz functions of higher order and those for the associated jet spaces. Their proofs widely exploit the concepts and methods of Local Approximation Theory. In particular, the key tools are the Yu. Brudnyi inequality relating local approximation to the corresponding difference characteristic, see Volume I, Theorem 2.37, and a Whitney type extension construction with Taylor polynomials replaced by polynomials of the best local approximation.

This approach is used throughout the chapter. It may be also successfully applied to the study of similar problems for other smoothness function spaces commonly used in analysis (Sobolev spaces of integer and noninteger smoothness, Besov and BMO spaces etc.). Unfortunately, this theme is too vast to discuss here, and we only formulate a sample of the corresponding extension results in Comments and present one of them (for the Morrey–Campanato spaces).

Section 9.2 is devoted to the study of traces to and extensions from closed subsets in $\mathbb{R}^n$ preserving Markov's type polynomial inequality. This class of sets is large enough and contains, e.g., closures of Lipschitz domains, the Ahlfors s-regular sets with $s > n-1$ and self-similar fractals with a separation condition such as Cantor's sets, Sierpiński's gasket, Antoine's necklace etc. (see Volume I, Theorem 4.21 and the related text for the description of these and similar objects). Along with the aforementioned result for the Morrey–Campanato spaces we give in this section a complete description of the trace space of $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ with an arbitrary k-majorant ω to a closed subset of this class.

In Section 9.3, we study similar problems for the significant in several fields of analysis class of sets called *uniform or* (ε,δ)*-domains*. In particular, uniform domains play an important role in the theory of quasiconformal mappings where this concept first appeared.

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As it was explained in Section 2.6 of Volume I, this class contains, along with Lipschitz, also domains with much more complicated (fractal) boundaries, e.g., a domain bounded by the von Koch snowflake. In spite of their complicated geometry, Whitney's cover associated to a uniform domain has several important characteristics allowing us to use the Whitney extension operator. This fact was discovered by P. Jones who proved in this way the extension theorems for Sobolev and BMO spaces on uniform domains. The extension results for general smoothness spaces on uniform domains including also the above mentioned ones were proved in Shvartsman's 1984 Thesis. They require a much more detailed information on the geometry of Whitney's cubes. A special case of his result related to the spaces $\Lambda^{k,\omega}(\mathbb{R}^n)$ is presented in Section 9.3.

9.1 Traces to closed subsets: criteria

9.1.1 Preliminaries

In the proofs of the main results, several facts presented in Chapter 2 of Volume I will be intensively exploited. For convenience of the reader we formulate them now in a form adapted to our consideration.

A. We begin with Theorem 2.37 of Volume I connecting the *k-th modulus of continuity* and *local best approximation*. The former is defined for a function $f \in \ell_\infty(G)$ where $G \subset \mathbb{R}^n$ is a domain and any $t > 0$ by

$$\omega_k(t\,; f)_G := \sup_{\|h\| \le t} \|\Delta_h^k f\|_{\ell_\infty(G_{kh})}\,; \tag{9.1}$$

here $G_y := \{x \in G\,; [x, x+y] \subset G\}$ and $\|h\|$ stands for the standard Euclidean norm of $\mathbb{R}^n$. *We omit the subscript if* $G = \mathbb{R}^n$.

The function $t \mapsto \omega_k(t\,; f)$, $t > 0$, is equivalent to a *k-majorant*, i.e., a function of the class Ω_k consisting of nondecreasing functions $\omega : (0, +\infty) \to \mathbb{R}_+$ such that $t \mapsto \frac{\omega(t)}{t^k}$, $t > 0$, is nonincreasing, see for details Volume I, subsection 2.1.3, Theorem 2.7 and Definition 2.8.

In turn, *local polynomial approximation of order k* is given for $f \in \ell_\infty^{loc}(\mathbb{R}^n)$ and a cube $Q \subset \mathbb{R}^n$ by

$$E_k(Q\,; f) := \inf\{\|f - g\|_{\ell_\infty(Q)}\,; p \in \mathcal{P}_{k-1,n}\}. \tag{9.2}$$

Let us note that here the degree of approximating polynomials equals the order of local approximation less 1. In particular, the right-hand side in (9.2) equals $\|f\|_{\ell_\infty(Q)}$ for $k = 0$.

A special case of Theorem 2.37 of Volume I relates the notions introduced above by the following inequality:

$$E_k(Q\,; f) \le w(k,n)\,\omega_k\left(\frac{r_Q}{k}\,; f\right)_Q, \tag{9.3}$$

where r_Q is the radius (half side length) of cube Q.

Since the k-th difference annihilates polynomials of degree $k-1$, the converse inequality is also true:

$$2^{-k}\omega_k\left(\frac{r_Q}{k}\,;f\right)_Q \le E_k(Q\,;f). \tag{9.4}$$

B. We will also intensively exploit properties of Whitney's covers presented in Lemma 2.14 and Corollary 2.15 of Volume I. Let us first recall the notations used previously and introduce several new ones.

An *n-cube* is a subset of $\mathbb{R}^n$ homothetic to the unit cube $Q_0 := [0,1]^n$. Regarding such a cube (denoted by Q and also Q', K etc.) as a closed ℓ_∞-ball we, by r_Q and c_Q, denote its radius and center. Further, λQ with $\lambda > 0$ is the cube of the same center c_Q and radius λr_Q (i.e., λ-homothetic to Q with respect to its center).

Finally, $Q_r(x)$ stands for a cube centered at x and of radius r.

Convention. Hereafter $\|\cdot\|$ stands for the ℓ_∞-norm and $\|\cdot\|_2$ for the ℓ_2-norm of $\mathbb{R}^n$. In these notations,

$$Q_r(x_0) = \{x \in \mathbb{R}^n\,;\, \|x - x_0\| \le r\}.$$

Moreover, we measure distances in $\mathbb{R}^n$ by the ℓ_∞-norm, e.g., for $S, S' \subset \mathbb{R}^n$,

$$d(S,S') := \inf\{\|x - x'\|\,;\, x \in S\,,\, x' \in S'\}.$$

We also denote by $|S|$ the Lebesgue n-measure of S.

The set of all cubes in $\mathbb{R}^n$ is denoted by $\mathcal{K}(\mathbb{R}^n)$ (or, briefly, $\mathcal{K}$) while $\mathcal{K}_S$ and $\mathcal{K}(S)$ are given by

$$\begin{aligned} \mathcal{K}_S &:= \{Q \in \mathcal{K}\,;\, c_Q \in S \text{ and } S \cap Q^c \ne \emptyset\} \text{ and}\\ \mathcal{K}(S) &:= \{Q \in \mathcal{K}\,;\, Q \subset S\}. \end{aligned} \tag{9.5}$$

Now given a closed subset $S \subset \mathbb{R}^n$, we denote by $\mathcal{W}_S$ the Whitney cover of its complement $S^c := \mathbb{R}^n \setminus S$. Let us recall that $\mathcal{W}_S$ consists of *dyadic* cubes, i.e., of the form $2^{-j}(Q_0 + k)$ for some $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^n$, satisfying

$$2Q \in \mathcal{K}(S^c) \quad \text{and} \quad 5Q \notin \mathcal{K}(S^c). \tag{9.6}$$

Now we list the basic properties of the Whitney cover, see Volume I, Section 2.2, especially Lemma 2.14 and Corollary 2.15 there, for details. In the formulation below, Q^* stands for λQ with $\lambda = \frac{9}{8}$ and Q, K are cubes of $\in \mathcal{W}_S$.

Proposition 9.1. (a) *Interiors of distinct cubes of $\mathcal{W}_S$ do not intersect.*

(b) *For every $Q \in \mathcal{W}_S$,*

$$\frac{1}{5}r_Q \le d(Q,S) \le 5r_Q. \tag{9.7}$$

(c) *$Q \cap K \neq \emptyset$ if and only if $Q^* \cap K^* \neq \emptyset$. Moreover, for some $c(n) > 0$ and every such pair Q, K,*

$$|Q^* \cap K^*| \geq c \min\{|Q^*|\,,\,|K^*|\}. \tag{9.8}$$

(d) *If $Q \cap K \neq \emptyset$, then*

$$\frac{1}{4} r_Q \leq r_K \leq 4 r_Q. \tag{9.9}$$

(e) *The order (multiplicity) of the cover*

$$\mathcal{W}_S^* := \{Q^*\,;\, Q \in \mathcal{W}_S\}$$

is bounded by a constant depending only on n.

Let us finally recall, see Volume I, Lemma 2.16, that there exists a C^∞ partition of unity $\{\varphi_Q\}_{Q \in \mathcal{W}_S}$ subordinate to the cover $\mathcal{W}_S^*$ which satisfies for every $\alpha \in \mathbb{Z}_+^n$ and $Q \in \mathcal{W}_S$ the inequality

$$\sup_{\mathbb{R}^n} |D^\alpha \varphi_Q| \leq c(\alpha, n)\, r_Q^{-|\alpha|}. \tag{9.10}$$

9.1.2 $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ spaces

We present a description of the trace space $\dot{\Lambda}^{k,\omega}|_S$ for arbitrary closed $S \subset \mathbb{R}^n$. Let us recall that the Lipschitz space $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ is defined by the semi-norm

$$|f|_{\Lambda^{k,\omega}(\mathbb{R}^n)} := \sup_{t>0} \frac{\omega_k(t; f)}{\omega(t)}$$

where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a function of the class Ω_k, see Volume I, Definitions 2.4 and 2.8 for details. Throughout this chapter we simplify this notation writing simply $\dot{\Lambda}^{k,\omega}$ if there is no danger of ambiguity.

The criterion presented below was due to Yu. Brudnyi and P. Shvartsman [BSh-1982]. It exploits polynomials of local best approximation, since Taylor polynomials cannot be used for this purpose. In fact, the spaces under consideration may contain nowhere differentiable functions (e.g., $\dot{\Lambda}^{2,\omega}(\mathbb{R})$ with $\omega(t) := t$ contains the Weierstrass nowhere differentiable function $\sum_{n=0}^{\infty} 2^{-n} \cos 2^n x$).

Definition 9.2. Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a function from Ω_k and $k \geq 1$ be an integer. A collection of polynomials $P := \{p_Q\,;\, Q \in \mathcal{K}_S\} \subset \mathcal{P}_{k-1,n}$ is said to be an (ω, k)-chain on S if, for some constant $C > 0$ and every pair $Q \subset Q'$ of cubes from $\mathcal{K}_S$,

$$\sup_Q |p_Q - p_{Q'}| \leq C\omega(r_{Q'}). \tag{9.11}$$

The set of all these chains is denoted by $\mathrm{Ch}(\omega, k; S)$. If a chain P belongs to this linear space, then $|P|_{\mathrm{Ch}}$ stands for the infimum of constants C in (9.11).

Theorem 9.3. *Let f be a locally bounded function on a closed set $S \subset \mathbb{R}^n$. Then f belongs to the trace space $\dot{\Lambda}^{k,\omega}|_S$ if and only if there exists an (ω, k)-chain $P := \{p_Q; Q \in \mathcal{K}_S\}$ such that for every $Q \in \mathcal{K}_S$,*

$$f(c_Q) = p_Q(c_Q). \tag{9.12}$$

Moreover, for some constants independent of f,

$$|f|_{\Lambda^{k,\omega}|_S} \approx \inf |P|_{\mathrm{Ch}},$$

where P runs over all (ω, k)-chains on S satisfying (9.12).

Proof. (Necessity) Let $f \in \dot{\Lambda}^{k,\omega}|_S$ and g be a function of $\dot{\Lambda}^{k,\omega}$ such that $g|_S = f$. Given a cube $Q \in \mathcal{K}_S$, we denote by p_Q a polynomial of degree $k-1$ such that

$$\max_Q |g - p_Q| = E_k(Q\,; g).$$

We then set $\widetilde{p}_Q := p_Q - p_Q(c_Q) + f(c_Q)$ and check that $\{\widetilde{p}_Q\,; Q \in \mathcal{K}_S\}$ is an (ω, k)-chain on S. Since condition (9.12) holds for $\widetilde{p}_Q$, this will prove this part of the theorem.

Let $Q \subset Q'$ be cubes from $\mathcal{K}_S$. Then

$$\max_Q |\widetilde{p}_Q - \widetilde{p}_{Q'}| \le 2\Big(\max_{Q'} |g - p_{Q'}| + \max_Q |g - p_Q|\Big) \le 4E_k(Q'\,; g).$$

By virtue of Theorem 2.37, see also (9.3), the right-hand side is at most $4w(k,n)\omega_k(g\,; Q')$ which, in turn, is bounded by $4w(k,n)|g|_{\Lambda^{k,\omega}(\mathbb{R}^n)} \cdot \omega(r_{Q'})$.

Hence, $P := \{\widetilde{p}_Q\,; Q \in \mathcal{K}_S\}$ is an (ω, k)-chain on S satisfying (9.12) and such that

$$|P|_{\mathrm{Ch}} \le C(k,n)|g|_{\Lambda^{k,\omega}(\mathbb{R}^n)}.$$

Taking the infimum over all such g we also get

$$\inf |P|_{\mathrm{Ch}} \le C(k,n)|f|_{\Lambda^{k,\omega}|_S}. \tag{9.13}$$

(Sufficiency) Let $f \in \ell_\infty^{\mathrm{loc}}(S)$ and $P := \{p_Q\,; Q \in \mathcal{K}_S\}$ be an (ω, k)-chain satisfying (9.12). We should extend f to a function from $\dot{\Lambda}^{k,\omega}$ and prove for this extension the inequality converse to (9.13). An extension operator used for this aim is obtained by modifying Whitney's extension procedure with the polynomials of local approximation replacing Taylor polynomials.

Specifically, given the cover $\mathcal{W}_S$ of S^c and the partition of unity $\{\varphi_Q\}_{Q \in \mathcal{W}_S}$ subordinate to $\mathcal{W}_S^*$, see subsection 9.1.1, a_Q denotes the nearest to c_Q point of S and $\widehat{Q}$ stands for the translate of cube Q centered at a_Q. Since the distance from S to Q is bounded from above by $5r_Q$, see Proposition 9.1 (a), we also have

$$\mathrm{dist}(Q, \widehat{Q}) \le 6r_Q = 3\,\mathrm{diam}\, Q. \tag{9.14}$$

Using these cubes we define an extension $\tilde{f}$ of f by

$$\tilde{f}(x) := \begin{cases} f(x), & \text{if } x \in S, \\ \sum_{Q \in \mathcal{W}_S} p_{\widehat{Q}} \varphi_Q, & \text{if } x \in S^c. \end{cases} \tag{9.15}$$

To prove that $\tilde{f}$ belongs to $\dot{\Lambda}^{k,\omega}$, it suffices (according to (9.4)) to establish for an arbitrary cube $Q \in \mathbb{R}^n$ the inequality

$$E_k(Q\,; \tilde{f}) \le C(k,n)\omega(r_Q)|P|_{\mathrm{Ch}}. \tag{9.16}$$

This will be achieved by a chain of auxiliary results; in their proofs we will use the following partition of the set $\mathcal{K}(\mathbb{R}^n)$.

A cube $Q \subset \mathbb{R}^n$ is of the *first class* denoted by $\mathcal{K}^{\mathrm{I}}$, if the following conditions hold:

(a) $Q \subset S^c$;

(b) $Q \subset K^*$ where $K \in \mathcal{W}_S$ is a (unique) Whitney cube containing the center of Q.

The remaining cubes constitute the second class:

$$\mathcal{K}^{\mathrm{II}} := \mathcal{K}(\mathbb{R}^n) \backslash \mathcal{K}^{\mathrm{I}}.$$

Lemma 9.4. *Let $Q \in \mathcal{K}^{\mathrm{II}}$. There is a numerical constant c such that*

$$\operatorname{dist}(Q, S) \le c \operatorname{diam} Q, \tag{9.17}$$

and for every cube $Q' \in \mathcal{W}_S$ intersecting Q,

$$\operatorname{diam} Q' \le c \operatorname{diam} Q. \tag{9.18}$$

Proof. If $Q \in \mathcal{K}^{\mathrm{II}}$, then Q intersects either S or $Q \backslash K^* \ne \emptyset$ where K is a Whitney cube containing c_Q.

In the first case, (9.17) is trivial. Now, let $z_0 \in Q \backslash K^*$, and $x_0 \in S$, $y_0 \in K$ be such that

$$\operatorname{dist}(S, K) = \|x_0 - y_0\|.$$

Then the left-hand side of (9.17) is bounded by

$$\begin{aligned} \|x_0 - z_0\| &\le \|x_0 - y_0\| + \|y_0 - z_0\| \le \operatorname{dist}(S, K) + \|y_0 - c_Q\| + \|c_Q - z_0\| \\ &\le \frac{5}{2} \operatorname{diam} K + \operatorname{diam} K + r_Q, \end{aligned}$$

see Proposition 9.1 (b). But $c_Q \in K$ and therefore

$$\operatorname{diam} K \le 2\|c_Q - z_0\| \le \operatorname{diam} Q.$$

Combining these estimates we have

$$\operatorname{dist}(Q, S) \le 5 \operatorname{diam} Q.$$

This proves (9.17) for the second case as well.

Further, let $Q' \in \mathcal{W}_S$ and $Q' \cap Q \neq \emptyset$. Pick a point $z_0 \in Q' \cap Q$. Applying Proposition 9.1 (b) and (9.17) we have

$$\operatorname{diam} Q' \le 10 \operatorname{dist}(Q', S) \le 10\big(\operatorname{diam} Q + \operatorname{dist}(Q, S)\big) \le 60 \operatorname{diam} Q.$$

The result is proved. □

The next lemma is a consequence of a general result presented in Appendix G of Volume I, see Corollary G.2 there.

Lemma 9.5. *Let C be a compact convex body in $\mathbb{R}^n$ and $S \subset C$ be a closed subset of C of a positive Lebesgue n-measure $|S|$. Then for every polynomial $p \in \mathcal{P}_{k,n}$ the inequality*

$$\max_C |p| \le \gamma \left(\frac{|C|}{|S|} \right)^k \max_S |p|$$

holds with a constant γ depending only on k and n.

Now the subsequent derivation of the basic inequality (9.16) is divided into two parts.

Claim I. Inequality (9.16) holds for every cube of the second class.

Proof. Without loss of generality we assume that

$$|P|_{\mathrm{Ch}} = 1. \tag{9.19}$$

We fix a chain P in (9.11) such that polynomials $p_Q \in P$ will satisfy, for $Q \subset Q'$,

$$\sup_Q |p_Q - p_{Q'}| \le 2\omega(r_{Q'}).$$

Now we fix a cube $Q \in \mathcal{K}^{\mathrm{II}}$) and first assume that its center $c_Q \in S$. Further we define two subfamilies of Whitney's cubes associated to Q by setting

$$\begin{aligned}
&\mathcal{W}_S'(Q) := \big\{Q' \in \mathcal{W}_S\,;\, Q' \cap Q \neq \emptyset\big\},\\
&\mathcal{W}_S''(Q) := \big\{Q'' \in \mathcal{W}_S\,;\ \text{there exists } Q' \in \mathcal{W}_S'(Q) \text{ such that } Q' \cap Q'' \neq \emptyset\big\}.
\end{aligned}$$

Clearly, $\mathcal{W}_S'(Q) \subset \mathcal{W}_S''(Q)$ and by (9.18) and Proposition 9.1, there is a constant $c_1 = c_1(n)$ such that

$$\bigcup \mathcal{W}_S''(Q) \subset c_1 Q; \tag{9.20}$$

recall that $\cup\mathcal{F}$ stands for the union of sets from a family $\mathcal{F}$.

Let now $\tilde{c} = \tilde{c}(n) \ge c_1(n)$ be a constant specified later. Then we set $\widetilde{Q} := \tilde{c}Q$ and write, for the extension $\tilde{f}$ given by (9.15),

$$\begin{aligned} E_k(Q\,;\tilde{f}) &\le \max_Q |f - p_{\widetilde{Q}}| \\ &\le \max\Big\{\max_{Q\cap S} |f - p_{\widetilde{Q}}|,\ \sup_{K\in\mathcal{W}'_S(Q)} \max_K |\tilde{f} - p_{\widetilde{Q}}|\Big\}. \end{aligned} \tag{9.21}$$

To estimate the first maximum we fix a point $x \in Q\cap S$ and denote by $Q(x)$ a cube centered at x and such that $Q(x) \subset \widetilde{Q}$. Then by Definition 9.2 of an (ω, k)-chain, and conditions (9.12) and (9.19) , we have

$$\big|f(x) - p_{\widetilde{Q}}(x)\big| = \big|p_{Q(x)} - p_{\widetilde{Q}}\big|(x) \le 2\omega(r_{\widetilde{Q}}) = 2\omega(\tilde{c}r_Q).$$

Since $\omega(t)/t^k$ is nonincreasing, i.e., $\omega(ct) \le c^k\omega(t)$, this gives

$$\max_{Q\cap S} |f - p_{\widetilde{Q}}| \le O(1)\omega(r_Q) \tag{9.22}$$

where $O(1)$ denotes a constant depending only on k and n.

Further, we estimate the supremum over $\mathcal{W}'_S(Q)$ in the right-hand side of (9.21). Let K be a cube from $\mathcal{W}'_S(Q)$. By (9.15) and the properties of a partition of unity, we get

$$\max_K |\tilde{f} - p_{\widetilde{Q}}| = \max_K \Big| \sum_{Q'\in\mathcal{W}'_S(K)} \varphi_{Q'}(p_{\widehat{Q}'} - p_{\widetilde{Q}})\Big| \le \sup_{Q'\in\mathcal{W}'_S(K)} \max_K |p_{\widehat{Q}'} - p_{\widetilde{Q}}|.$$

To estimate the maximum over K we apply Lemma 9.5 to the polynomial $p := p_{\widehat{Q}'} - p_{\widetilde{Q}}$. Since $Q' \in \mathcal{W}'_S(K)$, i.e., $Q' \in \mathcal{W}_S$ and $K \cap Q' \ne \phi$, and $\widehat{Q}'$ is the translate of Q' centered at a point of S, the estimates of Proposition 9.1 evaluate the size of K as $\operatorname{diam} K \le 4 \operatorname{diam} Q'$. Hence, we conclude that

$$\sup_K |p| \le \sup_{10Q'} |p| \le O(1) \sup_{Q'} |p|.$$

Together with the previous estimate this gets

$$\max_K |\tilde{f} - p_{\widetilde{Q}}| \le O(1) \sup_{Q'\in\mathcal{W}'_S(K)} \max_{Q'} |p_{\widehat{Q}'} - p_{\widetilde{Q}}|.$$

Since $K \in \mathcal{W}'_S(K) \subset \mathcal{W}''_S(Q)$, inequalities (9.21) and (9.22) lead to the inequality

$$E_k(Q\,;\tilde{f}) \le O(1)\Big\{\omega(r_Q) + \sup_{Q''\in\mathcal{W}''_S(Q)} \max_{Q''} |p_{\widehat{Q}''} - p_{\widetilde{Q}}|\Big\}. \tag{9.23}$$

To estimate the maximum over Q'' we note that for some constant $c_2 = c_2(n)$,

$$c_2Q'' \supset \widehat{Q}'' \quad \text{and} \quad c_2\widehat{Q}'' \supset Q'',$$

by Proposition 9.1 (b) and, moreover, $c_1 Q \supset Q''$ by (9.20). Choosing the constant $\tilde{c}$ from (9.21) to be equal to $c_1 c_2$, we then obtain the embedding

$$\widehat{Q}'' \subset \tilde{c}Q =: \widetilde{Q}.$$

This embedding, Lemma 9.5 and Definition 9.2 of an (ω, k)-chain give

$$\begin{aligned}\max_{Q''} |p_{\widehat{Q}''} - p_{\widetilde{Q}}| &\le O(1) \sup_{\widehat{Q}''} |p_{\widetilde{Q}''} - p_{\widetilde{Q}}| \\ &\le O(1)\omega(r_{\widetilde{Q}}) = O(1)\omega(\tilde{c} r_Q) \le O(1)\omega(r_Q).\end{aligned}$$

Together with (9.23) this yields

$$E_k(Q\,; \tilde{f}) \le O(1)\omega(r_Q),$$

and Claim I is proved for cubes $Q \in \mathcal{K}^{\mathrm{II}}$ centered at S.

Now let $Q \in \mathcal{K}^{\mathrm{II}}$ but $c_Q \notin S$. Let Q' be the smallest cube centered at a_Q and containing Q (here a_Q is a point of S closest to c_Q). Then by (9.17), $\operatorname{diam} Q' \le O(1) \operatorname{diam} Q$, and the previous inequality gives

$$E_k(Q\,; \tilde{f}) \le E_k(Q'\,; \tilde{f}) \le O(1)\omega(r_{Q'}) \le O(1)\omega(r_Q).$$

The proof of Claim I is complete. □

Claim II. Let Q be a cube of the first class and L be the unique cube of $\mathcal{W}_S$ containing its center. Then

$$E_k(Q\,; \tilde{f}) \le O(1)\Big(\frac{r_Q}{r_L}\Big)^k \omega(r_L). \tag{9.24}$$

As above, we normalize the chain P by condition (9.19).

Before proving Claim II we show how to derive from this the desired inequality (9.16) for $Q \in \mathcal{K}^{\mathrm{I}}$. Due to the definition of the class $\mathcal{K}^{\mathrm{I}}$ preceding Lemma 9.4 the cube $Q \subset L^*$, in particular, $r_Q \le \frac{9}{8} r_L$, and the right-hand side of (9.24) is bounded by

$$O(1)(r_Q)^k \frac{\omega(r_L)}{(r_L)^k} \le O(1)(r_Q)^k \frac{\omega(r_Q)}{(r_Q)^k} = O(1)\omega(r_Q).$$

Thus, (9.16) is true for this and therefore for every Q of $\mathbb{R}^n$.

It remains to establish Claim II. The key point is the inequality

$$E_k(Q\,; \tilde{f}) \le O(1)\Big(\frac{r_Q}{r_L}\Big)^k \sup_K \|p_{\widehat{K}} - p_{\widehat{L}}\|_{C(L)} \tag{9.25}$$

where K runs over the set

$$\mathcal{W}'''(Q) := \big\{K \in \mathcal{W}_S\,;\, K^* \cap Q \ne \emptyset\big\}.$$

This would imply Claim II in the following way. We first check that every cube K from $\mathcal{W}_S$ such that $K^* \cap Q \neq \emptyset$ satisfies for some constant $c = c(n) > 0$ the embeddings

$$c\widehat{K} \supset L \cup \widehat{L},\ cL \supset \widehat{K},\ c\widehat{L} \supset \widehat{K}.$$

To derive these, we use assertions (b) and (c) of Proposition 9.1 which immediately imply:

$$K \cap L \neq \emptyset \quad \text{and} \quad r_K \approx r_L \tag{9.26}$$

with the constants of equivalence depending only on n, and then use the definition of the cubes $\widehat{K}$ and $\widehat{L}$, see (9.14), and again Proposition 9.1 (b).

Applying these embeddings, Lemma 9.5, inequality (9.25) and the definition of an (ω, k)-chain, we then obtain the desired inequality (9.24) by the next estimate:

$$\begin{aligned} E_k(Q\,;\tilde{f}) &\leq O(1)\Big(\frac{r_Q}{r_L}\Big)^k \sup_{K \in \mathcal{W}_S'''(Q)} \big\{\|p_{\widehat{K}} - p_{c\widehat{L}}\|_{C(\widehat{K})} + \|p_{\widehat{L}} - p_{c\widehat{L}}\|_{C(L)}\big\} \\ &\leq O(1)\Big(\frac{r_Q}{r_L}\Big)^k \omega(cr_{\widehat{L}}) \leq O(1)(r_Q)^k \frac{\omega(r_L)}{(r_L)^k}\,. \end{aligned}$$

It remains to prove (9.25). Since $Q \subset S^c$, the function $\tilde{f}|_Q$ belongs to $C^\infty(Q)$. Therefore, the Jackson type approximation theorem, see any book on Approximation Theory, yields

$$E_k(Q\,;\tilde{f}) \leq O(1)(r_Q)^k \sup_{|\alpha|=k} \|D^\alpha \tilde{f}\|_{C(Q)}.$$

By the definition of $\tilde{f}$, for every $x \in Q$ and $|\alpha| = k$,

$$D^\alpha \tilde{f}(x) = D^\alpha\bigg(\sum_{K \in \mathcal{W}_S} \varphi_K p_{\widehat{K}}\bigg) = D^\alpha\bigg(\sum_{K \in \mathcal{W}_S} \varphi_K (p_{\widehat{K}} - p_{\widehat{L}})\bigg)(x).$$

This, the Leibnitz rule and embedding $\operatorname{supp}\varphi_K \subset K^*$ give

$$\|D^\alpha \tilde{f}\|_{C(Q)} \leq O(1) \sum_{K \in \mathcal{W}'''(Q)} \sum_{\alpha = \beta + \gamma} \|D^\beta \varphi_K\|_{C(Q)} \cdot \big\|D^\gamma (p_{\widehat{K}} - p_{\widehat{L}})\big\|_{C(Q)}.$$

Since L contains the center c_Q of cube Q (of the first class), it is contained in L^*. Therefore the Markov inequality of Lemma 2.35 of Volume I, Lemma 9.5 and (9.26), yield

$$\begin{aligned} \big\|D^\gamma (p_{\widehat{K}} - p_{\widehat{L}})\big\|_{C(Q)} &\leq O(1)(r_{L^*})^{-|\gamma|}\, \|p_{\widehat{K}} - p_{\widehat{L}}\|_{C(L^*)} \\ &\leq O(1)(r_L)^{-|\gamma|}\, \|p_{\widehat{K}} - p_{\widehat{L}}\|_{C(L)}. \end{aligned}$$

The previous three inequalities, inequality (9.26) and the estimate

$$\|D^{\beta}\varphi_K\|_{C(\mathbb{R}^n)} \le O(1)(r_K)^{-|\beta|} \le O(1)(r_L)^{-|\beta|},$$

see (9.10), yield (9.25).

The proof of the sufficiency for Theorem 9.3 is completed. $\square$

The following version of Theorem 9.3 concerning *simultaneous* extensions will be used in the forthcoming applications. Hence, we are looking for a linear bounded extension operator from $\dot{\Lambda}^{k,\omega}|_S$ into $\dot{\Lambda}^{k,\omega}$ for S being a closed subset of $\mathbb{R}^n$.

Theorem 9.6. *The space $\dot{\Lambda}^{k,\omega}|_S$ admits a simultaneous extension to $\dot{\Lambda}^{k,\omega}$ if and only if there is a map L from $\dot{\Lambda}^{k,\omega}|_S \times \mathcal{K}_S$ into the linear space* $\mathrm{Ch}(\omega, k\,;\, S)$ *of (ω, k)-chains on S such that*

(a) *L is linear in the first argument;*

(b) *for every polynomial $p_Q(f) := L(f\,;\,Q)$,*

$$p_Q(c_Q) = f(c_Q).$$

Moreover, the norm of the corresponding linear extension operator is equivalent to

$$\|L\| := \sup\Big\{\big|\{p_Q(f)\}_{Q\in\mathcal{K}_S}\big|_{\mathrm{Ch}}\,;\, |f|_{\Lambda^{k,\omega}|_S} \le 1\Big\}$$

with the constants of equivalence depending only on k and n.

Proof. (Sufficiency) Replacing in formula (9.15) for the extension operator polynomials $p_{\widehat{Q}}$ by polynomials $p_{\widehat{Q}}(f)$, we obtain the required linear extension operator $\mathcal{E} : \dot{\Lambda}^{k,\omega}|_S \to \dot{\Lambda}^{k,\omega}$. The proof of sufficiency in Theorem 9.3 immediately gives the estimate

$$|\mathcal{E}f|_{\Lambda^{k,\omega}(\mathbb{R}^n)} \le c(k,n)\big|\{p_Q(f)\}_{Q\in\mathcal{K}_S}\big|_{\mathrm{Ch}}$$

which implies the inequality

$$\|\mathcal{E}\| \le c(k,n)\|L\|.$$

(Necessity) Let $\mathcal{E} : \dot{\Lambda}^{k,\omega}|_S \to \dot{\Lambda}^{k,\omega}$ be a linear bounded extension operator. To introduce the required map $L : \dot{\Lambda}^{k,\omega}|_S \times \mathcal{K}_S \to \mathcal{P}_{k-1,n}$ we denote by Pr_Q, for every cube $Q \subset \mathbb{R}^n$, a linear projection of the space $C(Q)$ onto the space $\mathcal{P}_{k-1,n}\big|_Q$ whose norm is bounded by the constant $d(k,n) := \sqrt{\dim \mathcal{P}_{k-1,n}}$, see Kadets and Snobar [KS-1971]. Now we use the so-called Lebesgue inequality implying

$$\|f - Pr_Q f\|_{C(Q)} \le (1 + d(k,n))E_k(Q\,;\,f). \tag{9.27}$$

In fact, let $p^* \in \mathcal{P}_{k-1,n}$ be such that

$$\|f - p^*\|_{C(Q)} = E_k(Q\,;\,f).$$

Then the left-hand side of (9.27) is bounded by

$$\|f - p^*\|_{C(Q)} + \big\|Pr_Q(f - p^*)\big\|_{C(Q)},$$

and the result follows.

We now define the required map L by

$$L(f\,;\,Q) := Pr_Q\mathcal{E}f - (Pr_Q\mathcal{E}f)(c_Q) + f(c_Q).$$

Then L is linear in f. Let us check that $\{L(f\,;\,Q)\}_{Q\in\mathcal{K}_S}$ belongs to $\mathrm{Ch}(\omega, k\,;\,S)$. In fact, for cubes $Q \subset Q'$ from $\mathcal{K}_S$ and f from the unit ball of $\dot{\Lambda}^{k,\omega}|_S$, we have by (9.27) and the definition of the seminorm of $\dot{\Lambda}^{k,\omega}$:

$$\begin{aligned}\max_Q\big|L(f\,;\,Q) - L(f\,;\,Q')\big| &\le 2\Big(\max_Q\big|Pr_Q\mathcal{E}f - \mathcal{E}f\big| + \max_{Q'}\big|Pr_{Q'}\mathcal{E}f - \mathcal{E}f\big|\Big)\\ &\le 4(1 + d(k,n))E_k(Q'\,;\mathcal{E}f).\end{aligned}$$

By Theorem 2.37 of Volume I, the left-hand side is bounded from above by

$$c(k,n)\|\mathcal{E}\| \cdot |f|_{\Lambda^{k,\omega}|_S} \cdot \omega(r_{Q'}).$$

Hence, $P := \{L(f\,;\,Q)\}_{Q\in\mathcal{K}_S}$ is an (ω, k)-chain on S satisfying, clearly, condition (b) and the inequality

$$|P|_{\mathrm{Ch}} \le c(k,n)\|\mathcal{E}\|.$$

This completes the proof. □

As a consequence of the results proved one can characterize in the same fashion spaces $C^\ell\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_S$ or their normed counterparts assuming that

$$C_\omega := \sup_{t>0}\frac{1}{\omega(t)}\int_0^t \frac{\omega(u)}{u}\,du < \infty. \tag{9.28}$$

The typical example of a k-majorant satisfying (9.28) is a power function $t \mapsto t^\lambda$, $t > 0$, where $0 < \lambda \le k$. Because of that such an ω is called a *quasipower k-majorant.*

We restrict ourselves to only one consequence of Theorem 9.3 and leave the proofs of other results of this kind to the reader.

Corollary 9.7. *Assume that a k-majorant ω satisfies condition* (9.28)*. A function f bounded on a closed set $S \subset \mathbb{R}^n$ belongs to $C^k\dot{\Lambda}^{s,\omega}|_S$ if and only if there is an $(\widetilde{\omega}, \ell + k)$-chain $P := \{p_Q\,;\,Q \in \mathcal{K}_S\}$, where $\widetilde{\omega} : t \mapsto t^\ell\omega(t)$, such that for every $Q \in \mathcal{K}_S$,*

$$f(c_Q) = p_Q(c_Q). \tag{9.29}$$

Moreover, the trace seminorm of f is equivalent to $|P|_{Ch}$ with constants of equivalence depending only on $\ell + k$, C_ω and n.

Proof. According to Theorem 2.10 of Volume I the seminorm

$$|f|_{C^\ell\Lambda^{k,\omega}(\mathbb{R}^n)} := \max_{|\alpha|=\ell}\sup_{t>0}\frac{\omega_k(t\,;\,D^\alpha f)}{\omega(t)}$$

is equivalent under condition (9.28) to that of the space $\dot{\Lambda}^{\ell+k,\widetilde{\omega}}(\mathbb{R}^n)$ with constants of equivalence depending only on C_ω, $\ell+k$ and n. Applying Theorem 9.3 to the latter space we obtain the result. □

9.1.3 Spaces $J^\ell\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$

Unlike the traces of spaces $C^\ell\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ with nonquasipower ω, those under consideration can be described for arbitrary ω in terms of polynomials chains. Before formulating the result due to Yu. Brudnyi and Shvartsman [BSh-1998] we recall that $J^\ell\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ is the linear space of ℓ-jets $\vec{f} := \{f_\alpha\ :\ \mathbb{R}^n \to \mathbb{R}\}_{|\alpha|\le\ell}$ such that for every $\vec{f}$ there exists a (unique) function $F_{\vec{f}} \in C^\ell(\mathbb{R}^n)$ satisfying

$$f_\alpha = D^\alpha F_{\vec{f}} \quad \text{for all} \quad |\alpha| \le \ell.$$

This space is equipped with the seminorm

$$|\vec{f}|_{J^\ell\Lambda^{k,\omega}(\mathbb{R}^n)} := \max_{|\alpha|=\ell}\sup_{t>0}\frac{\omega_k(t\,;\,f_\alpha)}{\omega(t)} \tag{9.30}$$

that turns it into a complete seminormed space.

The nonhomogeneous version, i.e., the space $J^\ell\Lambda^{k,\omega}(\mathbb{R}^n)$, is determined by the norm

$$\|\vec{f}\|_{J^\ell\Lambda^{k,\omega}(\mathbb{R}^n)} := \sup_{\mathbb{R}^n}|f_\alpha| + |\vec{f}|_{J^\ell\Lambda^{k,\omega}(\mathbb{R}^n)}. \tag{9.31}$$

We will consider only the case of homogeneous spaces; the proof of the result for the normed spaces requires trivial changes.

Theorem 9.8. *Let $\vec{f} := (f_\alpha)_{|\alpha|\le\ell}$ be an ℓ-jet on a closed set $S \subset \mathbb{R}^n$. Then $\vec{f}$ belongs to the trace space $J^\ell\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_S$ if and only if there exist a constant $\lambda > 0$ and a family $\{p_Q\}_{Q\in\mathcal{K}_S} \subset \mathcal{P}_{\ell+k-1,n}$ such that*

(i) *For every $|\alpha| \le \ell$ and $Q \in \mathcal{K}_S$,*

$$D^\alpha p_Q = f_\alpha \quad on \quad S.$$

(ii) *For every pair $Q' \subset Q$ of cubes from $\mathcal{K}_S$,*

$$\sup_{Q'}|p_Q - p_{Q'}| \le \lambda(\|c_Q - c_{Q'}\| + r_{Q'})^\ell\omega(r_Q). \tag{9.32}$$

Moreover, the trace norm of $\vec{f}$ is equivalent to inf λ *with constants of equivalence depending only on $\ell + k$ and n.*

Remark 9.9. Since the sum in the brackets is at most r_Q, conditions (ii) imply that the family $\{p_Q\}$ is an $(\widetilde{\omega}, k+\ell)$-chain on S with $\widetilde{\omega}(t) := t^\ell \omega(t)$.

Proof. (Necessity) Hereafter we denote for brevity $J^\ell \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ by $\dot{X}$; hence $|\cdot|_X$ denotes its seminorm and $\dot{X}|_S$ stands for the trace space in question.

Assuming that $|\vec{f}|_{\dot{X}|_S} < 1$ we should prove assertions (i) and (ii) with $\lambda = \lambda(\ell+k, n)$. By definition there exists a function $f \in C^\ell \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ such that for every $|\alpha| \le \ell$,

$$D^\alpha f = f_\alpha \quad \text{on} \quad S \tag{9.33}$$

and, moreover,

$$|f|_X \le 1. \tag{9.34}$$

To define the desired family of polynomials $\{p_Q\}$ we first, given $Q \in \mathcal{K}_S$ and $|\alpha| \le \ell$, introduce a polynomial p_Q^α of degree $k-1$ satisfying the inequality

$$\sup_Q |D^\alpha f - p_Q^\alpha| \le w(k,n)\, \omega_k(r_Q\,;\, D^\alpha f),$$

see Volume I, Theorem 2.37 on its existence. Due to (9.34) the right-hand side is bounded by $w(k,n)\,\omega(r_Q)$. Hence,

$$\sup_Q |D^\alpha f - p_Q^\alpha| \le O(1)\omega(r_Q), \tag{9.35}$$

where hereafter $O(1)$ stands for constants depending only on $\ell + k$ and n.

Now the required family of polynomials is introduced by

$$p_Q(x) := \sum_{|\alpha| \le \ell} \frac{f_\alpha(c_Q)}{\alpha!}(x - c_Q)^\alpha + R_Q(x, c_Q), \tag{9.36}$$

where the remainder is given by

$$R_Q(x,c) := \ell \sum_{|\alpha|=\ell} \frac{(x-c)^\alpha}{\alpha!} \int_0^1 (1-t)^{\ell-1} [p_Q^\alpha(c + t(x-c)) - p_Q^\alpha(c)]\, dt.$$

The remainder is clearly a polynomial of degree $\ell+k-1$. Further, a straightforward computation shows that $D_x^\alpha R_Q(x,c)|_{x=c} = 0$ if $|\alpha| \le k$; hence, condition (i) holds for this family.

To prove condition (ii) we first prove an auxiliary inequality for cubes $Q' := Q_{r'}(c)$, $Q := Q_r(c)$ with the common center and $r' \le r$. Using for f the Taylor formula at point c and equalities (9.33), (9.36) we obtain

$$\sup_{Q'} |f - p_Q| \le (r')^\ell \sum_{|\alpha|=\ell} \frac{1}{\alpha!}\, 2 \sup_{Q'} |\varphi_\alpha|$$

with $\varphi_\alpha := D^\alpha f - p_Q^\alpha$.

Estimating the right-hand side by (9.35) we then get the required inequality

$$\sup_{Q'} |f - p_Q| \le O(1)(r')^\ell \omega(r). \tag{9.37}$$

Now let $Q' := Q_{r'}(c') \subset Q := Q_r(c)$ be an arbitrary pair of cubes in $\mathcal{K}_S$ and $Q'' := Q_\rho(c)$, where $\rho := \|c - c'\| + r'$ $(\le r)$. Since $Q' \subset Q'' \subset Q$ and the last two cubes have the common center, inequality (9.37) implies

$$\sup_{Q'} |p_Q - p_{Q'}| \le \sup_{Q'} |f - p_{Q'}| + \sup_{Q''} |f - p_Q|$$
$$\le O(1)[(r')^\ell \omega(r') + \rho^\ell \omega(r)] \le O(1)(\|c - c'\| + r')^\ell \omega(r).$$

This proves assertion (ii).

(Sufficiency) Let an ℓ-jet $\vec{f} := (f_\alpha)_{|\alpha| \le \ell}$ be defined on S and let a family $\{p_Q\,;\, Q \in \mathcal{K}_S\} \subset \mathcal{P}_{\ell+k-1,n}$ satisfy assumptions (i), (ii) of the theorem. We should find a function $f : \mathbb{R}^n \to \mathbb{R}$ such that

(A) $f \in C^\ell \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ and
$$|f|_{C^\ell \Lambda^{k,\omega}(\mathbb{R}^n)} \le O(1)\lambda,$$
where λ is the constant in (ii);

(B) for every $|\alpha| \le \ell$,
$$D^\alpha f = f_\alpha \quad \text{on} \quad S.$$

We define the required f using the extension operator similar to that in (9.15). Specifically, we set

$$f(x) := \begin{cases} f_0(x), & \text{if } x \in S, \\ \sum\limits_{Q \in \mathcal{W}_S} (p_{\widehat{Q}} \varphi_Q)(x), & \text{if } x \in S^c. \end{cases} \tag{9.38}$$

Let us recall that $\widehat{Q}$ is the cube congruent to Q and centered at the point of S nearest to c_Q, and, moreover, $\{\varphi_Q\}_{Q \in \mathcal{W}_S}$ is a C^∞ partition of unity subordinate to the cover $\mathcal{W}_S^*$ whose functions satisfy the inequalities

$$\sup_{\mathbb{R}^n} |D^\alpha \varphi_Q| \le c(\alpha, n) r_Q^{-|\alpha|}$$

for every multi-index $\alpha \in \mathbb{Z}_+^n$, see (9.10).

We begin with the proof of claim (B). To accomplish this we need several auxiliary results presented below.

Lemma 9.10. *Let $Q' := Q_{r'}(c') \subset Q_r(c) =: Q$ be cubes in $\mathcal{K}_S$. Then for every α polynomials of the family $\{p_Q\}$ satisfy*

$$\sup_{Q'} |D^\alpha (p_Q - p_{Q'})| \le c(\alpha, n)\lambda[(r')^{\ell - |\alpha|} + r^{\ell - |\alpha|}]\,\omega(r).$$

Proof. If $Q'' := Q_{2r}(c')$, then $Q' \subset Q \subset Q''$ and the left-hand side is bounded by the Markov inequality as

$$c(\alpha, n)\big[\,(r')^{-|\alpha|} \sup_{Q'} |p_{Q'} - p_{Q''}| + r^{-|\alpha|} \sup_{Q} |p_Q - p_{Q''}|\,\big].$$

Using the inequality in (ii) we estimate the first supremum by $\lambda(r')^k\omega(2r)$ and the second by $r^k\omega(2r)$. Since $\omega(2r) \le 2^k\omega(r)$, the result follows. □

Now we define for every $|\alpha| \le k$ a function $\widetilde{f}_\alpha : \mathbb{R}^n \to \mathbb{R}$ that will be proved to be the mixed α derivative of the function f introduced by (9.38). Specifically, we set

$$\widetilde{f}_\alpha(x) := \begin{cases} f_\alpha(x), & \text{if } x \in S, \\ (D^\alpha f)(x), & \text{if } x \in S^c. \end{cases} \tag{9.39}$$

In the formulation of the next lemma we set

$$d(y) := d(y, S)\,(:= \inf_{x\in S} \|y - x\|).$$

Lemma 9.11. *Let $Q := Q_r(c) \in \mathcal{K}_S$ and $y \in Q$. Then for every $|\alpha| \le \ell$,*

$$|(\widetilde{f}_\alpha - D^\alpha p_Q)(y)| \le O(1)(d(y)^{\ell-|\alpha|} + r^{\ell-|\alpha|})\,\omega(r). \tag{9.40}$$

Moreover, the inequality holds also for every $|\alpha| > \ell$ if $y \in Q \setminus S$.

Proof. Let first $|\alpha| \le \ell$ and $y \in Q \cap S$, i.e., $d(y) = 0$. If $Q'' := Q_{2r}(y)$, then $Q \subset Q''$ and assumption (ii) and Lemma 9.10 imply

$$\begin{aligned} |(\widetilde{f}_\alpha - D^\alpha p_Q)(y)| &= |D^\alpha(p_{Q''} - p_Q)(y)| \le \sup_Q |D^\alpha(p_{Q''} - p_Q)| \\ &\le O(1)\lambda(r^{\ell-|\alpha|} + (2r)^{\ell-|\alpha|})\,\omega(2r) \le O(1)\lambda r^{\ell-|\alpha|}\,\omega(r). \end{aligned}$$

Hence, (9.40) holds for this case.

If now $y \in Q \setminus S$, i.e., $0 < d(y) < r$, then the sum in (9.38) defining $(D^\alpha f)(y)$ is extended to those Whitney cubes K whose associated functions φ_K are nonzero at y. It is essential that all of them are contained in dilation of a fixed cube given by

$$Q_y := Q_r(\widetilde{y}), \quad \text{where} \quad r = d(y)\,(:= d(y, S)) \tag{9.41}$$

and $\widetilde{y}$ is the closest to y point of S. Moreover, the radii of these cubes are equivalent to $d(y)$.

These facts will be established later but now we set for a while

$$Q' := Q_y, \quad Q'' := Q_{2r}(\widetilde{y})$$

and then write for an arbitrary α,

$$(\widetilde{f}_\alpha - D^\alpha p_Q)(y) = (\widetilde{f}_\alpha - D^\alpha p_{Q'})(y) + D^\alpha(p_{Q'} - p_{Q''})(y) + D^\alpha(p_{Q''} - p_Q)(y).$$

Since Q', $Q \subset Q''$, the last two terms are estimated as required in (9.40) by the direct use of Lemma 9.10. Hence, it remains to estimate the first term.

We will show that for the chosen y and α,

$$|\widetilde{f}_\alpha - D^\alpha p_{Q'}|(y) \le O(1)\lambda d(y)^{\ell-|\alpha|}\,\omega(r) \tag{9.42}$$

with $Q' := Q_y$ and in this way complete the proof of the lemma.

To accomplish this we introduce a set of Whitney cubes $K \in \mathcal{W}_S$ whose extensions $K^* := \frac{9}{8}K$ contain the point y; we denote this set by $\mathcal{S}_y$. Since every K with $\varphi_K(y) > 0$ belongs to $\mathcal{S}_y$ (for $\operatorname{supp}\varphi_K \subset K^*$), it suffices to deal with $\mathcal{S}_y$.

Due to the properties of Whitney cubes, see, e.g., Proposition 9.1 (e),

$$\operatorname{card}\mathcal{S}_y \le c(n). \tag{9.43}$$

Now we show that for every cube $K \in \mathcal{S}_y$,

$$r_K \approx d(y) \tag{9.44}$$

with numerical constants of equivalence.

Due to Proposition 9.1(b) and the fact that $y \in K$ and $\widetilde{y} \in S$ we have

$$r_K \le 4d(S,K) \le 4\|y - \widetilde{y}\| =: 4d(y).$$

Conversely, let $x' \in K$ and $x'' \in S$ be such that $\|x' - x''\| = d(S,K)$. Then

$$d(y) \le \|\widetilde{y} - x''\| + \|x' - x''\| + \|x'' - y\| \le \|y - x''\| + d(S,K) + r_K.$$

Since the metric projection is 1-Lipschitz, the first term on the right-hand side is at most $\|y - x'\| \le r_K$. Further, by Proposition 9.1 (b) the second term is bounded by $4r_K$. Hence,

$$d(y) \le 6r_K$$

and (9.44) is done. In the same way, we prove that for $K \in \mathcal{S}_y$,

$$\widehat{K} \subset 8Q_y, \tag{9.45}$$

where $\widehat{K}$ is recalling to be a congruent to K cube centered at a point of S closest to c_K.

Actually, by the property of metric projection

$$\|\widetilde{y} - c_{\widehat{K}}\| \le \|y - c_K\| \le r_K.$$

Moreover, due to Proposition 9.1(b),

$$r_{\widehat{K}} := r_K \le 4d(K,S) \le 4d(y).$$

Hence, the cube $4Q_y$ contains the center of $c_{\widehat{K}}$ of $\widehat{K}$ and $8Q_y$ contains $\widehat{K}$.

Now we proceed with the proof of (9.42) setting as there $Q' := Q_y$. Since $\{\varphi_K\}_{K\in\mathcal{W}_S}$ is a partition of unity, the definitions of f and $\widetilde{f}_\alpha$ lead to the inequality

$$|\widetilde{f}_\alpha - D^\alpha p_{Q'}| \le \left| D^\alpha \left(\sum_{K\in\mathcal{W}_S} \varphi_K(p_{\widehat{K}} - p_{Q'}) \right) \right|.$$

Differentiating by the Leibnitz rule and noting that $0 \le \varphi_K \le 1$, $\operatorname{supp}\varphi_K \subset K^*$ and $|D^\alpha \varphi_K| \le c(\alpha,n) r_K^{-|\alpha|}$, see (9.10), we bound the right-hand side here by

$$\begin{aligned} &\sum_{K\in\mathcal{S}_y} \sum_{\alpha=\beta+\gamma} (r_K)^{-|\beta|} |D^\gamma(p_{\widehat{K}} - p_{Q'})|(y) \\ &\le O(1)(\operatorname{card}\mathcal{S}_y) \max_{K\in\mathcal{S}_y} \sum_{\alpha=\beta+\gamma} \sup_{Q'} |D^\gamma(p_{\widehat{K}} - p_{Q'})|. \end{aligned}$$

Exploiting (9.43)-(9.45) and setting $Q'' := 8Q'$ we further obtain

$$\begin{aligned} |\widetilde{f}_\alpha - D^\alpha p_{Q'}|(y) \le O(1) \max_{K\in\mathcal{S}_y} &\sum_{\alpha=\beta+\gamma} d(y)^{-|\beta|} \\ &\times \big[\sup_{Q'} |D^\gamma(p_{\widehat{K}} - p_{Q''})| + \sup_{Q'} |D^\gamma(p_{Q''} - p_{Q'})|\big]. \end{aligned} \tag{9.46}$$

Due to Lemma 9.10 the second term in the square brackets is bounded by

$$O(1)(r_{Q'}^{\ell-|\gamma|} + r_{Q''}^{\ell-|\gamma|})\omega(r_{Q''}) \le O(1)\lambda d(y)^{\ell-|\gamma|}\omega(d(y)).$$

Since $d(y) \le r$, we then have

$$\sup_{Q'} |D^\gamma(p_{Q''} - p_{Q'})| \le O(1) d(y)^{\ell-|\gamma|}\omega(r). \tag{9.47}$$

In turn, due to the Remez-type inequality of Lemma 9.5 and (9.44) the first term in the square brackets is bounded by

$$\begin{aligned} \sup_{Q''} |D^\gamma(p_{\widehat{K}} - p_{Q''})| &\le O(1) \left(\frac{|Q''|}{|\widehat{K}|} \right)^{\ell+k-1} \cdot \sup_{\widehat{K}} |D^\gamma(p_{\widehat{K}} - p_{Q''})| \\ &\le O(1) \sup_{\widehat{K}} |D^\gamma(p_{\widehat{K}} - p_{Q''})|. \end{aligned}$$

Now we estimate the right-hand side of (9.47) by applying subsequently the Markov inequality, assertion (ii) of the theorem and (9.44) to bound it by

$$O(1)\lambda r_{\widehat{K}}^{-|\gamma|}(r_{\widehat{K}} + r_{Q''})^\ell \omega(r_{Q''}) \le O(1)\lambda d(y)^{\ell-|\gamma|}\omega(r).$$

Inserting this and (9.47) into (9.46) we finally obtain

$$
\begin{aligned}
|\widetilde{f}_\alpha - D^\alpha p_{Q'}|(y) &\le O(1) \max_{\alpha=\beta+\gamma} \max_{K\in\mathcal{S}_y} \left(\sum_{\alpha=\beta+\gamma} d(y)^{|\beta|} d(y)^{\ell-|\gamma|} \right) \omega(r) \\
&\le O(1) d(y)^{\ell-|\alpha|}\omega(r). \qquad \square
\end{aligned}
$$

Now, we will show that the extension f given by (9.38) has all derivatives up to order k on $\mathbb{R}^n$ and that

$$D^\alpha f = \widetilde{f}_\alpha \quad \text{for all} \quad |\alpha| \le \ell. \tag{9.48}$$

Clearly, it suffices to check this only at points of S. We will prove this by induction on $|\alpha|$ starting with the trivial case $|\alpha| = 0$.

Let (9.48) be true at points of S for all α with $|\alpha| = j \le \ell - 1$. Setting $\alpha^i := \alpha + e_i$, where $\{e_i\}_{1\le i\le n}$ is the standard basis of $\mathbb{R}^n$, we show that for every $x \in S$ and $y \in \mathbb{R}^n$ tending to x,

$$\widetilde{f}_\alpha(y) = f_\alpha(x) + \sum_{i=1}^{n} f_{\alpha^i}(x)(y_i - x_i) + o(\|x-y\|). \tag{9.49}$$

This clearly means that $D_i\widetilde{f}_\alpha$ exists at x and equals $f_{\alpha^i}(x)\,(:= \widetilde{f}_{\alpha^i}(x)$ by (9.39)). Together with the induction hypothesis this yields

$$(D^{\alpha^i} f)(x) = \widetilde{f}_{\alpha^i}(x) \quad \text{for} \quad 1 \le i \le n,$$

i.e., (9.48) holds at the points of S for $|\alpha| = j+1$ as well.

To prove (9.49) we set $Q := Q_r(x)$, where $r := \|y-x\|$, and apply inequalities (9.40) and $d(y) \le \|x-y\|$ to obtain

$$|\widetilde{f}_\alpha - D^\alpha p_Q|(y) \le O(1)\lambda r^{\ell-|\alpha|}\omega(r).$$

Since $\ell - |\alpha| \ge 1$ and $\omega(r) = o(1)$ as $r \to 0$, the right-hand side is also $o(r)$. Moreover, by assertion (i) of the theorem $D^{\alpha^i} p_Q(x) = f_{\alpha^i}(x)$, $1 \le i \le k$, and therefore the difference

$$\widetilde{f}_\alpha(y) - f_\alpha(x) - \sum_{i=1}^{n} f_{\alpha^i}(x)(y_i - x_i) = \widetilde{f}_\alpha(y) - (D^\alpha p_Q)(x) - \langle(\operatorname{grad} D^\alpha p_Q)(x), y-x\rangle$$

is estimated by

$$
\begin{aligned}
&|\widetilde{f}_\alpha - D^\alpha p_Q|(y) + |(D^\alpha p_Q)(x) - \langle(\operatorname{grad} D^\alpha p_Q)(x), y-x\rangle| \\
&\le o(r) + \sum_{|\beta|\ge 2} \frac{1}{\beta!} |(D^{\alpha+\beta} p_Q)(x)|\, r^{|\beta|}.
\end{aligned}
$$

Hence, to obtain the desired result it remains to show that for every $|\beta| \ge 2$,

$$r^{|\beta|}(D^{\alpha+\beta}p_Q)(x) = o(r) \quad \text{as} \quad r := \|y - x\| \to 0. \tag{9.50}$$

If $|\alpha + \beta| \le k$, the left-hand side equals $r^{|\beta|} f_{\alpha+\beta}(x)$ and (9.50) holds. Now, let $|\alpha+\beta| > k$ and $Q_i := Q_{r_i}(x)$, where $r_i := 2^i r$, $i \in \mathbb{Z}_+$. Assuming that $r < 1/2$ we define an integer $j = j(r)$ by the condition $r_j \le 1 < r_{j+1}$. Then the left-hand side of (9.50) is at most

$$r^{|\beta|} \sum_{i=0}^{j-1} |D^{\alpha+\beta}(p_{Q_i} - p_{Q_{i+1}})|(x) + r^{|\beta|} \cdot |D^{\alpha+\beta} p_{Q_j}|(x).$$

The last term is $o(r)$ as $r \to 0$ while the previous sum is estimated by the Markov inequality and assertion (ii) of the theorem by

$$O(1)\lambda r^{|\beta|} \sum_{i=0}^{j-1} r_i^{-|\alpha|-|\beta|} \sup_{Q_i} |p_{Q_i} - p_{Q_{i+1}}| \le O(1)\lambda r^{|\beta|} \sum_{i=0}^{j-1} r_i^{\ell-|\alpha|-|\beta|} \omega(r_{i+1}).$$

Since ω is nondecreasing and $r_j \le 1$, the last sum is bounded by

$$O(1)\lambda \int_r^2 t^{\ell-|\alpha|} \frac{\omega(t)}{t^{|\beta|}} \frac{dt}{t}.$$

As $k - |\alpha| \ge 1$, the inequalities obtained imply that

$$r^{|\beta|}|D^{\alpha+\beta}p_Q|(x) \le O(1)\lambda r^{|\beta|} \int_r^2 \frac{\omega(t)}{t^{|\beta|}}\, dt + o(r) \quad \text{as} \quad r \to 0.$$

Finally, since $|\beta| > 1$, we have for a fixed $\eta > 0$,

$$\varlimsup_{r\to 0} r^{|\beta|-1} \int_r^2 \frac{\omega(t)}{t^{|\beta|}}\, dt \le \varlimsup_{r\to 0} r^{|\beta|-1} \int_r^\eta \frac{\omega(t)}{t^{|\beta|}}\, dt + \varlimsup_{r\to 0} r^{|\beta|-1} \int_\eta^2 \frac{\omega(t)}{t^{|\beta|}} \le \frac{\omega(\eta)}{|\beta|-1}.$$

Since $\omega(\eta) \to 0$ as $\eta \to 0$, the left-hand side is zero and therefore

$$r^{|\beta|}(D^{\alpha+\beta}p_Q)(x) = o(r) \quad \text{as} \quad r \to 0,$$

as required.

Hence, from the induction hypothesis for $|\alpha| \le j$ $(\le \ell - 1)$ follows (9.49) and therefore the f is an ℓ-times differentiable function and

$$D^\alpha f = f_\alpha \quad \text{on} \quad S \quad \text{for all} \quad |\alpha| \le \ell.$$

It remains to show that f belongs to the space $C^\ell \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ and its norm in this space is bounded by $O(1)\lambda$. In other words, we should prove that

$$\max_{|\alpha|=\ell} \sup_{t>0} \frac{\omega_k(t\,; D^\alpha f)}{\omega(t)} \le O(1)\lambda.$$

Because of the inequality

$$\omega_k(t\,;\,g) \le O(1)\sup\{E_k(Q\,;\,g)\,;\,|Q| = t^n\}, \tag{9.51}$$

see (9.4), it suffices to estimate the corresponding local approximation of $D^\alpha f$ with $|\alpha| = \ell$. This will be done by the argument already used for the proof of Theorem 9.3. Specifically, we as there decompose the set $\mathcal{K}$ of all cubes in $\mathbb{R}^n$ into subsets $\mathcal{K}^I$ and $\mathcal{K}^{II}$, where the former contains all cubes Q such that

$$Q \subset S^c \quad \text{and} \quad Q \subset K^*\,(:= \frac{9}{8}K);$$

here $K \in \mathcal{W}_S$ is a (unique) Whitney cube containing its center c_Q.

Lemma 9.12. *If $Q \in \mathcal{K}^{II}$, then for $|\alpha| = \ell$,*

$$E_k(Q\,;\,D^\alpha f) \le O(1)\,\omega(r_Q).$$

Proof. Let first $c_Q \in S$. Since $D^\alpha p_Q$ is a polynomial of degree $k-1$, inequality (9.40) of Lemma 9.11 implies that

$$E_k(Q\,;\,D^\alpha f) \le \sup_Q |D^\alpha f - D^\alpha p_Q| \le O(1)\lambda\,\omega(r_Q). \tag{9.52}$$

Now let $c_Q \notin S$ and x_Q be a nearest to c_Q point of S. If $\widetilde{Q}$ is the smallest cube centered at x_Q and containing Q, then $\widetilde{Q} \in \mathcal{K}^{II}$ and by Lemma 9.4 from the proof of Theorem 9.3,

$$\operatorname{diam}\widetilde{Q} \le O(1)\operatorname{diam} Q.$$

This and inequality (9.52) with $\widetilde{Q}$ in place of Q imply

$$E_k(Q\,;\,D^\alpha f) \le E_k(\widetilde{Q}\,;\,f) \le O(1)\lambda\,\omega(r_{\widetilde{Q}}) \le O(1)\lambda\,\omega(r_Q). \qquad \square$$

Now let $Q \in \mathcal{K}^I$ and let $\widetilde{Q}$ be defined as above.

Lemma 9.13. *If $|\alpha| = \ell$, then*

$$E_k(Q\,;\,D^\alpha f) \le O(1)\lambda\,(r_Q)^k\,\frac{\omega(r_{\widetilde{Q}})}{r_{\widetilde{Q}}}. \tag{9.53}$$

Proof. Since $Q \subset S^c$ by definition, $D^\alpha f|_Q \in C^\infty(Q)$, see (9.38). Applying subsequently the inequalities of Theorems 2.37 of Volume I, see, e.g., (9.3), and Theorem 2.7 (e) of Volume I, we then obtain

$$E_k(Q\,;\,D^\alpha f) \le O(1)\sup_{x,x+kh\in Q}|\Delta_h^k D^\alpha f|(x) \le O(1)(r_Q)^k \sum_{|\beta|=k}\sup_Q |D^{\alpha+\beta} f|.$$

Since $|\alpha+\beta| = \ell + k$ and $\deg p_{\widetilde{Q}} = k + \ell - 1$, inequality (9.40) for $y \in Q\,(\subset S^c)$ yields

$$|D^{\alpha+\beta} f|(y) = |D^{\alpha+\beta} f - D^{\alpha+\beta} p_{\widetilde{Q}}|(y) \le O(1)\lambda[d(y)^{-k} + (r_{\widetilde{Q}})^{-k}]\,\omega(r_{\widetilde{Q}}).$$

Along with the previous estimate this would lead to the required result, if we could prove that

$$r_{\widetilde{Q}} \le O(1)d(y) \quad \text{for} \quad y \in Q. \tag{9.54}$$

To this end we write for the center of $\widetilde{Q}$ (a point of S nearest to c_Q) and a point $z \in Q$,

$$\|z - c_{\widetilde{Q}}\| \le r_Q + d(c_Q) \le 2r_Q + d(y).$$

Since $\widetilde{Q}$ is the smallest cube centered at $c_{\widetilde{Q}}$ and containing Q, this, in turn, implies

$$r_{\widetilde{Q}} \le 2r_Q + d(y).$$

It remains to prove that $r_Q \le O(1)d(y)$ to obtain (9.54).

To this aim we let $\widehat{Q}$ denote a cube from $\mathcal{W}_S$ such that

$$Q \subset (\widehat{Q})^* \quad \text{and} \quad c_Q \in \widehat{Q};$$

it exists by the definition of $\mathcal{K}^I$. If the chosen point y belongs to another Whitney cube, say $P \in \mathcal{W}_S$, then $P \cap \widehat{Q} \ne \emptyset$ and therefore $P^* \cap (\widehat{Q})^* \ne \emptyset$, see Proposition 9.1 (c). Assertion (d) of this proposition then yields

$$\operatorname{diam} P \approx \operatorname{diam} \widehat{Q}$$

with numerical constants of equivalence.

Finally, we conclude that

$$\begin{aligned} r_Q &\le r_{(\widehat{Q})^*} \le O(1)\operatorname{diam} Q \le O(1)\operatorname{diam} P \\ &\le O(1)d(P,S) \le O(1)d(y,S) := O(1)\,d(y). \end{aligned}$$

This proves (9.54) and the lemma. □

Now we derive from this lemma the required estimate for $Q \in \mathcal{K}^I$. Since $r_Q \le r_{\widetilde{Q}}$ and the function $t \mapsto \frac{\omega(t)}{t^k}$, $t > 0$, is nonincreasing, the right-hand side of (9.53) is at most

$$O(1)\lambda\,(r_Q)^k\,\frac{\omega(r_Q)}{(r_Q)^k} = O(1)\lambda\,\omega(r_Q).$$

This and Lemma 9.12 for every cube Q of $\mathbb{R}^n$ imply that

$$E_k(Q\,;\,D^\alpha f) \le O(1)\lambda\,\omega(r_Q)$$

which along with the inequality

$$\omega_k(t\,;\,D^\alpha f) \le O(1) \sup_{|Q|=t^n} E_k(Q\,;\,D^\alpha f),$$

see (9.51), gives the desired estimate.

Summing up, we have proved that there exists an extension $f \in C^\ell \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ with seminorm $O(1)\lambda$ such that for every $|\alpha| \le \ell$,

$$D^\alpha f = f_\alpha \quad \text{on} \quad S.$$

This proves sufficiency and Theorem 9.8. □

In the formulation of the next result, we again by $\dot{X}$ denote $J^\ell \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ and use the linear space of polynomial chains $Ch(\omega, d, S)$ consisting of families $\{p_Q\}_{Q\in\mathcal{K}_S} \subset \mathcal{P}_{d-1,n}$ satisfying the condition

$$\max_{Q'} |p_Q - p_{Q'}| \le \lambda\,\omega(r_Q)$$

for every pair $Q' \subset Q$ of cubes in $\mathcal{K}_S$.

Theorem 9.14. *Let $\widetilde{\omega}(t) := t^\ell \omega(t)$, $t > 0$, where ω is a k-majorant. Assume that there exists a linear map $T : \dot{X}|_S \to Ch(\widetilde{\omega}, \ell + k, S)$ and $T\vec{f}$ satisfies for every $\vec{f}$ conditions* (i) *and* (ii) *of Theorem* 9.8 *for some $\lambda = \lambda(\vec{f})$ such that $\lambda(\vec{f}) \le \gamma|\vec{f}|_{X|_S}$ with a constant $\gamma > 0$ independent of $\vec{f}$.*

Then there exists a linear extension operator from $\dot{X}|_S$ into $\dot{X}$ with norm bounded by $O(1)\gamma$.

Proof. The extension operator in (9.38) written for the family $T\vec{f} \in Ch(\widetilde{\omega}, \ell+k, S)$ is clearly linear. Since $T\vec{f}$ satisfies conditions (i), (ii), the value of this operator at $\vec{f}$ is bounded by $O(1)\lambda(\vec{f}) \le O(1)\gamma\,|\vec{f}|_{\dot{X}|_S}$. Hence, its norm is at most $O(1)\gamma$. □

Remark 9.15. It is interesting to compare the result of Theorem 9.8 for $k = 1$, i.e., for $\dot{C}^{\ell,\omega}(\mathbb{R}^n)$, with the Whitney-Glaeser Theorem 2.19 of Volume I. In this case, condition (i) determines a unique polynomial p_Q of degree k that can be presented as

$$p_Q(x) = \sum_{|\alpha|\le k} \frac{f_\alpha(c_Q)}{\alpha!}(x - c_Q)^\alpha,$$

i.e., p_Q coincides with the Taylor polynomial $T^k_{c_Q}\vec{f}$ used in Theorem 2.19 of Volume I. Since p_Q now depends only on c_Q, condition (ii) may be presented in the following equivalent form:

For every pair $x, y \in S$ and for $z \in \mathbb{R}^n$,

$$|T^k_x\vec{f} - T^k_y\vec{f}|(z) \le \lambda(\|x - y\| + \|x - z\|)^\ell \omega(\|x - y\| + \|x - z\|). \tag{9.55}$$

To obtain this from (ii) it suffices to take there

$$Q' := Q_{\|x-z\|}(z) \subset Q := Q_{\|x-y\|+\|x-z\|}(y).$$

Conversely, if (9.55) holds and we choose $Q' := Q_{r'}(x) \subset Q := Q_r(y)$, then for every $z \in Q'$,

$$\|x-y\| + \|x-z\| \le \|x-y\| + r' \le r$$

and therefore (9.55) implies

$$\max_{Q'} |T_x^k \vec{f} - T_y^k \vec{f}| \le \lambda(\|x-y\| + r')^\ell \omega(r).$$

Let us also note that $\vec{f} \mapsto T_x^k \vec{f}$ is a linear map. Hence, Theorem 9.8 and Theorem 9.14 imply the following result.

A function $f : S \to \mathbb{R}$ belongs to the trace space $\dot{C}^{k,\omega}(\mathbb{R}^n)|_S$ with norm bounded by $O(1)\lambda$ if and only if condition (9.55) *holds.*

Moreover, there exists a linear extension operator from $\dot{C}^{k,\omega}(\mathbb{R}^n)|_S$ into $\dot{C}^{k,\omega}(\mathbb{R}^n)$ with norm bounded by $O(1)$.

Using the Markov inequality one can prove equivalence of condition (9.55) to the Taylor chain condition of the Whitney-Glaeser Theorem 2.19 of Volume I.

Remark 9.16. To formulate a version of Theorem 9.8 or Theorem 9.14 for the normed space $C^\ell \Lambda^{k,\omega}(\mathbb{R}^n)$ it suffices to add to conditions (i), (ii) one more, namely,

(iii) $\max_{|\alpha| \le \ell} \sup_S |f_\alpha| \le \lambda.$

The proofs follow with small changes the same arguments and may be remained to the reader.

9.1.4 Local versions of the extension results

We present the local versions of Theorems 9.3 and 9.6 leaving the similar formulations and proofs for Theorem 9.8 and Theorem 9.14 to the reader.

Let $\mathcal{K}_S^\delta \subset \mathcal{K}_S$ be the family of cubes centered at S and of side lengths less than $\delta > 0$. Further, a family

$$P := \{p_Q\}_{Q \in \mathcal{K}_S^\delta} \subset \mathcal{P}_{k-1,n}$$

is said to be a *polynomial* (ω, k)-*chain of size* δ if for some $c > 0$ and every pair $Q \subset Q'$ of cubes from $\mathcal{K}_S^\delta$,

$$\max_Q |p_{Q'} - p_Q| \le c\omega(r_{Q'}).$$

The linear space of these chains is denoted by $\mathrm{Ch}(\omega, k, \delta\,;\, S)$; a seminorm of this space is defined by

$$|P|_{\mathrm{Ch}}^\delta := \inf c.$$

Finally, S_δ denotes the δ-neighborhood of S and $\dot{\Lambda}^{k,\omega}\big|_S^\delta$ denotes the trace space given by

$$\dot{\Lambda}^{k,\omega}\big|_S^\delta := \{g|_S\,;\ g \in \dot{\Lambda}^{k,\omega}(S_\delta)\}$$

equipped with the standard trace seminorm

$$|f|_{\Lambda^{k,\omega}|_S^\delta} := \inf\big\{|g|_{\Lambda^{k,\omega}(S_\delta)}\,;\ g|_S = f\big\}.$$

Theorem 9.17. *Let f be a locally bounded function on a closed set $S \subset \mathbb{R}^n$. Then f belongs to the trace space $\dot{\Lambda}^{k,\omega}\big|_S^\delta$ if and only if there is an (ω,k)-chain $P := \{p_Q\}_{Q\in\mathcal{K}_S^\delta}$ such that for every cube $Q \in \mathcal{K}_S^\delta$,*

$$f(c_Q) = p_Q(c_Q).$$

Moreover, for some constants of equivalence independent of f,

$$|f|_{\Lambda^{k,\omega}|_S^\delta} \approx \inf |P|_{\mathrm{Ch}}^\delta.$$

Proof. It suffices to repeat line by line the derivation of Theorem 9.3. The only change is the formula for the extension operator , see (9.15), which should be modified to

$$\tilde{f}(x) := \begin{cases} f(x), & \text{if } x \in S, \\ \sum\limits_{Q\in\mathcal{W}_S^\delta} p_{\widehat{Q}}\varphi_Q, & \text{if } x \in S_\delta\backslash S, \end{cases}$$

where $\mathcal{W}_S^\delta$ is the subset of the family of Whitney's cubes $\mathcal{W}_S$ containing only those of side lengths less than δ. □

The analogous argument leads to the local version of Theorem 9.6.

Theorem 9.18. *The trace space $\dot{\Lambda}^{k,\omega}|_S$ admits a simultaneous extension into the space $\dot{\Lambda}^{k,\omega}(S_\delta)$ if and only if there is a map $L : \dot{\Lambda}^{k,\omega}\big|_S^\delta \times \mathcal{K}_S^\delta$ into the linear space $\mathrm{Ch}(\omega, k, \delta\,;\ S)$ such that*

(a) *L is linear in the first argument;*

(b) *for every polynomial $p_Q(f) := L(f\,;\ Q)$,*

$$p_Q(c_Q) = f(c_Q).$$

Moreover, the norm of the corresponding linear extension operator is equivalent to the norm

$$\|L\| := \sup\big\{\big|\{p_Q(f)\}_{Q\in\mathcal{K}_S^\delta}\big|_{\mathrm{Ch}}^\delta\big\}$$

where the supremum is taken over all f satisfying $|f|_{\Lambda^{k,\omega}|_S} \le 1$.

9.2 Traces to Markov sets

As the first application of the previous results we prove several theorems concerning smooth extensions from a rather large class of closed subsets in $\mathbb{R}^n$. The class introduced and studied by Johnson and Wallin [JW-1984, Ch. 2] contains, e.g., closures of bounded Lipschitz domains, sufficiently massive Ahlfors regular sets and self-similar fractals.

9.2.1 Markov sets

We present several results of the book [JW-1984] in a form convenient for our aim.

Definition 9.19. A closed subset $S \subset \mathbb{R}^n$ is said to be Markov if for some constant $c = c(n, S) \geq 1$ and every cube $Q \in \mathcal{K}_S$ and polynomial p of degree 1

$$\|\nabla p\| \leq \frac{c}{r_Q} \max_{Q \cap S} |p| . \tag{9.56}$$

We denote the class of sets satisfying (9.56) by $\mathrm{Mar}(\mathbb{R}^n)$. Its definition is clearly independent of the choice of norm for the vector $\nabla p := (D_1 p, \ldots, D_n p) \in \mathbb{R}^n$.

Example 9.20. The classical Markov inequality for univariate polynomials implies that a closed cube is a Markov set. More generally, let $S \subset \mathbb{R}^n$ be Ahlfors n-regular meaning (according to Definition 4.14 of Volume I) that for some constant $c > 0$ and all cubes $Q \in \mathcal{K}_S$ the Lebesgue n-measure of $Q \cap S$ satisfies

$$|Q \cap S| \geq c|Q|.$$

Let us show that S is a Markov set. In fact, from the classical Markov inequality and that of Lemma 9.5 we get for $p \in \mathcal{P}_{1,n}$,

$$\|\nabla p\| \leq \frac{1}{r_Q} \max_{Q} |p| \leq \frac{\gamma(n)}{r_Q} \cdot \frac{|Q|}{|Q \cap S|} \max_{Q \cap S} |p|$$

and (9.56) follows.

In particular, the closure of a bounded Lipschitz domain is Markov.

The subsequent two results describe the basic characteristics of Markov sets.

Theorem 9.21. *Let* $S \in \mathrm{Mar}(\mathbb{R}^n)$*. Then the following holds:*

(a) *For every cube* $Q \in \mathcal{K}_S$ *and polynomial* p *of degree* $k \geq 1$,

$$\max_{Q} |p| \leq c \max_{Q \cap S} |p| \tag{9.57}$$

where $c = c(k, n, S)$.

(b) *Let μ be a doubling measure on S regarded as a metric subspace of $\mathbb{R}^n$, and $0 < a \le \infty$. Then for every $Q \in \mathcal{K}_S$ and $p \in \mathcal{P}_{k,n}$,*

$$\max_{Q\cap S} |p| \le c \left\{ \frac{1}{\mu(Q)} \int_Q |p|^a \, d\mu \right\}^{\frac{1}{a}}, \tag{9.58}$$

where c depends also on the dilation constant of μ and on $a^* := \min(1, a)$.

Hereafter we, for brevity, write $\mu(Q)$ instead of $\mu(Q \cap S)$.

Proof. (a) We begin with

Lemma 9.22. *Assertion* (a) *is true if and only if for every $Q \in \mathcal{K}_S$ and $p \in \mathcal{P}_{k,n}$,*

$$\max_{Q\cap S} \|\nabla p\| \le \frac{c}{r_Q} \max_{Q\cap S} |p|. \tag{9.59}$$

Here and throughout this section $c, c_1, \dots$ denote positive constants depending only on k, n and S. They may change from line to line.

Proof. We first derive (9.57) from (9.59). Let $Q := Q_r(x_0) \in \mathcal{K}_S$, i.e., $x_0 \in S$ and $r < \operatorname{diam} S$, and $p(x) := \sum_{|\alpha|\le k} p_\alpha (x - x_0)^\alpha$. It is true that

$$\nu_Q(p) := \sum_{|\alpha|\le k} |p_\alpha| r^{|\alpha|} \le c \max_{Q\cap S} |p|. \tag{9.60}$$

In fact, repeatedly applying (9.59) we get

$$|p_\alpha| = \frac{|D^\alpha p(x_0)|}{\alpha!} \le c r^{-|\alpha|} \max_{Q\cap S} |p|$$

and (9.60) follows.

This and the Mean Value Theorem then imply

$$\max_Q |p| \le |p(x_0)| + r \max_Q \|\nabla p\| \le |p_0| + k\sqrt{n} \sum_{|\alpha|\le k} |p_\alpha| r^{|\alpha|} \le c \max_{Q\cap S} |p|,$$

as required.

Conversely, the classical Markov inequality and (9.57) lead to the required result:

$$\max_{Q\cap S} \|\nabla p\| \le \frac{c}{r} \max_Q |p| \le \frac{c_1}{r} \max_{Q\cap S} |p|. \qquad \square$$

According to the lemma we must prove that inequality (9.57) holds for $Q_r(x_0) \in \mathcal{K}_S$ and $p(x) := \sum_{|\alpha|\le k} p_\alpha (x - x_0)^\alpha$ with $k \ge 2$ assuming that it or, equivalently, inequality (9.59) is true for polynomials of degree 1.

Due to the evident inequality $\max_Q |p| \le \nu_Q(p)$ it suffices to find a point $\hat{x} \in Q \cap S$ such that

$$\nu_Q(p) \le c\, |p(\hat{x})|. \tag{9.61}$$

We may assume that

$$|p_0| \le \sum_{\alpha \ne 0} |p_\alpha| r^{|\alpha|}, \tag{9.62}$$

since otherwise (9.61) holds with $\hat{x} = x_0$ and $c = 2$.

We prove (9.61) by induction on k. Applying the induction hypothesis to the components of ∇p (polynomials of degree $k-1$) and taking into account (9.62), we have for $Q' := \frac{1}{2}Q = Q_{\frac{1}{2}r}(x_0)$,

$$\frac{\nu_Q(p)}{r} \le \sum_{i=1}^n \nu_Q(D_i p) \le c \max_{Q' \cap S} \|\nabla p\|.$$

Then we take a point $y \in Q' \cap S$ such that

$$\|\nabla p(y)\| \ge c^{-1} \frac{\nu_Q(p)}{r}$$

and introduce a cube

$$Q'' := Q_{\lambda r}(y) \subset Q_r(x_0)$$

where $\lambda \in (0, \frac{1}{2}]$ will be chosen later.

Further, we set

$$p_1(x) := (x-y) \cdot \nabla p(y), \ \ x \in \mathbb{R}^n.$$

Since p_1 is of degree 1 and

$$|\nabla p_1| = |\nabla p(y)| \ge c^{-1} \frac{\nu_Q(p)}{r},$$

the definition of Markov sets implies for some $\hat{x} \in Q'' \cap S$,

$$|p_1(\hat{x})| \ge c_1 \lambda r \|\nabla p_1\| \ge \frac{c_1}{c} \lambda \nu_Q(p) =: c_2 \lambda \nu_Q(p).$$

Using the Taylor expansion at the point y we then get for this $\hat{x}$,

$$p(\hat{x}) = p(y) + p_1(\hat{x}) + p_2(\hat{x})$$

where the remainder satisfies

$$|p_2(\hat{x})| \le c(\lambda r)^2 \max_Q \left(\sum_{|\alpha|=2} |D^\alpha p| \right) \le c(\lambda r)^2 \frac{\nu_Q(p)}{r^2} = c\lambda^2 \nu_Q(p).$$

Now we choose λ so small that the right-hand side here becomes less than $c_2 \frac{\lambda}{4} \nu_Q(p)$. We may also assume that

$$|p(y)| \le c_2 \cdot \frac{\lambda}{4} \cdot \nu_Q(p),$$

since otherwise (9.61) holds with $\hat{x} = y$.

Combining the estimates obtained we finally get

$$|p(\hat{x})| \geq |p_1(\hat{x})| - |p(y)| - |p_2(\hat{x})| \geq c_2 \nu_Q(p) \cdot \frac{\lambda}{2}.$$

This proves (9.61) and, hence, assertion (a) of the theorem.

(b) To prove (9.58) we choose a point $y \in Q' \cap S$ (where $Q' := \frac{1}{2}Q$) such that

$$\max_{Q \cap S} |p| \leq c|p(y)| ;$$

its existence follows from (9.57).

Then for every point $x \in Q'' := Q_{\lambda r}(y)$ where $\lambda \in (0, \frac{1}{2}]$ will be chosen later we get by (9.59) and the embedding $Q'' \subset Q$,

$$|p(x) - p(y)| \leq \lambda r \max_{Q} \|\nabla p\| \leq c \cdot \lambda r \cdot \frac{1}{r} \max_{Q \cap S} |p| \leq c_1 \lambda \, |p(y)|.$$

Choosing now $\lambda = \frac{1}{2c_1}$ we conclude that

$$\min_{Q''} |p| \geq \frac{1}{2} |p(y)| \geq \frac{c}{2} \max_{Q \cap S} |p|.$$

Now we get from here for $0 < a \leq 1$,

$$\left\{ \frac{1}{\mu(Q)} \int_Q |p|^a d\mu \right\}^{\frac{1}{a}} \geq \frac{1}{2} |p(y)| \left\{ \frac{1}{\mu(Q)} \int_{Q''} d\mu \right\}^{\frac{1}{a}} \geq \tilde{c} \left\{ \frac{\mu(\lambda Q_r(y))}{\mu(2Q_r(y))} \right\}^{\frac{1}{a}} \max_{Q \cap S} |p|.$$

Since the fraction in the curly brackets is estimated from below by a constant depending only on λ and the dilation constant of the measure μ, see Volume I, Definition 3.87, we obtain the required inequality (9.58) for $0 < a \leq 1$.

The remaining case of $1 < a \leq \infty$ follows from the case $a = 1$ by the Hölder inequality. □

Remark 9.23. The inverse Hölder inequality (9.58) with a fixed $a < \infty$ and a doubling measure μ determines a class of subsets larger than $\mathrm{Mar}(\mathbb{R}^n)$. For instance, $S := \mathrm{conv}\{(0,0), (1,0)\} \subset \mathbb{R}^2$ is not Markov, since for $p(x) := x_2$ its gradient $\nabla p = (0,1)$ while $\max_{Q \cap S} |p| = 0$. However, (9.58) holds for all $p \in \mathcal{P}_{k,n}$ and any doubling measure on S.

The next result describes some geometric characteristics of Markov sets.

Proposition 9.24. (a) *If sets S_i, $1 \leq i \leq l$, are Markov, then their direct product also belongs to* $\mathrm{Mar}(\mathbb{R}^n)$ *where* $n := \sum_{i=1}^{l} n_i$.

(b) *If* $S_i \in \mathrm{Mar}(\mathbb{R}^n)$, $1 \leq i \leq l$, *then* $\bigcup_{i=1}^{l} S_i \in \mathrm{Mar}(\mathbb{R}^n)$.

(c) *A set $S \subset \mathbb{R}^n$ is Markov if and only if the following condition does not hold:*

For every $\varepsilon > 0$ there exists a cube $Q := Q_r(x_0) \in \mathcal{K}_S$ so that $Q \cap S$ is contained in a strip $\{x \in \mathbb{R}^n\,;\, |e \cdot (x - x_0)| < \varepsilon r\}$ where $e \in \mathbb{R}^n$ is some unit vector.

(d) *An Ahlfors d-regular set $S \subset \mathbb{R}^n$ with $d > n-1$ belongs to* $\operatorname{Mar}(\mathbb{R}^n)$.

Proof. Assertions (a), (b) are evident.

To prove (c) we, on the contrary, assume that for any $\varepsilon > 0$ we can find a cube $Q := Q_r(x_0) \in \mathcal{K}_S$ and a unit vector $e \in \mathbb{S}^{n-1}$ such that

$$S \cap Q \subset \{x \in \mathbb{R}^n\,;\, |e \cdot (x - x_0)| < \varepsilon r\}.$$

Then for the polynomial $p(x) := r \cdot (x - x_0)$ and an arbitrary $\varepsilon > 0$ we have

$$r = r\,\|\nabla p\,\| > \frac{1}{\varepsilon} \max_{Q \cap S} |p\,|.$$

Hence, S does not satisfy Definition 9.19 (of Markov sets).

(d) According to Definition 4.14 and Proposition 4.15, of Volume I S is an Ahlfors d-regular set if for some $c_1, c_2 > 0$ and every $Q \in \mathcal{K}_S$ the Hausdorff d-measure of $Q \cap S$ satisfies

$$c_1 r_Q^d \le \mathcal{H}_d(Q \cap S) \le c_2 r_Q^d. \tag{9.63}$$

We should derive from here that $S \in \operatorname{Mar}(\mathbb{R}^n)$ provided that $d > n-1$.

This fact is an immediate consequence of the following A. and Yu. Brudnyis theorem [BB-2007a, Thm. 2.6].

Theorem 9.25. *Let S be a measurable subset of a fixed cube $\widehat{Q} \subset \mathbb{R}^n$ of positive Hausdorff d-measure where $d > n-1$. Assume that for some $c > 0$ and all $Q \in \mathcal{K}_S$,*

$$\mathcal{H}_d(Q \cap S) \le c r_Q^d. \tag{9.64}$$

Then for every polynomial $p \in \mathcal{P}_{k,n}$ and $a \in (0, +\infty]$,

$$\sup_{\widehat{Q}} |p\,| \le \gamma \left\{ \frac{1}{\mathcal{H}_d(S)} \int_S |p\,|^a d\mathcal{H}_d \right\}^{\frac{1}{a}} \tag{9.65}$$

where γ is independent of S and p but depend increasingly on $\left(\frac{|\widehat{Q}|}{\mathcal{H}_d(S)^{\frac{n}{d}}}\right)^{\frac{1}{a^*}}$ where $\frac{1}{a^*} := \max\{\frac{1}{a}, 1\}$ and on c.

Proof. We set for brevity

$$\|p\,;S\|_a := \left(\frac{1}{\mathcal{H}_d(S)}\int_S |p\,|^a\,d\mathcal{H}_d\right)^{\frac{1}{a}} \quad \text{and} \quad \|p\,;\widehat{Q}\|_\infty := \sup_{\widehat{Q}} |p\,|.$$

Since the above functions are invariant with respect to dilations and translations of $\mathbb{R}^n$, without loss of generality we may and will assume that $\widehat{Q} = [0,1]^n$.

Let $\Sigma(c,\lambda)$, $c,\lambda > 0$, be the class of subsets $S \subset \widehat{Q}$ satisfying (9.64) and the inequality

$$\{\mathcal{H}_d(S)\}^{n/d} \geq \lambda. \tag{9.66}$$

We must show that there is a positive constant $\gamma = \gamma(n,d,a,k,c,\lambda)$ such that for every $p \in \mathcal{P}_{k,n}$,

$$\|p\,;\widehat{Q}\|_\infty \leq \gamma\,\|p\,;S\|_a. \tag{9.67}$$

Let γ_0 be the optimal constant in (9.67). Since the class $\Sigma(c,\lambda)$ increases as λ decreases, γ_0 increases in $\left(\frac{1}{\lambda}\right)^{\frac{1}{a^*}}$, as required in the theorem.

If, on the contrary, the constant in (9.67) does not exist, one can find sequences of real polynomials $\{p_j\}$ of degrees k and sets $\{S_j\} \subset \Sigma(c,\lambda)$ so that

$$||p_j;\widehat{Q}||_\infty = 1 \text{ for all } j \in \mathbb{N}, \tag{9.68}$$

$$\lim_{j\to\infty} ||p_j;S_j||_a = 0. \tag{9.69}$$

Since all (quasi-) norms on the space $\mathcal{P}_{k,n}$ are equivalent, (9.68) implies existence of a subsequence of $\{p_j\}$ that converges uniformly on $\widehat{Q}$ to a polynomial $p \in \mathcal{P}_{k,n}$. Assume without loss of generality that $\{p_j\}$ itself converges uniformly to p. Then (9.68), (9.69) imply for this p that

$$||p\,;\widehat{Q}||_\infty = 1, \tag{9.70}$$

$$\lim_{j\to\infty} ||p\,;S_j||_a = 0. \tag{9.71}$$

From this we derive the next

Lemma 9.26. *There is a sequence of closed subsets $\{\sigma_j\}$ of $\widehat{Q}$ such that for every j greater than a fixed j_0 the following is true:*

$$\{\mathcal{H}_d(\sigma_j)\}^{n/d} \geq \frac{\lambda}{2^{n/d}}. \tag{9.72}$$

Moreover,

$$\max_{\sigma_j} |p\,| \to 0 \text{ as } j \to \infty. \tag{9.73}$$

Proof. Let first $a < \infty$. By the (probabilistic) Chebyshev inequality

$$\mathcal{H}_d(\{x \in S_j \ ; \ |p(x)| \le t\}) \ge \mathcal{H}_d(S_j) - \frac{\mathcal{H}_d(S_j)}{t^a} \|p\,; S_j\|_a^a.$$

Pick here $t = t_j := \|p\,; S_j\|_a^{1/2}$. Then by (9.71) the left-hand side is at least $\frac{1}{2}\mathcal{H}_d(S_j)$, for j sufficiently large. Denoting the set in the curly brackets by σ_j we also have

$$\max_{\sigma_j} |p\,| = t_j \to 0 \text{ as } j \to \infty,$$

as required.

If now $a = \infty$, we simply set $\sigma_j := S_j$ to obtain (9.72) and (9.73). □

We apply now the Hausdorff compactness theorem to find a subsequence of $\{\sigma_j\}$ converging to a closed subset σ of $\widehat{Q}$ in the Hausdorff metric. We assume without loss of generality that $\{\sigma_j\} \to \sigma$. By (9.73), σ is a subset of the zero set of p. Since p is nontrivial by (9.70), the dimension of its zero set is at most $n-1$; hence $\mathcal{H}_d(\sigma) = 0$ as $d > n-1$. Then for every $\epsilon > 0$ one can find some finite cover of σ by closed cubes Q_i of radii r_i at most $r(\epsilon)$ so that

$$\sum_i r_i^d < \epsilon. \tag{9.74}$$

Let σ_δ be a δ-neighborhood of σ with $0 < \delta < r(\epsilon)$ satisfying

$$\sigma_\delta \subset \bigcup_i Q_i$$

and let j be so large that $\sigma_j \subset \sigma_\delta$. For every Q_i intersecting σ_j we choose a point $x_i \in Q_i \cap \sigma_j$ and consider a closed cube $\widetilde{Q}_i$ centered at x_i of radius twice as long as r_i. Then $Q_i \subset \widetilde{Q}_i$, and $\{\widetilde{Q}_i\}$ covers σ_j.

This and (9.64) give

$$\mathcal{H}_d(\sigma_j) \le \sum_i \mathcal{H}_d(\sigma_j \cap \widetilde{Q}_i) \le c\,2^d \sum_i r_i^d\,.$$

Together with (9.72) and (9.74) this implies that

$$\frac{1}{2}\lambda^{d/n} \le \mathcal{H}_d(\sigma_j) \le c\,2^d \sum_i r_i^d \le 2^d c\,\epsilon.$$

Letting $\epsilon \to \infty$ one gets the contradiction. □

To draw from here the consequence proving assertion (d) of Proposition 9.24 we apply (9.65) with $a = \infty$ and $\widehat{Q} = Q$ where $Q \in \mathcal{K}_S$ and S satisfies the condition of d-regularity with $d > n-1$. Then (9.64) follows with $c = c_2$ from the right-hand side inequality in (9.63). Therefore for every $p \in \mathcal{P}_{k,n}$,

$$\sup_Q |p| \le \gamma \max_{Q \cap S} |p| \tag{9.75}$$

where γ is independent of p and Q but depends increasingly on $\frac{|Q|}{\mathcal{H}_d(Q\cap S)^{\frac{n}{d}}}$. By the left inequality in (9.63) this fraction is bounded by $\left(\frac{1}{c_1}\right)^n$. Hence we may replace γ by a larger constant independent of p and Q. According to (9.57) and Lemma 9.22 this implies that $S \in \mathrm{Mar}(\mathbb{R}^n)$. □

Example 9.27. (a) It might be concluded from the previous results that Markov sets were massive. However, they may have an arbitrarily small Hausdorff dimension. The simplest example is a Cantor set in $\mathbb{R}^n$. Specializing the sequence of throwing cubes we may obtain the Cantor set C_n which is Ahlfors d-regular for as small $d > 0$ as we want. On the other hand, C_n is the direct product of $\frac{d}{n}$-regular one-dimensional Cantor sets. Since the latter belong to $\mathrm{Mar}(\mathbb{R})$ by Proposition 9.24 (d), C_n is Markov by assertion (a) of the same proposition.

In fact, most commonly used fractals including the Sierpiński gasket and the Antoine necklace are Markov as well. Indeed, let $S \subset \mathbb{R}^n$ be a compact self-similar fractals set, see Volume I, Definition 4.18, satisfying the condition

$$\dim(\operatorname{conv} S) = n. \tag{9.76}$$

Because of self-similarity, the local version of (9.76) also holds, i.e., for every cube Q centered at S the convex hull of $Q \cap S$ is also of dimension n. We derive from this that S is Markov. To this end we denote by $\sigma(Q \cap S)$ the minimal closed convex set containing $Q \cap S$ which is symmetric with respect to the center of the cube Q. The criterion of Proposition 9.24 (c) immediately implies the following:

A closed set $S \subset \mathbb{R}^n$ is Markov if and only if, for some constant $\lambda \in (0,1]$ and all cubes $Q \in \mathcal{K}_S$,

$$\lambda Q \subset \sigma(Q \cap S).$$

Since $\operatorname{conv}(Q \cap S) \subset \sigma(Q \cap S)$, the local version of (9.76) and a standard compactness argument imply the existence of $\lambda \in (0,1]$ for which this variant of the criterion holds. Hence, S is Markov.

Clearly, the aforementioned classical fractals satisfy (9.76) and therefore are Markov. Moreover, due to Theorem 4.21 of Volume I, self-similar fractals obeying (9.76) may have an arbitrarily small Hausdorff dimension.

(b) Let $S \subset \mathbb{R}^n$ be a compact Markov set and $\phi : U \to V \subset \mathbb{R}^n$ be a C^1-diffeomorphism defined on an open neighborhood of S. Then $\phi(S)$ is Markov.

To establish this, we first note that Theorem 9.21 (a) and Lemma 9.22 imply that $S \in \mathrm{Mar}(\mathbb{R}^n)$ if and only if for some constant $c > 1$ and every cube $Q \in \mathcal{K}_S$ and linear polynomial p,

$$\max_Q |p| \le c \max_{Q\cap S} |p|.$$

Hence, if, on the contrary, $\phi(S)$ is not Markov, there exist sequences of cubes $Q_i := Q_{r_i}(x_i) \in \mathcal{K}_{\phi(S)}$ and linear polynomials p_i, $i \in \mathbb{N}$, such that

$$\max_{Q_i} |p_i| = 1, \ i \in \mathbb{N}, \quad \text{and} \quad \lim_{i\to\infty} \Big(\max_{Q_i \cap \phi(S)} |p_i| \Big) = 0. \tag{9.77}$$

To derive a contradiction we set $\hat{x}_i := \phi^{-1}(x_i) (\in S)$ and denote by $\widehat{Q}_i$ the cube of radius r_i and center $\hat{x}_i$. Further, we define a linear polynomial $\hat{p}_i$ given for $x \in \mathbb{R}^n$ by

$$\hat{p}_i(x) := p_i\big(x_i + J(\hat{x}_i)(x - \hat{x}_i)\big)$$

where J is the Jacobi matrix of ϕ.

We will show that for some $c \in (0, 1)$,

$$0 < \varliminf_{i\to\infty} \max_{c\widehat{Q}_i} |\hat{p}_i| \quad \text{and} \quad \varlimsup_{i\to\infty} \max_{c\widehat{Q}\cap S} |\hat{p}_i| = 0. \tag{9.78}$$

Since $r_i \to 0$ by (9.77), the cubes $\widehat{Q}_i$ belong to $\mathcal{K}_S$ for sufficiently large indices and (9.78) implies the desired contradiction: $S \notin \mathrm{Mar}(\mathbb{R}^n)$.

Let K be a compact set in the domain of ϕ containing S. By the Taylor formula, for all $x, y \in K$,

$$\phi(x) = \phi(y) + J(y)(x - y) + \varepsilon(x, y)\|x - y\| \tag{9.79}$$

where $\varepsilon : K \times K \to \mathbb{R}^n$ converges uniformly to zero as $x - y \to 0$.

Further, ϕ is a bi-Lipschitz embedding; therefore there exist constants $0 < c < 1 < c'$ depending only on the compact set K and distortion of ϕ such that for all sufficiently large i,

$$c^{-1}\widehat{Q}_i \subset \phi^{-1}(Q_i) \subset c'\widehat{Q}_i.$$

This and Lemma 9.5 yield

$$\max_{\widehat{Q}_i} |\hat{p}_i| \ge c_1 \max_{\phi^{-1}(Q_i)} |\hat{p}_i|.$$

Using now (9.79), the equality $x_i = \phi(\hat{x}_i)$ and linearity of p_i, we write

$$\hat{p}_i(x) = p_i\big(\phi(x)\big) = \big[p_i\big(\varepsilon(x, \hat{x}_i)\big) - p_i(0)\big]\,\|x - \hat{x}_i\|.$$

Due to (9.77) we then get for the first term

$$\max_{\phi^{-1}(Q_i)} \big|p_i(\phi)\big| = \max_{Q_i} |p_i| = 1.$$

To estimate the similar maximum for the second term we use the classical Markov inequality to bound it by

$$\begin{aligned}
&\max_{c'\widehat{Q}_i} \big|p_i\big(\varepsilon(\cdot, \hat{x}_i)\big) - p_i(0)\big| \cdot \|x - \hat{x}_i\| \\
&\le c'\sqrt{n}\, r_i \max_{c'\widehat{Q}_i} \big|\nabla p_i \cdot \varepsilon(\cdot, \hat{x}_i)\big| \\
&\le c'\sqrt{n} r_i \,\frac{\sqrt{n}}{c' r_i} \max_{c'\widehat{Q}_i} |p_i| \cdot \max_{c'\widehat{Q}_i} \big\|\varepsilon(\cdot, \hat{x}_i)\big\|.
\end{aligned}$$

In view of Lemma 9.5 and (9.77) the first maximum is bounded by a constant depending only on n while the second tends to zero as $\widehat{Q}_i \to \hat{x}_i$, i.e., as $i \to \infty$. Combining these results we get

$$\varliminf_{i\to\infty} \max_{\phi^{-1}(Q_i)} |\hat{p}_i| \ge 1 - \varlimsup_{i\to\infty} \max_{\phi^{-1}(Q_i)} \Big(\big|p_i\big(\varepsilon(\cdot, \hat{x}_i)\big) - p_i(0)\big| \cdot \|x - \hat{x}_i\|\Big) \ge 1.$$

Due to Lemma 9.5 the maximum in the left-hand side is bounded by that over $c\widehat{Q}_i$ multiplied by a constant depending only on n and $\frac{c'}{c}$. Hence, the previous inequality implies the first one in (9.78).

In the same manner, we now have

$$\begin{aligned}
\varlimsup_{i\to\infty} \max_{c\widehat{Q}_i \cap S} |\hat{p}_i| &\le \varlimsup_{i\to\infty} \max_{\phi^{-1}(Q_i)\cap S} |\hat{p}_i| \\
&\le \varliminf_{i\to\infty} \Big(\max_{\phi^{-1}(\widehat{Q}_i)\cap S} |p_i(\phi)| \Big) + n \varlimsup_{i\to\infty} \max_{c'\widehat{Q}_i} \big\|\varepsilon(\cdot, \hat{x}_i)\big\|.
\end{aligned}$$

The maximum in the first term of the sum equals $\max\limits_{\widehat{Q}_i \cap \phi(S)} |p_i|$ and, by (9.77), tends to zero as $i \to \infty$. Since the same is true for the second term, the equality in (9.78) follows.

Problem. *Is the result true for ϕ being a Lipschitz homeomorphism?*

We guess that the answer, in general, is negative, but for ϕ sufficiently close to the identity map (with constants of "closeness" depending on S) we can prove the result.

(c) A Lipschitz manifold in $\mathbb{R}^n$ of dimension $\le n-1$ (see Volume I, subsection 4.5.2 for its definition) is not Markov. This follows from the fact that such a manifold locally is a graph of a Lipschitz map ϕ of an open set $U \subset \mathbb{R}^k$, $k \le n-1$, into $\mathbb{R}^n$. By the Rademacher theorem, see Volume I, Section 4.5, ϕ is differentiable almost everywhere on U. Then the condition of Proposition 9.24 (c) is violated at each point $(x, \phi(x))$ where ϕ is differentiable at x. We leave the details to the reader.

In particular, a Markov set $S \subset \mathbb{R}^n$ cannot contain an open in S subset which is bi-Lipschitz homeomorphic to a subset in $\mathbb{R}^d$, $1 \le d < n$.

(d) A Markov set $S \subset \mathbb{R}^n$ has the following uniqueness property:

If a polynomial equals zero in an open in S subset, then it equals zero identically on $\mathbb{R}^n$.

This clearly follows from inequality (9.57).

In turn, one can derive from here a useful characteristic of S:

If a C^∞ function equals zero on an open subset of S, then all its derivatives equal zero on this subset.

Finally, we introduce the notions of Markov and locally Markov sets for subsets of C^1-manifolds in $\mathbb{R}^n$.

Definition 9.28. A closed subset S of a d-dimensional C^1-manifold $\Sigma \subset \mathbb{R}^n$, $0 < d < n$, is said to be Markov (in Σ) if for every $Q \in \mathcal{K}_S$ and polynomial p in $x \in \mathbb{R}^n$,

$$\max_{Q\cap\Sigma} |p\,| \le c \max_{Q\cap S} |p\,| \tag{9.80}$$

where c depends only on d, S and the degree of p.

If there exists a positive number $r < \operatorname{diam} S$ such that (9.80) is valid for all $Q \in \mathcal{K}_S$ with $r_Q \le r$, then S is called *locally Markov.*

The simplest example of this kind is a Markov set in a d-dimensional affine plane L. In this case, (9.80) with $\deg p = 1$ may be regarded as the definition of a Markov set in L. Clearly, all the above presented characteristics of Markov sets in $\mathbb{R}^n$ hold for those in the plane.

A wide class of examples of locally Markov sets can be obtained from the previous one by the argument of Example 9.27 (b). Specifically, let $\Sigma \subset \mathbb{R}^n$ be a d-dimensional relatively compact connected C^1-manifold, $0 < d < n$, and ϕ be a C^1-diffeomorphism of a neighborhood of $\overline{\Sigma}$ into $\mathbb{R}^n$ so that $\phi(\Sigma) \subset \mathbb{R}^d$. Then $\phi^{-1}(S)$ is locally Markov in Σ whenever $S \subset \phi(\Sigma) \cap \mathbb{R}^d$ is Markov in $\mathbb{R}^d$. In fact, inequality (9.80) is true in this case for cubes $Q \in \mathcal{K}_{\phi^{-1}(S)}$ with $r_Q \le r$ for a fixed $r > 0$.

To deduce that in this setting $\phi^{-1}(S)$ is also (global) Markov one should impose additional restrictions to Σ. In particular, it suffices to assume that Σ satisfies the following doubling inequality:

For every $Q \in \mathcal{K}_S$ and polynomial p,

$$\sup_{2Q\cap\Sigma} |p| \leq c \sup_{Q\cap\Sigma} |p|$$

where c depends only on Q, Σ and degree of p.

This inequality is valid, e.g., for relatively compact connected real analytic submanifolds $\Sigma \subset \mathbb{R}^n$. The construction of nonanalytic manifolds Σ satisfying the doubling inequality is more complicated.

9.2.2 Traces of the space $C^l\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ to Markov sets

We first describe the trace of the homogeneous space $C^l\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ to a Markov set $S \subset \mathbb{R}^n$ via behavior of local approximation on S and then prove the corresponding extension result.

Let us recall that local best approximation of order k for a function $f \in l_\infty^{\text{loc}}(\mathbb{R}^n)$ is a set-function given for $S \subset \mathbb{R}^n$ by

$$E_k(S\,; f) := \inf_{p\in\mathcal{P}_{k-1,n}} \|f - p\|_{\ell_\infty(S)}. \tag{9.81}$$

Using this we define a space of bounded functions on S denoted by $\dot{\mathcal{E}}^{k,\omega}(S)$ given by the seminorm

$$|f|_{\mathcal{E}^{k,\omega}(S)} := \sup_{Q\in\mathcal{K}_S} \frac{E_k(Q\cap S\,; f)}{\omega(r_Q)}. \tag{9.82}$$

According to Theorem 2.37 of Volume I, see also (9.3), we have up to equivalence of the seminorms

$$\dot{\Lambda}^{k,\omega}(\mathbb{R}^n) = \dot{\mathcal{E}}^{k,\omega}(\mathbb{R}^n). \tag{9.83}$$

In view of Theorem 9.7 we also have up to equivalence of the seminorms

$$C^l\dot{\Lambda}^{k,\omega}(\mathbb{R}^n) = \dot{\Lambda}^{k+l,\widehat{\omega}}(\mathbb{R}^n), \tag{9.84}$$

where $\widehat{\omega}(t) := t^l\omega(t)$, $t > 0$, under the assumption on ω of being a quasipower k-majorant. Hence, the smoothness space $C^l\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ may be, in this case, described by behavior of local approximation of its members.

These results pose the following

Problem 9.29. *Find a class of closed subsets S in $\mathbb{R}^n$ such that*

$$\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_S = \dot{\mathcal{E}}^{k,\omega}(S).$$

The result presented below shows that Markov sets are contained in this class which, however, is not exhausted by these sets.

Theorem 9.30. *Let $S \subset \mathbb{R}^n$ be Markov. Then the following holds.*

(a) *Up to equivalence of the seminorms,*

$$\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_S = \dot{\mathcal{E}}^{k,\omega}(S). \tag{9.85}$$

In particular, for a quasipower k-majorant ω and $\widehat{\omega}(t) := t^l\omega(t)$, $t > 0$,

$$C^l\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_S = \dot{\mathcal{E}}^{k+l,\widehat{\omega}}(S).$$

(b) *In both cases, there exists a linear continuous extension operator from the trace space into the initial space.*

Proof. (a) If $f \in \dot{\Lambda}^{k,\omega}|_S$, then the trace of some $g \in \dot{\Lambda}^{k,\omega}$ to S equals f. (As before, we omit hereafter the symbol $\mathbb{R}^n$ in the notation of smoothness spaces on $\mathbb{R}^n$ writing, e.g., $\dot{\Lambda}^{k,\omega}$ instead of $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$.) Hence, for each $Q \in \mathcal{K}_S$,

$$E_k(Q \cap S\,; f) \le E_k(g; Q) \le \omega(r_Q)|g|_{\Lambda^{k,\omega}}.$$

Dividing by $\omega(r_Q)$ and taking infimum over g and supremum over Q we obtain the embedding

$$\dot{\Lambda}^{k,\omega}|_S \subset \dot{\mathcal{E}}^{k,\omega}(S) \tag{9.86}$$

with the embedding constant 1.

This clearly is true for arbitrary closed S.

Conversely, let $f \in \dot{\mathcal{E}}^{k,\omega}(S)$. We must show that f belongs to the trace space. To this end we use the criterion of Theorem 9.3 to construct a suitable (ω, k)-chain.

Given $Q \in \mathcal{K}_S$, let p_Q be a polynomial of degree $k-1$ such that

$$E_k(Q \cap S\,; f) = \sup_{Q\cap S} |f - p_Q|. \tag{9.87}$$

Further, we set $\widehat{p}_Q := p_Q + f(c_Q) - p_Q(c_Q)$. Due to the choice of p_Q,

$$\sup_{Q\cap S} |f - \widehat{p}_Q| \le 2E_k(Q \cap S\,; f).$$

Let us show that $P(f) := \{\widehat{p}_Q\}_{Q\in\mathcal{K}_S}$ is an (ω, k)-chain on S, see Definition 9.2. Since S is Markov, we have for some c and every pair of cubes $Q \subset Q'$ from $\mathcal{K}_S$,

$$\max_Q |\widehat{p}_Q - \widehat{p}_{Q'}| \le c \max_{Q\cap S} |\widehat{p}_Q - \widehat{p}_{Q'}| \le 4cE_k(Q' \cap S\,; f).$$

By (9.82), the right-hand side is at most $4c\,\omega(r'_Q)|f|_{\mathcal{E}^{k,\omega}(S)}$. Hence, the family $P(f) := \{\widehat{p}_Q\}_{Q\in\mathcal{K}_S}$ satisfies condition (9.11) of Definition 9.2 and its seminorm in the space of (ω, k)-chains S satisfies

$$|P(f)|_{\mathrm{Ch}} \le 4c\,|f|_{\mathcal{E}^{k,\omega}(S)}.$$

Since, moreover, $\widehat{p}_Q(c_Q) = f(c_Q)$, the assumptions of Theorem 9.3 hold for $P(f)$ and the function f belongs to $\dot{\Lambda}^{k,\omega}|_S$ and its trace seminorm is bounded by $C|P(f)|_{\mathrm{Ch}} \le 4Cc\,|f|_{\mathcal{E}^{k,\omega}(S)}$, where $C = C(n,k)$.

This proves the converse to (9.86) embedding and, hence, equality (9.85).

The second part of assertion (a) immediately follows from (9.84) and (9.85).

(b) In this case, we exploit a linear version of the above used criterion given by Theorem 9.6. According to this theorem and (9.85) a linear bounded extension operator from $\dot{\Lambda}^{k,\omega}|_S$ to $\dot{\Lambda}^{k,\omega}$ exists if for every $f \in \dot{\mathcal{E}}^{k,\omega}(S)$ there is an (ω,k)-chain $\{\widehat{p}_Q(f)\}_{Q\in\mathcal{K}_S}$ such that $\widehat{p}_Q(f)(c_Q) = f(c_Q)$ and the operator $f \mapsto \widehat{p}_Q(f)$ is linear.

To define such a chain we use the Kadets-Snobar theorem [KS-1971] asserting that there exists a projection onto every n-dimensional subspace of a Banach space of norm at most $\sqrt{n}$. Applying this to the subspace $\mathcal{P}_{k-1,n}|_S$ of the space $\ell_\infty(Q\cap S)$ where $Q \in \mathcal{K}_S$ and denoting by T_Q the corresponding projection of norm $\le \gamma_{k,n} := \sqrt{\dim \mathcal{P}_{k-1,n}}$ we then get, using the polynomial p_Q defined by (9.87),

$$\begin{aligned}\|f - T_Q f\|_{\ell_\infty(Q\cap S)} &\le \|f - p_Q\|_{\ell_\infty(Q\cap S)} + \|T_Q(f-p_Q)\|_{\ell_\infty(Q\cap S)}\\ &\le (\gamma_{k,n}+1)E_k(Q\cap S\,; f).\end{aligned}$$

Now by setting

$$\widehat{p}_Q(f) := T_Q f - (T_Q f)(c_Q) + f(c_Q)$$

we clearly define the chain $\{\widehat{p}_Q(f)\}_{Q\in\mathcal{K}_S}$ satisfying the assumptions of Theorem 9.6.

This proves (b) for the space $\dot{\Lambda}^{k,\omega}$ while the result for $C^l\dot{\Lambda}^{k,\omega}$ with a quasi-power k-majorant ω is a direct consequence of the previous one. □

Finally, we describe the traces of the smoothness spaces in question to Markov subsets of certain C^1-manifolds in $\mathbb{R}^n$.

Let $\Sigma \subset \mathbb{R}^n$ be a d-dimensional C^1 manifold, $0 < d < n$, given by the equations

$$x_i := p_i(x_1,\dots,x_d), \quad x = (x_1,\dots,x_n) \in \mathbb{R}^n, \quad d < i \le n, \tag{9.88}$$

where p_i is a polynomial of degree l_i.

Theorem 9.31. *Let $S \subset \Sigma$ be a Markov set in Σ and $\widehat{k} := (k-1)(\max l_i) + 1$. Then the following continuous embeddings hold:*

$$\dot{\Lambda}^{k,\omega}|_S \subset \dot{\mathcal{E}}^{k,\omega}(S) \subset \dot{\Lambda}^{\widehat{k},\omega}|_S. \tag{9.89}$$

Moreover, there exists a linear continuous extension operator from $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_S$ to $\Lambda^{\widehat{k},\omega}(\mathbb{R}^n)$.

Proof. The left embedding in (9.89) is true for arbitrary closed S, see (9.86). To prove the right embedding we first note that there exists a linear extension operator $T : \mathcal{P}_{k-1,n}|_\Sigma \to \mathcal{P}_{\widehat{k}-1,n}$ such that for every cube $Q \in \mathcal{K}_S$,

$$\max_Q |Tp| = \max_{Q\cap\Sigma} |p\,|. \tag{9.90}$$

This operator is defined for $x \in \mathbb{R}^n$ by

$$(Tp)(x) := p\,(x_1, \dots, x_d, p_{d+1}(x_1, \dots, x_d), \dots, p_n(x_1, \dots, x_d))$$

where p_i are polynomials in (9.88).
Clearly, Tp is a polynomial of degree at most $(k-1)\max(\deg p_i) =: \widehat{k}-1$. Further, by definition $(Tp)|_\Sigma = p\,|_\Sigma$ for every $x \in \mathbb{R}^n$ and therefore (9.90) holds.

Now we follow the argument of the previous theorem. Hence, to establish the right embedding in (9.89) we must find for every $f \in \dot{\mathcal{E}}^{k,\omega}(S)$ an $(\omega, \widehat{k})$-chain $\{\widehat{p}_Q\}_{Q\in\mathcal{K}_S}$ such that for every Q,

$$\widehat{p}_Q(c_Q) = f(c_Q). \tag{9.91}$$

Let $p_Q \in \mathcal{P}_{k-1,n}$ where $Q \in \mathcal{K}_S$ is such that

$$E_k(Q \cap S\,; f) = \sup_{Q\cap S} |f - p_Q|.$$

Using (9.90) and the definition of a Markov set in Σ, see (9.80), we then have for the family $\{Tp_Q\}_{Q\in\mathcal{K}_S} \subset \mathcal{P}_{\widehat{k}-1,n}$ and every pair $Q \subset Q'$ of cubes from $\mathcal{K}_S$,

$$\max_Q |Tp_Q - Tp_{Q'}| \le c \cdot \max_{Q\cap S} |p_Q - p_{Q'}| \le 2c\,\omega(r_{Q'})|f|_{\mathcal{E}^{k,\omega}(S)}.$$

Setting as in the proof of Theorem 9.30

$$\widehat{p}_Q := Tp_Q - (Tp_Q)(c_Q) + f(c_Q)$$

we conclude that $\{\widehat{p}_Q\}_{Q\in\mathcal{K}_S}$ is an $(\omega, \widehat{k})$-chain on S satisfying (9.91)and derive from this fact the right embedding in (9.89).

As for a simultaneous extension of $\dot{\Lambda}^{k,\omega}|_S$ to $\dot{\Lambda}^{\widehat{k},\omega}$, it suffices to linearize the operator $f \mapsto \{\widehat{p}_Q(f)\}_{Q\in\mathcal{K}_S}$ using the Kadets-Snobar theorem. □

For S being a Markov subset in an affine plane Theorem 9.31 gives

$$\dot{\Lambda}^{k,\omega}|_S = \dot{\mathcal{E}}^{k,\omega}(S)$$

along with the assertion on a simultaneous extension from the trace space.

In the case of a polynomial surface of degree greater than 1, a version of this result takes place for the nonhomogeneous spaces $\Lambda^{k,\omega}(\mathbb{R}^n)$ and $\mathcal{E}^{k,\omega}(\mathbb{R}^n)$ under some restriction on ω. In its formulation, the space $\mathcal{E}^{k,\omega}(\mathbb{R}^n)$ is defined by a norm given for $f \in \ell_\infty(S)$ by

$$\|f\|_{\mathcal{E}^{k,\omega}(S)} := \|f\|_{\ell_\infty(S)} + |f|_{\mathcal{E}^{k,\omega}(S)}.$$

Corollary 9.32. *Assume that $\widehat{k} > k$ and a k-majorant ω satisfies the condition*

$$\sup_{t>0} \left\{ \frac{t^k}{\omega(t)} \int_t^\infty \frac{\omega(s)}{s^{k+1}}\, ds \right\} < \infty. \tag{9.92}$$

Then up to equivalence of the norms

$$\Lambda^{k,\omega}|_S = \mathcal{E}^{k,\omega}(S) \tag{9.93}$$

and there exists a simultaneous extension from the trace space to $\Lambda^{k,\omega}(\mathbb{R}^n)$.

Proof. Due to (9.89) for a function $f \in \ell_\infty(S)$ from the trace space

$$|f|_{\mathcal{E}^{k,\omega}(S)} \le O(1)|f|_S,$$

where $|\cdot|_S$ stands for the trace seminorm of $\dot{\Lambda}^{k,\omega}|_S$ and $O(1)$ denotes a constant independent of f. Since, moreover, $\|f\|_{\ell_\infty(S)}$ is bounded by the trace norm of the space in (9.93), the norm of $\mathcal{E}^{k,\omega}(S)$ is majorated by the norm of $\Lambda^{k,\omega}|_S$. Hence, the space in the left-hand side of (9.93) is continuously embedded into that in the right-hand side.

To prove the converse embedding we exploit Corollary F.8 of Appendix F of Volume I. According to this result there exists a constant $c > 1$ such that for every cube Q, function $f \in \ell_\infty(Q)$ and $t \in (0, r_Q)$,

$$\omega_k(t\,; f - p_Q)_Q \le c\, t^k \int_t^{\tilde{c} r_Q} \frac{\omega_{\widehat{k}}(s\,; f)_Q}{s^{k+1}}\, ds \tag{9.94}$$

where $0 < t \le \tilde{c} r_Q := \frac{2\sqrt{n}}{k}\, r_Q$ and p_Q is a polynomial of degree $\widehat{k} - 1$ such that

$$\sup_Q |f - p_Q| = E_{\widehat{k}}(Q\,; f).$$

The latter clearly implies that

$$\max_Q |p_Q| \le 2 \sup_Q |f|.$$

This and the classical Markov inequality then give for $0 < t \le r_Q$,

$$\omega_k(t\,; p_Q)_Q \le t^k \max_{|\alpha|=k} \left\{ \max_Q |D^\alpha p_Q| \right\} \le c(k,n) t^k\, \frac{\|f\|_{\ell_\infty(\mathbb{R}^n)}}{(r_Q)^k}. \tag{9.95}$$

Taking the limit as $r_Q \to \infty$, we conclude from (9.94) and (9.95) that for $f \in \Lambda^{\widehat{k},\omega}$ and $t > 0$,

$$\omega_k(t\,; f) \le c t^k \left(\int_t^\infty \frac{\omega(s)}{s^{k+1}}\, ds \right) |f|_{\Lambda^{\widehat{k},\omega}}.$$

Dividing both sides by $\omega(t)$, taking supremum over $t > 0$ and using condition (9.92) on ω we obtain for $f \in \Lambda^{\widehat{k},\omega}$ the inequality

$$|f|_{\Lambda^{k,\omega}} \le O(1)|f|_{\Lambda^{\widehat{k},\omega}}$$

which clearly implies the continuous embedding

$$\Lambda^{\widehat{k},\omega}|_S \subset \Lambda^{k,\omega}|_S.$$

In turn, the continuous embedding of (9.89) for the corresponding homogeneous spaces implies that

$$\mathcal{E}^{k,\omega}(S) \subset \Lambda^{\widehat{k},\omega}|_S.$$

Together with the previous result this leads to the required converse embedding and proves equality (9.93).

It remains to show existence of a simultaneous extension operator from $\mathcal{E}^{k,\omega}(S)$ into $\Lambda^{k,\omega}(\mathbb{R}^n)$. To this end we use in our settings the linear continuous extension operator denoted now by Ext which is given by formula (9.15). Then Ext continuously maps $\dot{\mathcal{E}}^{k,\omega}(S)$ into $\dot{\Lambda}^{\widehat{k},\omega}|_S$, see Theorem 9.31. In view of the above proved equality it is also true that

$$\Lambda^{\widehat{k},\omega}|_S = \Lambda^{k,\omega}|_S.$$

Hence, it suffices to check that for $f \in \ell_\infty(S)$,

$$\|\operatorname{Ext} f\|_{\ell_\infty(\mathbb{R}^n)} \le O(1)\|f\|_{\ell_\infty(S)}. \tag{9.96}$$

But according to (9.15) the left-hand side of (9.96) is bounded by

$$\begin{aligned}&\left(\sum_{Q\in\mathcal{W}_S}\varphi_Q\right)\sup\{|Tp_{\widehat{Q}}(x)|\,;\,x\in\operatorname{supp}\varphi_Q\,,\,Q\in\mathcal{W}_S\}\\ &=\sup\{|Tp_{\widehat{Q}}(x)|\,;\,x\in\operatorname{supp}\varphi_Q\,,\,Q\in\mathcal{W}_S\}\end{aligned}$$

where $p_{\widehat{Q}}$ is a polynomial of degree $k-1$ such that

$$\sup_{\widehat{Q}\cap S}|f-p_{\widehat{Q}}| = E_k(\widehat{Q}\cap S\,;f)$$

and $T : \mathcal{P}_{k-1,n}|_S \to \mathcal{P}_{\widehat{k}-1,n}$ is the extension operator from the proof of Theorem 9.31.

As above, the latter implies

$$\max_{\widehat{Q}\cap S}|p_{\widehat{Q}}| \le 2\|f\|_{\ell_\infty(S)},$$

while (9.14) and Lemma 9.5 give

$$\max_{\operatorname{supp}\varphi_Q}|p_{\widehat{Q}}| \le \max_{6\widehat{Q}}|p_{\widehat{Q}}| \le c(k,n,S)\max_{\widehat{Q}\cap S}|p_{\widehat{Q}}|.$$

Combining all these inequalities we prove (9.96) and the corollary. □

As the only consequence of the previous corollary we single out results concerning Sobolev and Besov spaces over $\ell_\infty(\mathbb{R}^n)$. In the latter case, ω is a power function, say, $\omega(t) := t^\sigma$, $t > 0$, where $\sigma > 0$ and we denote by k the smallest integer greater than σ.

Corollary 9.33. (a) *Let S be a Markov set in an affine plane of $\mathbb{R}^n$. Then up to equivalence of the seminorms*

$$\dot{B}^\sigma_\infty|_S = \dot{\mathcal{E}}^{k,\omega}(S), \quad \dot{W}^l_\infty|_S = \dot{\mathcal{E}}^{l,\widetilde{\omega}}(S)$$

where $\widetilde{\omega}(t) := t^l$, $t > 0$.

Moreover, there exist simultaneous extensions from the trace spaces to the initial spaces.

(b) *If S is a Markov set in the nonlinear polynomial surface* (9.88)*, then the previous assertions are true for the Banach spaces $B^\sigma_\infty|_S$ and $\mathcal{E}^{k,\omega}(S)$.*

Proof. Let us recall that $\dot{B}^\sigma_\infty$ is equal, by definition, to the space $C^{k'}\dot{\Lambda}^{2,\omega'}$ where k' is the largest integer less than σ and $\omega'(t) := t^{\sigma-k'}$, $t > 0$, see, e.g., equation (2.23) in Volume I. It follows from inequalities (2.13) and (2.14) of Theorem 2.7 of Volume I that up to equivalence of the seminorms

$$\dot{B}^\sigma_\infty = \dot{\Lambda}^{k,\omega}.$$

Further, according to Remark 2.33 of Volume I the homogeneous Sobolev space $\dot{W}^l_\infty$ isometrically coincides with the space $C^{l-1,1}$ $(:= C^{l-1}\dot{\Lambda}^{1,1})$. According to Theorem 2.7 of Volume I the latter space equals $\dot{\Lambda}^{l,\widetilde{\omega}}$. Hence, up to equivalence of the seminorms

$$\dot{W}^l_\infty = \dot{\Lambda}^{l,\widetilde{\omega}}.$$

Using these two equalities and Corollary 9.32 we prove assertion (a).

To prove (b) it suffices to note that the majorant $\omega(t) := t^\sigma$ satisfies condition (9.92) of Corollary 9.32 for $k > \sigma$. □

Remark 9.34. The trace-extension results of this subsection hold also for the nonhomogeneous counterparts of the smoothness spaces involved. We should only note that the extension operator Ext^S_k given by (9.15) continuously maps $\ell_\infty(S)$ into $\ell_\infty(\mathbb{R}^n)$, see the proof of inequality (9.96).

It is worth noting that this fact implies a kind of universality of Ext^S_k; for instance, this operator simultaneously extends all trace spaces $B^\sigma_\infty|_S$ with $0 < \sigma < k$.

9.2.3 Traces of Morrey-Campanato spaces to Markov sets

The family of function spaces in question is defined by means of local polynomial approximation in an *integral metric*. The relation of the trace-extension problem for these spaces to that for Lipschitz spaces of higher order was established by

Yu. Brudnyi [Br-1970b] in the case of Ahlfors n-regular domains. To formulate a general result of this kind we need the general concept of local approximation concerning functions from the space $L_q(d\mu)$, $0 < q \le \infty$, where μ is a Borel measure on $\mathbb{R}^n$.

Definition 9.35. The normalized local best approximation of order k for a function $f \in L_q^{\mathrm{loc}}(d\mu)$ is a set function given on subsets $S \subset \mathbb{R}^n$ of finite μ-measure by

$$\mathcal{E}_k(S\,; f; L_q(d\mu)) := \inf \left\{ \left(\frac{1}{\mu(S)} \int_S |f - m|^q \, d\mu \right)^{\frac{1}{q}} ; m \in \mathcal{P}_{k-1,n} \right\}. \quad (9.97)$$

For $L_q(d\mu) = L_q(\mathbb{R}^n)$, i.e., for μ being the Lebesgue n-measure, we shorten this notation by writing $\mathcal{E}_k(\cdot\,; f\,; L_q)$ instead of $\mathcal{E}_k\big(\cdot\,; f\,; L_q(\mathbb{R}^n)\big)$.

Now the *Morrey-Campanato space* $\dot{M}_q^{k,\omega}(\mathbb{R}^n)$ is determined by the seminorm

$$|f|_{M_q^{k,\omega}(\mathbb{R}^n)} := \sup_{Q \in \mathcal{K}(\mathbb{R}^n)} \left\{ \frac{\mathcal{E}_k(f; Q; L_q)}{\omega(r_Q)} \right\}; \quad (9.98)$$

here $\omega : (0, +\infty) \to \mathbb{R}_+$ is a monotone function (it may be decreasing).

For the special case of $\omega(t) := t^{-\sigma}$ where $0 < \sigma < \frac{n}{q}$, and $k = 0$, the space was introduced by Morrey [Mo-1938]. Since $\mathcal{E}_0(f; Q; L_q) := \left\{ \frac{1}{|Q|} \int_Q |f|^q \, dx \right\}^{\frac{1}{q}}$, this space denoted by $\mathcal{M}_q^{-\sigma}(\mathbb{R}^n)$ is defined by the norm (quasinorm for $q < 1$)

$$\|f\|_{\mathcal{M}_q^{-\sigma}(\mathbb{R}^n)} := \sup_Q \left\{ |Q|^{\frac{\sigma}{n} - \frac{1}{q}} \int_Q |f|^q \, dx \right\}^{\frac{1}{q}}.$$

Further, for $k = 1$, $0 < q < \infty$ and $\omega(t) = 1$, $t > 0$, the space $\dot{M}_q^{k,\omega}(\mathbb{R}^n)$ coincides up to equivalence of the (quasi-) norms with BMO, the space introduced by John and Nirenberg [JN-1961]. Let us recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ is defined by the seminorm

$$|f|_{\mathrm{BMO}(\mathbb{R}^n)} := \sup_{Q \in \mathcal{K}(\mathbb{R}^n)} \frac{1}{|Q|} \int_Q |f - f_Q| \, dx$$

where f_Q is the integral mean of f on Q, and the coincidence was established by John-Nirenberg [JN-1961] for $1 \le q < \infty$ and by Strömberg [Str-1979] for $q < 1$.

Finally, the space $\dot{M}_q^{k,\omega}(\mathbb{R}^n)$ for $\omega(t) = t^\sigma$, $0 \le \sigma \le k$, was introduced by Campanato [Cam-1964]. He, in particular, proved that for a noninteger σ this space coincides with the Besov space $\dot{B}_\infty^\sigma(\mathbb{R}^n)$ (this independently was proved by Meyers [Me-1964] for $0 < \sigma < 1$). Actually, this fact holds also for an integer σ as it follows from the next results established by Yu. Brudnyi [Br-1965a].

If ω is a quasipower k-majorant and $1 \le q < \infty$, then up to equivalence of the seminorms

$$\dot{M}_q^{k,\omega}(\mathbb{R}^n) = \dot{\Lambda}^{k,\omega}(\mathbb{R}^n). \quad (9.99)$$

We will show that this kind of L_q-stability result is also true for the traces of Morrey-Campanato spaces to Markov sets. For this goal we generalize definition (9.98) introducing the space $\dot{M}^{k,\omega}_{q,\mu}(S)$ where μ is a Borel measure on the subset $S \subset \mathbb{R}^n$. This is defined by the seminorm

$$|f|_{M^{k,\omega}_{q,\mu}}(S) := \sup_Q \frac{\mathcal{E}_k(Q\,;f\,;L_q(d\mu))}{\omega(r_Q)} \tag{9.100}$$

where Q runs over all cubes centered at S and of radius at most $4\operatorname{diam} S$ (all distances are measured in the ℓ_∞-norm of $\mathbb{R}^n$).

The main result of this subsection is as follows.

Theorem 9.36. *Let $S \subset \mathbb{R}^n$ be a Markov set, μ be a doubling measure on $\mathbb{R}^n$ with support S, and ω be a quasipower k-majorant. Then up to equivalence of the seminorms*

$$\dot{M}^{k,\omega}_{q}(\mathbb{R}^n)|_S = \dot{M}^{k,\omega}_{q,\mu}(S). \tag{9.101}$$

Moreover, for $1 \le q \le \infty$ there exists a simultaneous extension of the spaces $\dot{M}^{k,\omega}_{q,\mu}(S)$ into $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$.

Remark 9.37. (a) The immediate consequence of this result is the equality

$$\dot{M}^{k,\omega}_{q}(\mathbb{R}^n)|_S = \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_S$$

which generalizes (9.99) to a wide class of closed subsets on $\mathbb{R}^n$.

(b) For S being an Ahlfors σ-regular subset of $\mathbb{R}^n$ with $n-1<\sigma\le n$ (a special case of Markov sets, see Theorem 9.25) the result was proved by A. and Yu. Brudnyis [BB-2007a] (in this case, μ is the Hausdorff σ-measure). The proof of Theorem 9.36 follows the very same line.

(c) According to (9.101) the space $\dot{M}^{k,\omega}_{q,\mu}(S)$, up to equivalence of the seminorms, is independent of the choice of a doubling measure μ. Hence, stability in μ also holds in this case.

(d) For $q<1$ there is no *linear* continuous extension operator, but a nonlinear homogeneous continuous extension operator exists as can be seen from the proof.

Proof. Given $f \in \dot{M}^{k,\omega}_{q,\mu}(S)$ we must find a function $\widetilde{f} : S \to \mathbb{R}$ which equals f modulo μ-measure zero and admits an extension to a function from $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$.

We begin with

Lemma 9.38. *Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a quasipower k-majorant and $t_j := 2^j$, $j \in \mathbb{Z}_+$. Then for every pair of integers $-\infty < i < i' < \infty$ we have*

$$\sum_{j=i}^{i'} \omega(t_j) \le c(k,\omega)\,\omega(t_{i'}). \tag{9.102}$$

Proof. By the monotonicity of ω and (9.28) the sum is bounded by

$$\frac{1}{\ln 2}\int_{t_i}^{t_{i'+1}} \frac{\omega(u)}{u}\,du \le \frac{1}{\ln 2}\,C_\omega\,\omega(t_{i'+1}) \le \frac{C_\omega}{\ln 2}2^k\omega(t_{i'}). \qquad \square$$

Our next result reformulates Theorem 9.3 concerning the trace space of $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ to adapt it to our situation.

To formulate the result we need the next variant of Definition 9.2.

Let $S \subset \mathbb{R}^n$ and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be as above, and $\{t_i\}_{i\in\mathbb{Z}_+}$ be the sequence of Lemma 9.38. By $\widetilde{\mathcal{K}}_S$ we denote the set of cubes centered at S and of radius at most $4\,\mathrm{diam}\,S$.

Definition 9.39. A family $\Pi := \{P_Q\}_{Q\in\widetilde{\mathcal{K}}_S}$ of polynomials of degree $k-1$ is said to be an (ω, k, S)-chain if, for every pair of cubes $Q \subset Q'$ from $\widetilde{\mathcal{K}}_S$ which satisfy for some $i \in \mathbb{Z}$ the condition

$$t_i \le r_Q < r_{Q'} \le t_{i+2}, \tag{9.103}$$

the inequality

$$\max_{x\in Q} |P_Q(x) - P_{Q'}(x)| \le C\omega(r_{Q'}) \tag{9.104}$$

holds with a constant C independent of Q, Q' and i.

The linear space of such chains denoted by $\mathrm{Ch}(\omega, k, S)$ is equipped with the seminorm $|\Pi|_{\mathrm{Ch}} := \inf C$ where the infimum is taken over all constants C in (9.104).

Using this concept we now formulate and prove the desired variant of Theorem 9.3.

Proposition 9.40. (a) *A locally bounded function $f\colon S\to\mathbb{R}$ belongs to $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_S$ if and only if there exists an (ω, k, S)-chain $\Pi := \{P_Q\}_{Q\in\widetilde{\mathcal{K}}_S}$ such that for every $Q \in \widetilde{\mathcal{K}}_S$,*

$$f(c_Q) = P_Q(c_Q). \tag{9.105}$$

Moreover, the two-sided inequality

$$|\Pi|_{\mathrm{Ch}} \approx |f|_{\Lambda^{k,\omega}(\mathbb{R}^n)|_S}$$

holds with constants independent of f.

(b) *If, moreover, this chain depends on f linearly, then there is a linear extension operator $T_k : \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_S \to \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ such that*

$$||T_k|| \le O(1)|\Pi|_{\mathrm{Ch}}.$$

Hereafter $O(1)$ denotes a constant depending only on inessential parameters. It may change from line to line and even in a single line.

Proof. Theorems 9.3 and 9.6 imply this result under the assumption that inequality (9.104) holds for any pair of cubes $Q \subset Q'$ from $\mathcal{K}_S \subset \widetilde{\mathcal{K}}_S$. The restrictions (9.103) may not, however, be satisfied for this pair. In the forthcoming derivation we explain how these obstacles can be overcome.

First, let S be unbounded and so $\mathcal{K}_S = \widetilde{\mathcal{K}}_S$. We must show that if a (ω, k, S)-chain satisfies condition (9.104) under restriction (9.103), then (9.104) holds for *any* pair $Q \subset Q'$ from $\mathcal{K}_S$. Since the necessity of conditions (9.104) and (9.105) trivially follows from Theorem 9.3, the proposition for this case would be proved.

Let $Q \subset Q'$ be a pair of cubes from $\mathcal{K}_S$ of radii r and r', respectively. Then for some indices $i \le i'$,

$$t_i \le r \le t_{i+1} \quad \text{and} \quad t_{i'} \le r' \le t_{i'+1};$$

recall that $t_i := 2^i$, $i \in \mathbb{Z}$. If $i = i'$, then by (9.104),

$$\max_Q |P_Q - P_{Q'}| \le 2\omega(r')|\Pi|_{\mathrm{Ch}}$$

as required.

Now, let $i < i'$ and numbers r_j, $i \le j \le i' + 1$, be given by

$$r_i := r, \quad r_{i'+1} := 2r' \quad \text{and} \quad r_j := t_j \quad \text{for } i < j < i' + 1.$$

Let Q_j be a cube centered at c_Q of radius r_j, $i \le j < i' + 1$, and $Q_{i'+1}$ be a cube centered at $c_{Q'}$ of radius $r_{i'+1}$. Then $\{Q_j\}_{i \le j \le i'+1} \subset \mathcal{K}_S$ is an increasing sequence of cubes starting at $Q_i := Q$ and

$$\max_Q |P_Q - P_{Q'}| \le \sum_{j=i}^{i'} \max_{Q_{j+1}} |P_{Q_j} - P_{Q_{j+1}}|. \tag{9.106}$$

Since (9.103) holds for every pair $Q_j \subset Q_{j+1}$, $i \le j \le i'$, we may apply (9.104) to each of them. This and (9.102) estimate the right-hand side of (9.106) by

$$2|\Pi|_{\mathrm{Ch}} \sum_{j=i}^{i'} \omega(r_{j+1}) \le O(1)|\Pi|_{\mathrm{Ch}}\omega(t_{i'+2}) \le O(1)|\Pi|_{\mathrm{Ch}}\omega(r').$$

Thus inequality (9.104) holds for every pair $Q \subset Q'$ of cubes from $\mathcal{K}_S$.

Now, let $\operatorname{diam} S < \infty$. The previous argument proves the required inequality

$$\max_Q |P_Q - P_{Q'}| \le C\omega(r_{Q'})$$

for every pair $Q \subset Q'$ from $\widetilde{\mathcal{K}}_S$ under the restriction $r_{Q'} \le 2 \operatorname{diam} S$. If, moreover, $Q' \in \mathcal{K}_S$, then $S \cap (Q')^c \ne \emptyset$ and so $r'_Q \le \operatorname{diam} S$. Hence, in this case, (9.104) also holds for all cubes $Q \subset Q'$ from $\mathcal{K}_S$.

This completes the proof of the proposition. $\square$

Now we outline the proof of Theorem 9.36.

Given $f \in \dot{M}^{k,\omega}_{q,\mu}(S)$, where $S \subset \mathbb{R}^n$ is a Markov set, we will define a new function $\widetilde{f} : S \to \mathbb{R}$ such that

$$\widetilde{f}(x) = f(x) \quad \mu\text{-almost everywhere on } S. \tag{9.107}$$

We then apply Proposition 9.40 to show that $\widetilde{f}$ belongs to $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_S$. To construct a linear extension operator from $\dot{M}^{k,\omega}_{q,\mu}(S)$ into $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$, we must find an (ω, k, S)-chain linearly depending on f. In the definition of the desired chain we use the following construction.

For $Q := Q_r(x) \in \widetilde{\mathcal{K}}_S$ we set $S_r(x) := Q \cap S$; note that $S_r(x)$ is a ball in the set S regarded as a metric subspace of ℓ^n_∞. By the Kadets–Snobar theorem [KS-1971], there exists a linear projection from the space $L_1(S_r(x)\,; d\mu)$ onto its subspace $\mathcal{P}_{k-1,n}\big|_S$ whose norm is bounded by $\sqrt{d_{k,n}} := \sqrt{\dim \mathcal{P}_{k-1,n}}$. Denoting this projection by π_Q we hence have

$$\|\pi_Q\|_1 \le \sqrt{d_{k,n}}. \tag{9.108}$$

Using the family of polynomials $\{\pi_Q(f)\}_{Q \in \widetilde{\mathcal{K}}_S}$ we then show that:

(1) There exists an (ω, k, S)-chain $\widetilde{\Pi}(f) := \{\widetilde{P}_Q(f)\}_{Q \in \widetilde{\mathcal{K}}_S}$ linearly depending on f and such that
$$|\widetilde{\Pi}(f)|_{\mathrm{Ch}} \le O(1)|f|_{M^{k,\omega}_q(\mathbb{R}^n)|_S}. \tag{9.109}$$

(2) For every $Q \in \mathcal{K}_S$
$$\widetilde{f}(c_Q) = \widetilde{P}_Q(f)(c_Q). \tag{9.110}$$

Since the operator $f \mapsto \widetilde{P}_Q(f)$ is linear, Proposition 9.40 (b) will imply that the function $\widetilde{f} \in \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_S$ and there exists a linear extension operator $T_k : \dot{M}^{k,\omega}_{q,\mu}(S) \to \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ satisfying $\|T_k\| \le O(1)$. The fact that the restriction to S of every $f \in \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ belongs to $\dot{M}^{k,\omega}_{q,\mu}(S)$ follows then straightforwardly from Theorem 2.37 of Volume I, see (9.3), and definition (9.100).

To realize this program we need several auxiliary results. The main tool in their proofs is inequality (9.58) comparing integral norms of polynomials on Markov sets. We will use a special case of this inequality presented below.

Let S be a Markov set and μ be a doubling measure on S. Clearly, the closed ball $S_r(x)$ of the space $(S, \|\cdot\|_\infty)$ is also Markov and satisfies the basic inequality (9.56) for S of Definition 9.19 with the same constant. Then inequality (9.58) yields for $p \in \mathcal{P}_{k-1,n}$ and $0 < q \le \infty$,

$$\max_{S_r(x)} |p| \le O(1)\left\{\frac{1}{\mu(S_r(x))}\int_{S_r(x)} |p|^q\right\}^{\frac{1}{q}}. \tag{9.111}$$

In particular, these integral norms are equivalent for every such a pair with the constants of equivalence independent of x and r.

Lemma 9.41. *For every $Q = Q_r(x) \in \widetilde{\mathcal{K}}_S$,*

$$\left\{\frac{1}{\mu(S_r(x))}\int_{S_r(x)} |f - \pi_Q(f)|^q \, d\mu\right\}^{1/q} \leq O(1)\mathcal{E}_k(Q\,; f\,; L_q(d\mu)). \tag{9.112}$$

Proof. By P_Q we denote a polynomial of degree $k-1$ satisfying

$$\left\{\frac{1}{\mu(S_r(x))}\int_{S_r(x)} |f - P_Q|^q \, d\mu\right\}^{1/q} = \mathcal{E}_k(Q\,; f\,; L_q(d\mu)). \tag{9.113}$$

Since π_Q is a linear projection, we get

$$f - \pi_Q(f) = (f - P_Q) + \pi_Q(f - P_Q).$$

Applying then the triangle inequality we estimate the left-hand side in (9.112) as required but with the factor $(1 + \|\pi_Q\|_q)$ instead of $O(1)$. So it remains to show that $\|\pi_Q\|_q \leq O(1)$. For $q = 1$ this norm is bounded by $\sqrt{d_{k,n}}$ by (9.108). On the other hand, inequality (9.111) implies that

$$\|\pi_Q(g)\|_1 \approx \|\pi_Q(g)\|_q$$

with the constants of equivalence independent of g and Q. Thus by the Hölder inequality we have

$$\|\pi_Q(g)\|_q \leq O(1)\|\pi_Q(g)\|_1 \leq O(1)\|\pi_Q\|_1 \, \|g\|_1 \leq O(1)\|g\|_q,$$

as required. □

Lemma 9.42. *Let $Q = Q_r(x) \in \widetilde{\mathcal{K}}_S$. Then there exists the limit*

$$\widetilde{f}(x) := \lim_{Q \to x} P_Q(x) \tag{9.114}$$

and, moreover,

$$|\widetilde{f}(x) - P_Q(x)| \leq O(1)\omega(r)|f|_{M^{k,\omega}_{q,\mu}(S)}. \tag{9.115}$$

Proof. Let $i \in \mathbb{Z}$ be determined by

$$t_i < r \leq t_{i+1} \tag{9.116}$$

and let for integers $j \leq i$,

$$Q_j := Q_{t_j}(x), \quad P_j := P_{Q_j};$$

recall that $t_i := 2^i$. We also set $Q_{i+1} := Q$ and $P_{i+1} := P_Q$. Since S is Markov, inequality (9.111) implies that

$$|P_{j+1}(x) - P_j(x)| \leq O(1)|||P_{j+1} - P_j; S_j|||$$

where for brevity we set hereafter

$$|||g; S_j||| := \left\{ \frac{1}{\mu(S_j)} \int_{S_j} |g|^q \, d\mu \right\}^{1/q} \quad \text{and} \quad S_j := Q_j \cap S.$$

Adding and subtracting f and recalling the definition of P_j, see (9.113), we estimate the right-hand side of the last inequality by

$$O(1)\{\mathcal{E}_k(Q_j\,; f\,; L_q(d\mu)) + |||f - P_{j+1}; S_j|||\}.$$

By definition, the first term is bounded by $\omega(t_j)|f|_{M^{k,\omega}_{q,\mu}(S)}$, while the second one is at most

$$\left(\frac{\mu(S_{j+1})}{\mu(S_j)}\right)^{1/q} \mathcal{E}_k(Q_{j+1}\,; f\,; L_q(d\mu)) \le \left(\frac{\mu(S_{j+1})}{\mu(S_j)}\right)^{1/q} \omega(t_{j+1})|f|_{M^{k,\omega}_{q,\mu}(S)}.$$

Since, $\frac{t_{j+1}}{t_j} = 2$, the measure μ is doubling and ω is a k-majorant, we finally get

$$|P_{j+1}(x) - P_j(x)| \le O(1)\omega(t_j)|f|_{M^{k,\omega}_{q,\mu}(S)}.$$

This, Lemma 9.38 and the choice of i, see (9.116), yield

$$\begin{aligned} \sum_{j \le i} |P_{j+1}(x) - P_j(x)| &\le O(1)|f|_{M^{k,\omega}_{q,\mu}(S)} \sum_{j \le i} \omega(t_j) \le O(1)\omega(t_i)|f|_{M^{k,\omega}_{q,\mu}(S)} \\ &\le O(1)\omega(r)|f|_{M^{k,\omega}_{q,\mu}(S)}. \end{aligned}$$

This clearly implies that the limit

$$\widetilde{f}(x) := \lim_{Q \to x} P_Q(x) = P_{i+1}(x) + \sum_{j \le i} (P_j(x) - P_{j+1}(x))$$

exists and, moreover, that

$$|\widetilde{f}(x) - P_Q(x)| \le O(1)\omega(r)|f|_{M^{k,\omega}_{q,\mu}(S)}.$$

□

Lemma 9.43. *The assertion of the previous lemma holds for* $\pi_Q(f)$.

Proof. Since π_Q is a linear projection,

$$P_Q - \pi_Q(f) = \pi_Q(P_Q - f),$$

and then inequality (9.111) and Lemma 9.41 yield for $Q := Q_r(x)$,

$$\begin{aligned} |P_Q(x) - \pi_Q(f)(x)| &\le O(1)|||P_Q - \pi_Q(f); Q \cap S||| \\ &\le O(1)\big\{\mathcal{E}_k(Q\,; f\,; L_q(d\mu)) + |||f - P_Q; Q \cap S|||\big\} \\ &\le O(1)\mathcal{E}_k(Q\,; f\,; L_q(d\mu)) \le O(1)\omega(r)|f|_{M^{k,\omega}_{q,\mu}(S)}. \end{aligned}$$

This immediately implies that

$$\lim_{Q\to x} \pi_Q(f)(x) = \lim_{Q\to x} P_Q(x) = \widetilde{f}(x)$$

and gives the required estimate of $|\widetilde{f}(x) - \pi_Q(f)(x)|$ by the right-hand side of (9.115). $\square$

Now we assume for simplicity that $|f|_{M^{k,\omega}_{q,\mu}(S)} = 1$, so that for $Q \in \widetilde{\mathcal{K}}_S$,

$$\mathcal{E}_k(Q\,;f\,;L_q(d\mu)) \le \omega(r_Q). \tag{9.117}$$

Lemma 9.44. *Let $Q \subset K$ be cubes from $\widetilde{\mathcal{K}}_S$ of radii r and R, respectively, where $r < R \le 2\operatorname{diam} S$, and let $\widetilde{K} := 2K$. Then*

$$\mathcal{E}_1(Q\,;f\,;L_q(d\mu)) \le O(1)\left\{r\int_r^{2R} \frac{\omega(t)}{t^2}\,dt + \frac{r}{R}|||f;Q\cap\widetilde{K}|||\right\}. \tag{9.118}$$

Proof. Let $J \in \mathbb{N}$ be determined by the condition $R \le 2^J r < 2R$ and let Q_j be the cube centered at c_Q and of radius $r_j := 2^j r$, $j = 0, 1, \dots, J-1$, and $Q_J := \widetilde{K}$, so that $r_J := 2R$. Then $\{Q_j\}_{0\le j\le J}$ is an increasing sequence of cubes from $\widetilde{\mathcal{K}}_S$. We also set $P_j := P_{Q_j}$, $0 \le j \le J$, see (9.113) for the definition of $P_Q \in \mathcal{P}_{k-1,n}$. Then we get

$$\begin{aligned}&\mathcal{E}_1(Q\,;f\,;L_q(d\mu)) \le \mathcal{E}_1(Q\,;f - P_Q\,;L_q(d\mu))\\ &+\sum_{j=0}^{J-1}\mathcal{E}_1(Q\,;P_{j+1}-P_j\,;L_q(d\mu)) + \mathcal{E}_1(Q\,;P_{\widetilde{K}}\,;L_q(d\mu)).\end{aligned} \tag{9.119}$$

The first term of the sum clearly equals $\mathcal{E}_k(Q\,;f\,;L_q(d\mu))$, and therefore is bounded by

$$\omega(r) \le O(1)\,r\int_r^{2R}\frac{\omega(t)}{t^2}\,dt$$

as required.

To estimate the remaining terms we use two inequalities whose proofs are postponed to the end.

(A) If $p \in \mathcal{P}_{k-1,n}$ and $Q \in \widetilde{\mathcal{K}}_S$ is a cube of radius r, then

$$\mathcal{E}_1(Q\,;p\,;L_q(d\mu)) \le O(1) r \max_{|\alpha|=1} |||D^\alpha p; Q\cap S|||. \tag{9.120}$$

(B) If $\widetilde{Q} \in \widetilde{\mathcal{K}}_S$ is a cube of radius $\widetilde{r}$ containing Q, then

$$\max_{|\alpha|=1} |||D^\alpha p; Q\cap S||| \le O(1)\frac{1}{\widetilde{r}}|||p;\widetilde{Q}\cap S|||. \tag{9.121}$$

Using these inequalities to estimate the j-th term of the sum in (9.119) we get

$$r^{-1}\mathcal{E}_1(Q\,; P_{j+1} - P_j\,; L_q(d\mu)) \le O(1)\frac{1}{r_j}\,|||P_{j+1} - P_j; Q_j \cap S|||.$$

Adding and subtracting f inside the norm in the right-hand side, and recalling the definition of polynomials P_j we estimate this norm by

$$O(1)\frac{1}{r_j}\left(\mathcal{E}_k(Q_j\,; f\,; L_q(d\mu)) + \left(\frac{\mu(Q_{j+1}\cap S)}{\mu(Q_j \cap S)}\right)^{1/q}\mathcal{E}_k(Q_{j+1}\,; f\,; L_q(d\mu))\right)$$
$$\le O(1)\frac{\omega(r_j)}{r_j};$$

here $O(1)$ depends also on the doubling constant of μ.

Moreover, summing over j the estimates

$$\frac{\omega(r_j)}{r_j} \le O(1)\int_{r_j}^{r_{j+1}} \frac{\omega(t)}{t^2}\,dt, \quad 0 \le j \le J-1,$$

we then have

$$\sum_{j=0}^{J-1}\mathcal{E}_1(P_{j+1} - P_j; Q; L_q(d\mu)) \le O(1)\,r\cdot\int_r^{2R}\frac{\omega(t)}{t^2}\,dt.$$

Finally, using again (9.120) and (9.121) we bound the last term in (9.119) by

$$O(1)r\frac{|||P_{\widetilde{K}}; \widetilde{K}\cap S|||}{R} \le O(1)r\frac{2\,|||f; \widetilde{K}\cap S|||}{R}.$$

Combining these estimates we prove (9.119).

To complete the proof of the lemma it remains to prove assertions (A), (B). For (A) we have

$$\mathcal{E}_1(Q\,; p\,; L_q(d\mu)) \le \inf_{c\in\mathbb{R}}\|p - c\|_{C(Q)} \le O(1)r\max_{|\alpha|=1}\|D^\alpha p\|_{C(Q)},$$

where the first inequality is trivial and the second is proved as follows.

Using the homothety of $\mathbb{R}^n$ we may replace Q by the unit cube $Q_0 := [0,1]^n$. The functions in p of the both parts of this inequality are norms on the finite-dimensional factor-space $\mathcal{P}_{k-1,n}/\mathcal{P}_{0,n}$ and therefore they are equivalent with constants depending only on k and n. This implies the desired inequality.

Continuing the derivation we now use inequality (9.111) to have

$$||D^\alpha p||_{C(Q)} \le O(1)|||D^\alpha p; Q\cap S|||$$

and this completes the proof of (9.120).

Inequality (9.121) is proved similarly using the classical Markov inequality at the second step. □

Lemma 9.45. *$f = \widetilde{f}$ modulo μ-measure zero.*

Proof. Let $L(f)$ be the Lebesgue set of f, i.e., the set of points $x \in S$ such that

$$f(x) = \lim_{r\to 0} \frac{1}{\mu(S_r(x))} \int_{S_r(x)} f\, d\mu.$$

Since μ is a Borel doubling measure on S, the family of "balls" $\{S_r(x)\,;\, x \in S, 0 < r \leq 1\}$ satisfies, for all $x, y \in S$ and $0 < r \leq 1$, the conditions

(i) $S_r(x) \cap S_r(y) \neq \phi$ implies $S_r(y) \subset S_{3r}(x)$;

(ii) $\mu\big(S_{2r}(x)\big) \leq O(1)\mu\big(S_r(x)\big)$.

Therefore a Vitaly type covering argument may be applied to this setting, see, e.g., Stein [Ste-1993, Sec. I.3], to obtain

$$\mu(S \setminus L(f)) = 0.$$

It remains to show that

$$f(x) = \widetilde{f}(x) \text{ for } x \in L(f).$$

To this end we choose a cube $Q = Q_r(x) \in \widetilde{\mathcal{K}}_S$ with $0 < r < 1$ and set

$$f_r(x) := \frac{1}{\mu(S_r(x))} \int_{S_r(x)} f\, d\mu.$$

By the triangle inequality and inequality (9.111) we obtain for $f_r(x) - P_Q$,

$$|f_r(x) - P_Q(x)| \leq O(1)\{|||f - f_r(x); Q \cap S||| + \mathcal{E}_k(Q\,; f\,; L_q(d\mu))\}. \tag{9.122}$$

But $f \mapsto f_r$ is a projection of norm 1 from $L_1(S_r(x))$ onto the space $\mathcal{P}_{0,n}$ and the argument of Lemma 9.41 gives in this case

$$|||f - f_r(x); Q \cap X||| \leq O(1)\mathcal{E}_1(Q\,; f\,; L_q(d\mu)).$$

This, Lemma 9.44 and (9.117) imply that for a sufficiently small r and a fixed cube $K \subset Q$ of radius 1 the right-hand side of (9.122) is bounded by

$$\begin{aligned}
&O(1)\{\mathcal{E}_1(Q\,; f\,; L_q(d\mu)) + \mathcal{E}_k(Q\,; f\,; L_q(d\mu))\}\\
&\leq O(1)\left\{ r\left(\int_r^2 \frac{\omega(t)}{t^2}\,dt + |||f; K \cap S|||\right) + \omega(r)\right\}.
\end{aligned}$$

We conclude from here that for every $0 < \epsilon < 2$,

$$\begin{aligned}
&\lim_{r\to 0} |f_r(x) - P_Q(x)|\\
&\leq O(1)\,\overline{\lim_{r\to 0}}\left(\omega(r) + r\left(\int_r^\epsilon \frac{\omega(t)}{t^2}\,dt + \int_\epsilon^2 \frac{\omega(t)}{t^2}\,dt + |||f; K \cap S|||\right)\right)\\
&= O(1)\,\overline{\lim_{r\to 0}}\left(r\int_r^\epsilon \frac{\omega(t)}{t^2}\,dt\right) \leq O(1)\omega(\epsilon).
\end{aligned}$$

Letting $\epsilon \to 0$ and noting that $\lim_{r\to 0} f_r(x) = f(x)$ for a Lebesgue point x and $\lim_{Q\to x} P_Q(x) = \widetilde{f}(x)$ we complete the proof of the lemma. □

Now we finalize the proof of Theorem 9.36. To this end, we must find an (ω, k)-chain satisfying conditions (9.109) and (9.110). We construct this chain using the polynomials

$$\widetilde{P}_Q(f) := \pi_Q(f) - \pi_Q(f)(c_Q) + \widetilde{f}(c_Q),$$

where $Q \in \widetilde{\mathcal{K}}_S$ and π_Q is the Kadets–Snobar projection, see Lemma 9.41. Then $\widetilde{P}_Q(f)(c_Q) = \widetilde{f}(c_Q)$ and (9.110) is true for the family $\widetilde{\Pi}(f) := \{\widetilde{P}_Q\}_{Q\in\widetilde{\mathcal{K}}_S}$. We will show that (9.109) is also true for $\widetilde{\Pi}(f)$.

Let $Q \subset Q'$ be cubes from $\widetilde{\mathcal{K}}_S$ of radii $r < r'$, respectively, satisfying for some $i \in \mathbb{Z}$ the condition

$$2^i \le r < r' \le 2^{i+2}.$$

Due to the polynomial inequality (9.58) and Lemma 9.41 we then have

$$\begin{aligned}
&\max_Q |\pi_Q(f) - \pi_{Q'}(f)| \le O(1) \max_{S\cap Q} |\pi_Q(f) - \pi_{Q'}(f)| \\
&\le O(1) |||\pi_Q(f) - \pi_{Q'}(f); S \cap Q||| \\
&\le O(1) \left\{ \mathcal{E}_k(Q\,; f\,; L_q(d\mu)) + \left(\frac{\mu(Q' \cap S)}{\mu(Q \cap S)} \right)^{1/q} \mathcal{E}_k(Q'\,; f\,; L_q(d\mu)) \right\}.
\end{aligned}$$

Since μ is a doubling measure on S, the ratio of μ-measures is bounded by $O(1)$. Together with inequality (9.117) this implies that

$$\max_Q |\pi_Q(f) - \pi_{Q'}(f)| \le O(1)\omega(r').$$

Moreover, by Lemma 9.43,

$$|\widetilde{f}(c_Q) - \pi_Q(f)(c_Q)| \le O(1)\omega(r).$$

Due to the definition of $\widetilde{P}_Q(f)$ this implies the required inequality

$$\max_Q |\widetilde{P}_Q(f) - \widetilde{P}_{Q'}(f)| \le O(1)\omega(r').$$

The proof of Theorem 9.36 is complete. □

9.3 Simultaneous extensions from uniform domains

We continue to discuss the Restricted Extension Problem partially studied in Section 2.5 of Volume I for the space $\dot{C}^{l,\omega}(\mathbb{R}^n)$. Let us recall that this problem concerns two distinct approaches to the concept of smoothness for functions on a domain $G \subset \mathbb{R}^n$, the "*outer*" approach based on the traces to G and the "*inner*" one using the behavior of functions on G. The goal is, given a smoothness space, to find the maximal class of domains for which both of these ways yield the same result. There are few known results in this direction; all of them are presented in Chapter 2 of Volume I and concern functions of two variables, see Proposition 2.68 and Theorems 2.69 and 2.83 there. Since the problem is extremely difficult even for the space $C^{1,1}(\mathbb{R}^n)$ with $n \geq 3$, a more realistic aim is to find the largest possible class of domains possessing the required property.

Here we will show that the class of uniform domains introduced in Definition 2.81 of Volume I gives the desired extension result for the spaces $C^\ell \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ with the quasipower k-majorants ω. In fact, uniform domains form the largest known class which is universal with respect to the extension property in question, in the sense that the analogous (simultaneous) extension results hold for all classical smoothness spaces (BMO, Morrey–Campanato, Sobolev, Besov and the like) defined on these domains. Some of them have been proved by different methods based on the P. Jones seminal idea presented in his papers [Jon-1980] and [Jon-1981] on simultaneous extensions of BMO and Sobolev spaces on uniform domains. The local approximation approach described here is rather universal and provides extension results for all aforementioned spaces defined over a wide range of integrable spaces (such as Lorentz, Orlicz, Morrey, BMO etc.). Unfortunately, this vast theme cannot be presented here, mostly because analysis essentially prevails in this field over geometry dissonating with the declared theme of the present book.

9.3.1 Uniform domains

Uniform domains were introduced in Section 2.6 of Volume I in order to formulate two and prove one of the extension results for Sobolev spaces. The proofs we presented there concern only the bounded simply connected domains in the plane (quasidisks) whose geometric properties are relatively simple, see Volume I, Proposition 2.84. However, the general case requires a much more detailed study; the corresponding geometric facts are formulated and used in subsection 9.3.2 while most of their proofs are presented in subsection 9.3.3.

For convenience, we begin our discussion of the concept of a uniform domain by recalling the basic Definition 2.81 of Volume I, which we now split into two parts – Definition 9.46 of global uniformity and Definition 9.49 of local uniformity. The definition of the class of *uniform domains* is based on ideas of John [Jo-1961]. Roughly speaking, a domain is uniform if every two points could be joined by a curvilinear doubled cone of a suitable size. This cone is defined as follows.

Let $\gamma : [0, 1] \to \mathbb{R}^n$ be a rectifiable curve with endpoints x, y. A *cone with axis* γ *and parameter* $c > 0$ is the open set

$$Kn(\gamma\,;\,c) := \cup\big\{B_{c\rho(z)}(z)\,;\,z \in \gamma([0,1])\big\} \tag{9.123}$$

where

$$\rho(z) := \min\big\{\|x - z\|, \|y - z\|\big\}.$$

Such a cone is also defined for $y = \infty$ (i.e., γ is unbounded and locally rectifiable).

Definition 9.46. A domain $G \subset \mathbb{R}^n$ is said to be c-uniform if for every pair $x, y \in G$ there is a rectifiable curve γ joining x and y in G such that

(a) $\ell(\gamma) \le c\|x - y\|$;

(b) for the cone with axis γ and parameter c,

$$Kn(\gamma\,;\,c) \subset G. \tag{9.124}$$

The class of these domains is denoted by $\mathcal{U}(\mathbb{R}^n)$.

Remark 9.47. (a) (John) condition (b) of this definition can be written equivalently as

$$\rho(z) \le c\,\mathrm{dist}(z, \partial G), \quad z \in \gamma([0,1]). \tag{9.125}$$

(b) In turn, the first condition can be replaced by the equivalent condition (with, maybe, another c):

$$\mathrm{diam}\,\gamma \le c\|x - y\|.$$

Example 9.48. (a) According to Definition 2.63 of Volume I, uniform domains are quasiconvex. An example of the strip $\big\{x \in \mathbb{R}^2\,;\,|x_1| < 1\big\}$ which is not uniform, shows that the converse is not true.

Bounded domains with Lipschitz boundaries (named *Lipschitz domains* in Definition 2.72 of Volume I) are uniform. In fact, they are quasiconvex, see Volume I, formula (2.132), and the John condition now holds with cones whose axes are straight segments, see Volume I, Proposition 2.73.

(b) However, the boundary of a uniform domain may be quite pathological. For example, the classical von Koch snowflake bounds a uniform domain, but each of its subarcs has the Hausdorff dimension $\frac{\log 4}{\log 3} > 1$, see Volume I, Example 4.22. More generally, every quasidisk (the image of an open disk or half-plane under a quasiconformal mapping of $\mathbb{R}^2$ onto itself) is uniform (the Martio–Savras theorem [MS-1979]).

In the case $n > 2$, for every $1 < p < n$, one can construct a hypersurface in $\mathbb{R}^n$ of Hausdorff dimension p bounding a uniform domain.

(c) One of the typical singularities of quasidisks demonstrates a domain bounded by two spirals emanating from a fixed point. We consider a quantitative version of this example which will be exploited later.

Let L_0 be a piecewise linear curve in the plane subsequently connecting points $(-1,0), (-1,1)$, $(1,1)$, $(1,-1)$, $\left(-\frac{1}{2},-1\right)$ and $\left(-\frac{1}{2},0\right)$. For $i \in \mathbb{N}$, we set

$$G_0 := L_0 + Q_{\frac{1}{8}}(0), \; G_i := 2^{-i} G_0,$$

where λG denotes the λ-homothety with respect to $(0,0)$. Then the open connected domain

$$G := \bigcup_{i=0}^{\infty} G_i$$

is a spiral domain twisting around the origin whose coils are G_i. Its axis is the infinite piecewise linear curve

$$L := \bigcup_{i=0}^{\infty} L_i,$$

where $L_i := 2^{-i} L_0, \; i \in \mathbb{N}$.

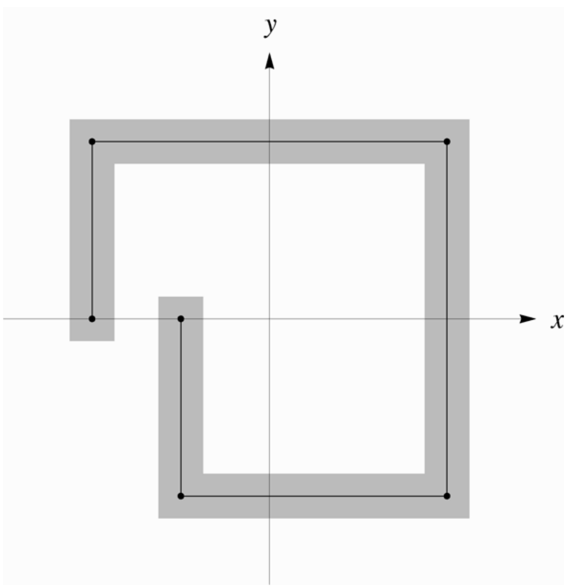

Figure 9.1: Broken line L_0 and closed domain G_0.

Let us show that $G \in \mathcal{U}$. Let $x, y \in G$ and $x \in G_i$, $y \in G_j$ for some $i \leq j$. We must check that there is a curve γ joining x and y in G and such that for some $c > 0$,

$$\ell(\gamma) \leq c\|x - y\|, \tag{9.126}$$

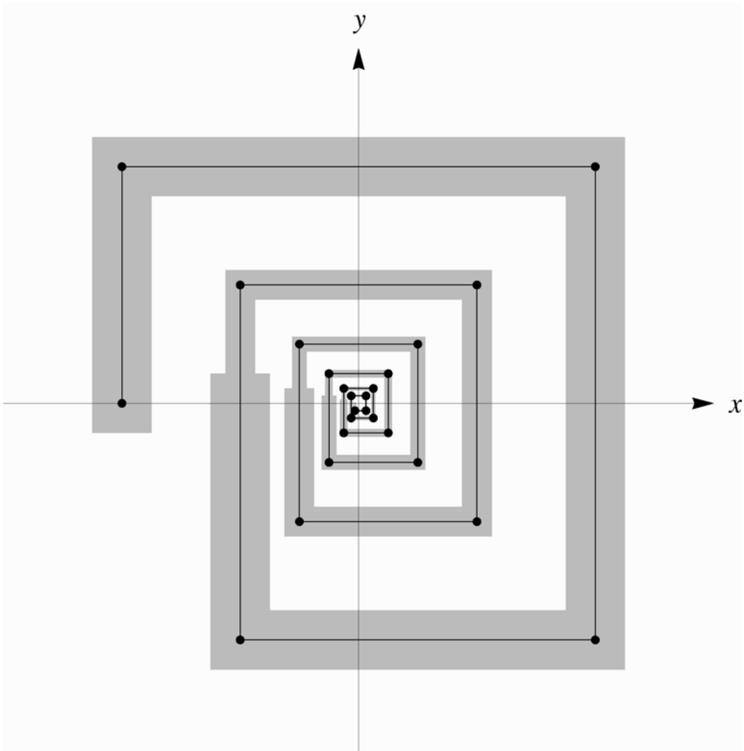

Figure 9.2: Broken line L and closed domain G.

and, moreover,

$$Kn(\gamma\,;\,c) \subset G. \tag{9.127}$$

Clearly, we can (and will) use here the uniform norm $\|x\|_\infty := \max_{1\le i\le n} |x_i|$ instead of the Euclidean one which is more suitable for the next derivation.

First, let $\|x-y\|_\infty \le 2^{-i-3}$. Then x and y belong to $G_i \cup G_{i+1}$ and this set is the union of rectangles whose number is bounded by a constant independent of i. Hence, conditions (9.126) and (9.127) trivially hold.

Now, let $\|x-y\|_\infty > 2^{-i-3}$. By x', y' we denote the points of the infinite spiral L closest to x and y, respectively and define the desired curve γ as being composed of three subarcs γ_i where $\gamma_1 := [x, x']$, $\gamma_3 := [y', y]$ and γ_2 is the connected part of the spiral L bounded by the endpoints x', y'.

Show that γ satisfies (9.126). By the definition of G,

$$\operatorname{diam}\Big(\bigcup_{k\geq i} G_k\Big) \approx 2^{-i}, \quad 1 \leq i \leq \infty. \tag{9.128}$$

Therefore, for every $z \in \gamma \cap G_k$, $i \leq k \leq j$, we have

$$\|x-z\|_\infty + \|y-z\|_\infty \leq 2\operatorname{diam}\Big(\bigcup_{k\geq i} G_k\Big) \leq O(1)2^{-i} \leq O(1)\|x-y\|_\infty.$$

This implies that $\operatorname{diam}\gamma \leq O(1)\|x-y\|_\infty$, and (9.126) is true, cf. Remark 9.47 (b).

Now we check (9.127). If $z \in \gamma_1$, then $z \in x' + Q_{2^{-i-3}}(0)$ by the definition of γ_1; moreover, $\|x-y\|_\infty > 2^{-i-3}$ and therefore $\operatorname{dist}(z, \partial G) \geq \|x-z\|_\infty = \min\big(\|x-z\|_\infty, \|y-z\|_\infty\big)$. The same argument works for $z \in \gamma_3$.

Now let $z \in \gamma_2$. Then $z \in L_k$ for some $i \leq k \leq j$ and therefore $z + Q_{2^{-k-3}}(0) \subset G$. We conclude from this that

$$\operatorname{dist}(z, \mathbb{R}^2 \backslash G) \geq 2^{-k-3}.$$

On the other hand, (9.128) implies that

$$\|x-y\|_\infty \leq \operatorname{diam}\Big(\bigcup_{\ell=j}^{k} G_\ell\Big) \leq O(1)2^{-k},$$

whence

$$\operatorname{dist}\big(z, \mathbb{R}^2\backslash G\big) \geq c\|x-y\|_\infty \geq c_1 \min\big(\|x-z\|_\infty, \|y-z\|_\infty\big)$$

for some constants $c, c_1 > 0$ independent of x, y, z.

Hence, (9.127) is also true, cf. Remark 9.47 (a).

We can complicate this example by gluing a finite member of smaller spiral domains of this type to the boundary of G, then repeating this process for the boundaries of these smaller spirals, and so on.

For applications to the extension problems the class $\mathcal{U}$ may be widened as follows.

Definition 9.49. A domain $G \subset \mathbb{R}^n$ is said to be (c_0, c_1)-uniform if for every pair $x, y \in G$ satisfying

$$\|x-y\| \leq c_0,$$

there exists a curve γ joining x and y in G and such that

(a) $\ell(\gamma) \leq c_1\|x-y\|$;

(b) $Kn(\gamma\,;\,c_1) \subset G$.

It is readily seen that a *bounded* c-uniform domain G is also (c_0, c_1)-uniform with $c_0 := \operatorname{diam} G$ and $c_1 := c$. On the other hand, the strip $\{x \in \mathbb{R}^n\,;\,|x_1| < 1\}$ and its bi-Lipschitz equivalents are typical examples of nonuniform and (c_0, c_1)-uniform domains.

Examples 9.48 (b) and (c) make an impression that the geometric structure of a uniform domain may be very intricate. The next result shows that this is not the case. It implies, for instance, that the boundary ∂G of a uniform domain cannot have outer cusps (but may have inner ones) and its Lebesgue n-measure is zero.

Proposition 9.50. *Let $G \subset \mathbb{R}^n$ be (c_0, c_1)-uniform. Then the following is true:*

(a) *G is locally (Ahlfors) n-regular meaning that for some $\delta > 0$ and every cube Q centered at G of diameter less than c_0,*

$$|Q \cap G| \geq \delta |Q|.$$

(b) $|\partial G| = 0$.

(c) *For every cube Q from (a),*

$$\operatorname{diam}(G \cap Q) \geq \delta_1 r_Q,$$

where $\delta_1 := \delta^{\frac{1}{n}}$.

Proof. (a) Since G is connected and $\operatorname{diam} Q < c_0 \leq \operatorname{diam} G$, there is a point $\tilde{x} \in G \cap \partial Q$; hence, c_Q and $\tilde{x}$ belong to G and $\|c_Q - \tilde{x}\| = r(< c_0)$.

Let γ be a curve connecting c_Q and $\tilde{x}$ in G and such that $Kn(\gamma\,;\,c_1) \subset G$. Let z be a point dividing γ into two subarcs equal in length. Then $\rho(z) \geq \frac{1}{2}\,\|c_Q - \tilde{x}\| = \frac{1}{2}\,r$ and $Q_{c_1\rho(z)}(z) \subset G$. Hence

$$|G \cap Q| \geq \left|Q \cap Q_{c_1\rho(z)}(z)\right| \geq (4c_1)^{-n} r^n = (4c_1)^{-n}|Q| =: \delta(c_1, n)|Q|.$$

The result is established.

(b) Assume, on the contrary, that for some cube $K \subset \mathcal{K}_G$,

$$|K \cap \partial G| > 0.$$

Then from the Lebesgue differentiation theorem applied to the characteristic function χ_G we obtain

$$\lim_{Q \to x} \frac{1}{|Q|} \int_Q \chi_G dx = \chi_G(x) = 0$$

for almost all $x \in K \cap \partial G$. On the other hand, for sufficiently small Q,

$$\frac{1}{|Q|} \int_Q \chi_G dx = \frac{|Q \cap G|}{|Q|} \geq \delta > 0,$$

a contradiction.

(c) By the Bieberbach–Urysohn isodiametric inequality, see, e.g., [BZ-1988, Sec. 4.2],

$$r_Q = |Q|^{\frac{1}{n}} \le \sqrt[n]{\delta^{-1}|G \cap Q|}.$$

The proof is complete. □

9.3.2 Simultaneous extensions of Lipschitz spaces from uniform domains

Here we prove the extension theorems for spaces $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ and (as a consequence) for more general spaces $C^l\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ using the local approximation approach. A generalization concerning analogs of these spaces defined over $L_p(\mathbb{R}^n)$, $0 < p < \infty$, is formulated similarly and can be proved by a modification of the argument presented in this subsection. Our main result is

Theorem 9.51. *Let $G \subset \mathbb{R}^n$ be c-uniform and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a quasipower k-majorant, i.e.,*

$$C_\omega := \sup_{t>0} \frac{1}{\omega(t)} \int_0^t \frac{\omega(u)}{u}\, du < \infty.$$

Then there exists a linear extension operator $\mathcal{E}_k : \dot{\Lambda}^{k,\omega}(G) \to \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ whose norm is bounded by a constant depending only on k, n, c and C_ω.

Proof. Without loss of generality we assume that the k-majorant ω satisfies

$$\lim_{t\to 0} \frac{\omega(t)}{t^k} = \infty. \tag{9.129}$$

In fact, otherwise $\omega(t) = O(t^k)$ and $\dot{\Lambda}^{k,\omega}(G)$ coincides with $\dot{C}^{k-1,1}(G)(= \dot{W}^k_\infty(G))$. Since G is quasiconvex, the existence of the required extension operator follows from Theorem 2.64 of Volume I. So we continue the proof under this condition.

As we will see, each function $f \in \dot{\Lambda}^{k,\omega}(G)$ is uniformly continuous on every subset $Q \cap G$ where $Q \in \mathcal{K}_G$. Hence, f can be uniquely extended by continuity to all of $\overline{G}$. Denoting by $\dot{\Lambda}^{k,\omega}(\overline{G})$ the space formed by these extensions we will use the sufficiency part of Theorem 9.6 to prove the desired extension result for this space. The main part of the consequent proof is the following fact.

For every cube $Q \in \mathcal{K}_{\overline{G}}$,

$$E_k(G \cap Q\,; f) \le O(1)\omega(r_Q)|f|_{\Lambda^{k,\omega}}(G). \tag{9.130}$$

As before, $O(1)$ stands for a constant depending only on inessential parameters (k, n, C_ω and c).

From here we derive, for a suitable linear projection $Pr_Q : C(Q \cap G) \to \mathcal{P}_{k-1,n}$, where $Q \in \mathcal{K}_{\overline{G}}$, the inequality

$$\|f - Pr_Q f\|_{C(Q\cap G)} \le O(1)\omega(r_Q)|f|_{\Lambda^{k,\omega}(G)}. \tag{9.131}$$

Then we complete the proof of the theorem using the argument of Theorem 9.6. In turn, due to (9.3), inequality (9.130) holds for every cube $Q \subset G$. Therefore we should prove the same only for cubes $Q \in \mathcal{K}_{\overline{G}}$ intersecting ∂G.

This is the content of the next result proved, in essence, by Shvartsman [Shv-1986, Prop. 1.58].

Proposition 9.52. *For every cube Q centered at $\overline{G}$ and intersecting ∂G,*

$$E_k(Q \cap G\,; f) \le O(1)\omega(r_Q)|f|_{\Lambda^{k,\omega}(G)}. \tag{9.132}$$

Proof. For simplicity, we denote by $\mathcal{W}(G)$ the Whitney cover of G (i.e., $\mathcal{W}(G):= \mathcal{W}_{G^c}$, according to the notation of Proposition 9.1), and by $\mathcal{W}^*(G)$ the cover of G by cubes $Q^* := \frac{9}{8}Q$, where $Q \in \mathcal{W}(G)$. Recall that all $Q^* \subset G$ and the order of the cover $\mathcal{W}^*(G)$ satisfies

$$\operatorname{ord} \mathcal{W}^*(G) \le c(n), \tag{9.133}$$

see Proposition 9.1 (e).

Due to Lemma 9.60 from the next subsection, for the cube Q there is a Whitney cube $I \in \mathcal{W}(G)$ such that

$$I \subset \ell Q \quad \text{and} \quad r_I \approx r_Q \tag{9.134}$$

with the constant $\ell > 1$ and constants of equivalence depending only on inessential parameters.

Since inequality (9.132) suffices to prove for ℓQ instead of Q, one can assume that $\ell = 1$. Now we use this I as the initial cube of a “chain” of Whitney cubes $Q_1 = I, Q_2, \dots, Q_m$ such that Q_i and Q_{i+1} touch, $1 \le i \le m$. The final cube $F = Q_m$ of this chain is introduced as follows.

Let $U_Q := \bigcup\{Q' \in \mathcal{W}(G)\,; Q' \cap Q \ne \emptyset\}$ be the set of Whitney cubes intersecting Q and let p_S where $S \subset G$ denote a polynomial of degree $k-1$ such that

$$E_k(S\,; f) = \|f - p_S\|_{C(S)}.$$

Since $G = \bigcup \mathcal{W}(G)$, we get

$$E_k\big(Q \cap G\,; f\big) \le E_k(U_Q\,; f) \le \sup\big\{\|f - p_{I^*}\|_{C(Q')}\,; Q' \in U_Q\big\}.$$

Hence, there is a cube from $\mathcal{W}(G)$ denoted by F such that $F \cap Q \ne \emptyset$ and

$$E_k(Q \cap G\,; f) \le 2\|f - p_{I^*}\|_{C(F)}. \tag{9.135}$$

This F will be the final cube of the desired chain.

To find the remaining cubes of the chain we apply Lemma 9.61, see the next subsection, to the pair I, F, assuming without loss of generality that $I \ne F$. This lemma asserts that if the Whitney cubes I, F are such that

$$\|c_I - c_F\| + r_F \le O(1)r_I, \tag{9.136}$$

then there exists a family of cubes $\{Q_i\}_{1 \le i \le m} \subset \mathcal{W}(G)$ such that

(a) $I = Q_1$ and $F = Q_m$;

(b) cubes Q_i and Q_{i+1} touch, $1 \le i < m$;

(c) for some constant $\gamma = O(1)$ and every $1 \le j < i \le m$,

$$Q_i \subset \gamma Q_j; \tag{9.137}$$

(d) each cube occurs in the chain at most $O(1)$ times.

We postpone the proof of (9.136) to the final part, and for now use the connected chain $\{Q_i\}$ to prove the inequality

$$E_k\big(Q \cap G\,; f\big) \le O(1) \sum_{i=1}^{m} E_k(Q_i^*\,; f). \tag{9.138}$$

Using (9.135) and (9.137) we first get

$$E_k\big(Q \cap G\,; f\big) \le 2\bigg(\sum_{i=1}^{m-1} \big\|p_{Q_{i+1}^*} - p_{Q_i^*}\big\|_{C(\gamma Q_i)} + \|f - p_{F^*}\|_{C(F)} \bigg).$$

The last term in the brackets is at most $E_k(Q_m^*\,; f)$ and it remains to estimate the i-th term there. We show that

$$\big\|p_{Q_{i+1}^*} - p_{Q_i^*}\big\|_{C(\gamma Q_i)} \le O(1)\big(E_k(Q_i^*\,; f) + E_k(Q_{i+1}^*\,; f)\big)$$

and in this way prove (9.138).

To this end, we first note that Q_i and Q_{i+1} touch and therefore

$$\big|Q_i^* \cap Q_{i+1}^*\big| \ge c|Q_i^*|$$

for some $c = c(n) > 0$, see Volume I, Corollary 2.15 or Proposition 9.1. Further we use the next inequality for a polynomial $p \in \mathcal{P}_{k,n}$ and a closed subset S of a convex set $C \subset \mathbb{R}^n$:

$$\max_C |p| \le \left(\frac{4n|C|}{|S|}\right)^k \max_S |p|, \tag{9.139}$$

see Lemma 9.5.

Applying these two facts to the polynomial $p := p_{Q_{i+1}^*} - p_{Q_i^*}$ we get

$$\|p\|_{C(\gamma Q_i)} \le (4n\gamma)^{k-1}\|p\|_{C(Q_i^*)} \le (4n\gamma)^{k-1}\left(\frac{4n}{c(n)}\right)^{k-1} \|p\|_{C(Q_i^* \cap Q_{i+1}^*)}.$$

Since the norm in the right-hand side is at most

$$\big\|f - p_{Q_i^*}\big\|_{C(Q_i^*)} + \big\|f - p_{Q_{i+1}^*}\big\|_{C(Q_{i+1}^*)} = E_k(Q_i^*\,; f) + E_k(Q_{i+1}^*\,; f),$$

inequality (9.138) is proved modulo the proof of (9.136).

To establish the latter inequality we apply Lemma 9.59 below which asserts that if a cube Q centered at G intersects ∂G and a cube $Q' \in \mathcal{W}(G)$ intersects Q, then

$$r_{Q'} \le 2r_Q. \tag{9.140}$$

This and (9.134) yield, for $Q' := F$,

$$\|c_I - c_F\| + r_F \le \|c_I - c_Q\| + \|c_Q - c_F\| + r_F \le r_Q + r_Q + r_F \le 6r_Q \le O(1)r_I,$$

as required.

Now we use inequality (9.138) to derive the final result. To this end, we denote by $[I, F]$ and $[I^*, F^*]$ the chain $\{Q_i\}_{1\le i\le m}$ and the family $\{Q_i^*\}_{1\le i\le m} \subset \mathcal{W}^*(G)$, respectively. Since every subfamily of the family having a point in common contains at most $\mu = \mu(n)$ cubes, see Proposition 9.1 (e), and each cube appears in the chain at most $O(1)$ times, every point is contained in at most $\widetilde{\mu} = O(1)$ cubes of the family, i.e.,

$$\operatorname{ord}[I^*, F^*] \le \widetilde{\mu} = O(1).$$

Now we apply to $[I^*, F^*]$ the following result of Brudnyi and Kotlyar [BKo-1970], but postpone its proof to the final part.

Lemma 9.53. *Let $\mathcal{Q}$ be a family of cubes of $\mathbb{R}^n$ homothetic to the unit cube $[0,1]^n$. Assume that $\operatorname{ord}\mathcal{Q} \le N$. Then $\mathcal{Q}$ is the union of at most $2^n(N-1)+1$ disjoint subfamilies.*

Using this we divide the family $[I^*, F^*]$ into $\lambda := 2^n(\widetilde{\mu}-1)+1 = O(1)$ disjoint subfamilies $\mathcal{C}_j$. Then the right-hand side of (9.138) is bounded by

$$O(1) \max_{1\le j\le \lambda} \sum_{K\in\mathcal{C}_j} E_k(f\,;\,K^*)$$

and we may, without loss of generality, assume that the family $[I^*, F^*]$ is itself disjoint.

Under this assumption we define the subdivision of the chain $[I, F]$ in the following fashion.

Let $1 = t_0 > t_1 > \cdots$ be a sequence in $(0, 1]$ such that

$$t_i/t_{i+1} = 2^k, \quad i \in \mathbb{Z}_+.$$

Given $i \in \mathbb{Z}_+$, define a subset $\mathcal{L}_i$ of the set $[I^*, F^*]$ by

$$\mathcal{L}_i := \Big\{K^* \in [I^*, F^*]\,;\, t_{i+1} < \frac{r_{K^*}}{3\gamma r_Q} \le t_i\Big\}, \tag{9.141}$$

where γ is defined in (9.137); some $\mathcal{L}_i$ may be empty. Since $r_{K^*} = \frac{9}{8} r_K \leq \frac{9}{8}\gamma 2r_Q < 3\gamma r_Q$ by (9.140) and $\lim_{i\to\infty} t_i = \lim_{i\to\infty} 2^{-ik} = 0$, we have

$$[I^*, F^*] = \bigsqcup_i \mathcal{L}_i.$$

Show now that

$$\ell := \sup_i \operatorname{card} \mathcal{L}_i = O(1).$$

In fact, by the definition of $\mathcal{L}_i$ and the choice of $\{t_i\}$, for every pair K, K' of cubes from $\mathcal{L}_i$, we get

$$2^{-k} = \frac{t_{i+1}}{t_i} \leq \frac{r_K}{r_{K'}} \leq \frac{t_i}{t_{i+1}} = 2^k.$$

Moreover, the cubes of $\mathcal{L}_i$ do not intersect and, by (9.137), all of them are subsets of a cube $c\gamma Q_j^*$, where Q_j^* stands for the cube in $\mathcal{L}_i$ of the maximal index. Then the comparison of volumes yields

$$\ell_i := \operatorname{card} \mathcal{L}_j \leq (4c\gamma)^n = O(1)$$

and the desired estimate for ℓ follows.

Let $\mathcal{L}_i := \{K_{ji}^*\}_{1\leq j\leq \ell}$, where the cubes of index $j > \ell_i$ are assumed to be empty sets. Then

$$[I^*, F^*] = \bigsqcup_{i\in\mathbb{Z}} \{K_{ji}^*\}_{1\leq j\leq \ell} =: \bigsqcup_{j=1}^{\ell} \mathcal{R}_j,$$

where we set

$$\mathcal{R}_j := \{K_{ji}^*\}_{i\in\mathbb{Z}_+}.$$

Hence, inequality (9.138) may be rewritten as

$$E_k(Q\cap G\,; f) \leq O(1)\sum_{j=1}^{\ell} \sum_{K\in\mathcal{R}_j} E_k(K^*\,; f) \leq O(1) \max_{1\leq j\leq \ell} \sum_{K\in\mathcal{R}_j} E_k(K^*\,; f).$$

Moreover, by definition, for every $1 \leq j \leq \ell$ and $i \in \mathbb{Z}_+$,

$$r_{K_{ji}^*} \leq 3\gamma t_i r_Q.$$

Finally, by Theorem 2.37 of Volume I, see (9.3),

$$E_k(K^*\,; f) \leq O(1)\omega(r_K)|f|_{\Lambda^{k,\omega}(G)}.$$

Combining all these estimates we finally get

$$E_k(f\,; Q\cap G) \leq O(1)\left(\sum_{i\in\mathbb{Z}_+} \omega(3\gamma t_i r_Q)\right)|f|_{\Lambda^{k,\omega}(G)}. \tag{9.142}$$

Since ω is a k-majorant, we get

$$\omega(3\gamma t_i r_Q) \leq O(1) \int_{t_i}^{t_{i+1}} \frac{\omega(sr_Q)}{s}\, ds.$$

Summing over $i \geq 0$ and applying (9.129) we estimate the sum in (9.142) by

$$O(1) \int_0^1 \frac{\omega(sr_Q)}{s}\, ds \leq O(1)\omega(r_Q).$$

This completes the proof of inequality (9.132).

Proof of Lemma 9.53. Assume first that the family of cubes $\mathcal{Q}$ is finite and use induction on card $\mathcal{Q}$. Since the case card $\mathcal{Q} \leq 2^n(N-1)+1$ is trivial, we should prove the result for card $\mathcal{Q} = \ell$ provided that it is true for all such families with $\ell - 1 \geq 2^n(N-1)+1$ cubes. Let $K_{\min}$ be a cube of minimal volume in $\mathcal{Q}$. By the induction hypothesis,

$$\mathcal{Q}\backslash\{K_{\min}\} = \bigsqcup_{i \leq m} \mathcal{Q}_i,$$

where every subfamily $\mathcal{Q}_i$ consists of pairwise disjoint cubes and $m := 2^n(N-1)+1$ (among these there may be empty subfamilies). Since $K_{\min}$ is minimal, every cube $K \in \mathcal{Q}$ intersecting $K_{\min}$ should contain at least one of its 2^n-vertices. As ord $\mathcal{Q} \leq N$, such a vertex can be covered by at most $N-1$ cubes from the set $\mathcal{Q}\backslash\{K_{\min}\}$. Therefore the number of cubes from this set intersecting $K_{\min}$ is at most $2^n(N-1)$. Since the number of subfamilies $\mathcal{Q}_i$ is $2^n(N-1)+1$, the cubes of one of these subfamilies do not intersect $K_{\min}$. Adding $K_{\min}$ to this $\mathcal{Q}_i$, one obtains the required partition of $\mathcal{Q}$ into $2^n(N-1)+1$ disjoint subfamilies.

Now, let card $\mathcal{Q} = \infty$. Define the intersection graph $\Gamma(\mathcal{Q})$ whose vertices are cubes $K \in \mathcal{Q}$, and K and K' are joined by an (unique) edge if $K \cap K' \neq \emptyset$. The validity of Lemma 9.53 for $\mathcal{Q}$ is equivalent to the existence of a partition of the set of vertices of $\Gamma(\mathcal{Q})$ into $\ell := 2^n(N-1)+1$ subsets such that edges join only vertices from distinct subsets. A graph with this property is said to be *ℓ-colored.* From the first part of the proof it follows that every finite subgraph of $\Gamma(\mathcal{Q})$ is ℓ-colored. The application of the Zorn lemma allows us to derive from this fact ℓ-colorability of $\Gamma(\mathcal{Q})$ (the so called de Bruin–Erdos theorem, see, e.g., the book [Har-1969] of Harary).

This proves the lemma. □

The proof of Proposition 9.52 is complete. □

Proceeding with the proof of Theorem 9.51 we now show that a function f from $\dot{\Lambda}^{k,\omega}(G)$ is locally uniformly continuous and may be continuously extended to the function $\bar{f} : \overline{G} \to \mathbb{R}$. In order to do so, we might extend the basic estimate (9.132) to the function $\bar{f}$ as follows.

Let p_Q be a polynomial of degree $k-1$ such that

$$E_k(Q\cap G\,;f) = \|f - p_Q\|_{C(G\cap Q)}. \tag{9.143}$$

Then, by the definition of $\bar{f}$,

$$E_k(Q\cap \overline{G}\,;\bar{f}) \le \|\bar{f} - p_Q\|_{C(Q\cap\overline{G})} = \|f - p_Q\|_{C(Q\cap G)} = E_k(Q\cap G\,;f)$$

and it remains to apply (9.132).

To prove locally uniform continuity we need

Lemma 9.54. *Let K be a cube of length side $2R$ centered at G and intersecting ∂G. Let $Q\subset K$ be a cube of length side $2r$. Then for $1\le s<k$,*

$$E_s(G\cap Q\,;f) \le O(1)r^s\left\{\left(\int_r^{2R}\frac{\omega(t)}{t^{s+1}}\,dt\right)|f|_{\Lambda^{k,\omega}(G)} + \left(\frac{1}{R}\right)^s\|f\|_{C(K\cap Q)}\right\}. \tag{9.144}$$

Proof. Let us define the natural number J by the condition

$$R \le 2^J r < 2R$$

and let Q_j be the cube of radius $r_j := 2^j r$ centered at the center c_Q of the cube Q, $j=0,1,\dots,J-1$; we also set $Q_J := K$. Further, we set for brevity $p_j := p_{Q_j}$. Under this notation we get

$$\begin{aligned} E_s(Q\cap G\,;f) \le \Big\{&E_s(Q\cap G\,;f - p_Q) \\ &+ \sum_{j=0}^{J-1} E_s(Q\cap G\,;p_j - p_{j+1}) + E_s(Q\cap G\,;p_K)\Big\}. \end{aligned} \tag{9.145}$$

The first term clearly equals $E_k(Q\cap G\,;f)$ and therefore is bounded by $\omega(r)|f|_{\Lambda^{k,\omega}(G)}$. By monotonicity of ω,

$$\omega(r) \le O(1)r^s\int_r^{2R}\frac{\omega(t)}{t^{s+1}}\,dt$$

and this leads to the required estimate of this term.

To estimate the remaining terms, we use inequalities whose proofs are postponed to the end.

Claim. (a) Let p be a polynomial of degree $k-1$ and Q be a cube of length side $2r$ centered at G. Then for $1\le s<k$,

$$E_s(Q\cap G\,;p) \le O(1)r^s\max_{|\alpha|=s}\|D^\alpha p\|_{C(Q\cap S)}. \tag{9.146}$$

(b) Let, in addition, $\widetilde{Q}$ be a cube of length side $2\tilde{r}$ centered at G and containing Q. Then

$$\max_{|\alpha|=s} \|D^\alpha p\|_{C(Q\cap G)} \le O(1)\tilde{r}^{-s}\|p\|_{C(\widetilde{Q}\cap G)}. \tag{9.147}$$

Using these inequalities to estimate the j-th term in (9.145) we get

$$r^{-s}E_s\big(Q\cap G\,; p_{j+1}-p_j\big) \le O(1)r_j^{-s}\|p_{j+1}-p_j\|_{C(Q_j\cap G)}.$$

The norm in the right-hand side is bounded by

$$E_k(Q_j\cap G\,; f) + E_k(Q_{j+1}\cap G\,; f) \le 2\omega(r_{j+1})|f|_{\Lambda^{k,\omega}(G)}.$$

Moreover, by monotonicity of $\omega \in \Omega_k$.

$$r_j^{-s}\omega(r_{j+1}) \le 2^k r_j^{-s}\omega(r_j) \le O(1)\int_{r_j}^{r_{j+1}} \frac{\omega(t)}{t^{s+1}}\,dt.$$

Summing the estimates obtained over j we then have

$$\sum_{j=0}^{J-1} E_s\big(Q\cap G\,; p_j - p_{j+1}\big) \le O(1)r^s\int_r^{2R}\frac{\omega(t)}{t^{s+1}}\,dt,$$

as required.

Finally, using the claim we bound the last term in (9.145) by

$$O(1)r^s\max_{|\alpha|=s}\big\|D^\alpha p_K\big\|_{C(Q\cap G)} \le O(1)r^s\,\frac{\|p_K\|_{C(K\cap G)}}{R^s}\,.$$

By the definition of p_K, the norm in the right-hand side is at most

$$E_k(K\cap G\,; f) + \|f\|_{C(K\cap G)} \le 2\|f\|_{C(K\cap G)}.$$

This leads to the desired estimate of the last term.

To complete the proof of Lemma 9.54, it remains to establish the Claim. For assertion (a) we use a Jackson type approximation theorem to have

$$E_s(Q\cap G\,; p) \le E_s(Q\,; p) \le O(1)r^s\max_{|\alpha|=s}\|D^\alpha p\|_{C(Q)}.$$

For the second inequality we proceed as follows. Using scaling we can replace Q from the proof by the unit cube $Q_0 = [0,1]^n$. Then the functions

$$p \mapsto E_s(Q_0\,; p) \quad \text{and} \quad p \mapsto \max_{|\alpha|=s}\|D^\alpha p\|_{C(Q_0)}$$

are norms on the finite-dimensional space $\mathcal{P}_{k-1,n}/\mathcal{P}_{s,n}$. Since all norms on such a space are equivalent, these two are equivalent with the constants of equivalence depending only on n and k.

Maintaining the derivation we now apply Lemma 9.5 to the polynomial $D^\alpha p$ of degree $k-s-1$ to obtain

$$\|D^\alpha p\|_{C(Q)} \le \left(\frac{4n}{\mu}\right)^{k-s-1} \|D^\alpha p\|_{C(Q\cap G)} \tag{9.148}$$

where $\mu := \frac{|Q\cap G|}{|Q|}$. By virtue of Proposition 9.50 (a), $\mu \ge c(G) > 0$, and we finally obtain inequality (9.146).

Inequality (9.147) is proved similarly. In fact, it suffices to replace $Q\cap G$ by $\widetilde{Q}$ and apply (9.146) to $\widetilde{Q}$.

The proof of the lemma is complete. □

Now we use this lemma for $s=1$ and note that

$$E_1(Q\cap G\,; f) = \frac{1}{2}\sup\{f(x)-f(y)\,;\, x,y\in Q\cap G\}.$$

Therefore we conclude that for $x,y\in Q\cap G$ satisfying $\|x-y\|<2r$,

$$|f(x)-f(y)| \le O(1)r\left(|f|_{\Lambda^{k,\omega}(G)}\int_r^{2R}\frac{\omega(t)}{t^2}\,dt + \frac{\|f\|_{C(K\cap G)}}{R}\right).$$

Since the right-hand side tends to zero as $r\to 0$, f is uniformly continuous on any bounded subset of G.

The proof of Theorem 9.51 is complete. □

We conclude with a local version of Theorem 9.51.

Theorem 9.55. *Let G be a (c_0,c_1)-uniform domain of $\mathbb{R}^n$, and ω be a quasi- power k-majorant, see* (9.129)*. Then for some $\delta>0$ there exists a linear extension operator $\mathcal{E}_k : \Lambda^{k,\omega}(G)\to\Lambda^{k,\omega}(G_\delta)$ whose norm is bounded by a constant depending on inessential parameters.*

Let us recall that G_δ is an open δ-neighborhood of G.

Proof. All geometric results of subsection 9.3.3 are proved for (c_0,c_1)-uniform domains. Therefore we can repeat line by line the argument of the proof of Theorem 9.51. However, the projections $Pr_Q : C(Q\cap G)\to\mathcal{P}_{k-1,n}$ can be defined only for cubes Q of side length at most c_0. Hence, the family $\{Pr_Q f\}$ with these Q forms an (ω,k)-chain of size c_0 linearly depending on f. It remains to apply the local version of Theorem 9.6 (i.e., Theorem 9.18) to obtain the required linear extension. □

Combining Theorems 9.51, 9.55 and 2.10 (of Volume I) we immediately get

Corollary 9.56. *Let ω be a quasipower s-majorant. Then the following is true:*

(a) *If G is c-uniform, then there exists a linear continuous extension operator from $\dot{C}^{k,s,\omega}(G)$ to $C^s\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$.*

(b) *If G is (c_0, c_1)-uniform, then such an operator acts from $C^s\Lambda^{k,\omega}(G)$ to $C^s\dot{\Lambda}^{k,\omega}(G_\delta)$, where G_δ is the open δ-neighborhood of G.*

Remark 9.57. For the nonhomogeneous space $C^s\Lambda^{k,\omega}(G)$ which is recalled to be defined by the norm

$$\|f\|_{\ell^\infty(G)} + \max_{|\alpha|=s} \sup_{t>0} \frac{\omega_k(t\,;\,D^\alpha f)}{\omega(t)}\,,$$

assertion (b) is true for the target space $C^s\Lambda^{k,\omega}(\mathbb{R}^n)$ substituted for that over G_δ. This can be done by multiplying the extension $\tilde{f} \in C^s\Lambda^{k,\omega}(G_\delta)$ of the function f by the C^∞ test-function which is 1 on G and 0 outside $G_{\frac{\delta}{2}}$.

Finally we show that Theorems 9.51 and 9.55 are not true for nonquasipower k-majorants.

Example 9.58. Let G be the spiral uniform domain G of Example 9.48 (c). Let us recall that $G = \bigcup\limits_{i\ge 0} G_i \subset \mathbb{R}^2$ where the i-th coil G_i of the spiral domain is given by

$$G_i := 2^{-i}\big(L_0 + Q_{\frac{1}{8}}(0)\big);$$

here L_0 is the piecewise linear curve with vertices $v_1 := (-1, 0), v_2 := (-1, 1), v_3 := (1, 1), v_4 := (1, -1), v_5 := \big(-\frac{1}{2}, -1\big), v_6 := \big(-\frac{1}{2}\,, 0\big)$ and $\lambda S := \{\lambda x\,;\, x \in S\}$.

Now we define a function f from the space $\dot{\Lambda}^{1,\omega}(G)$ where $\omega(t) \approx \big(\log \frac{1}{t}\big)^2$ as $t \to 0$ such that any continuous extension $\tilde{f}$ of this function to an arbitrary neighborhood G_δ, $\delta > 0$, does not belong to $\dot{\Lambda}^{1,\omega}(G_\delta)$ with this ω.

We first define this f on every set $G_i\backslash T_i$ where T_i is the "tail" of G_i, i.e., the square $T_i := Q_{2^{-i-2}}(w_i)$ centered at $w_i := 2^{-i}v_6$ and of radius 2^{-i-2}, by setting

$$f(x) := \frac{1}{i} \text{ for } x \in G_i\backslash T_i, \quad i = 1, 2, \dots;$$

we also set $f(x) := 1$ for $x \in G_0$.

Further, we smoothly extend f to the remaining set $\bigcup\limits_{i\ge 1} T_i$ as follows.

Let $\varphi : \mathbb{R}^2 \to [0, 1]$ be a C^∞ function which equals 1 on $[-1, 1]^2$ and 0 outside $[-2, 2]^2$. Scaling gives rise to a C^∞ function $\varphi_i : \mathbb{R}^2 \to [0, 1]$ which equals 1 on the square $\frac{1}{2}\,T_i$ and 0 outside T_i. (Here $\frac{1}{2}\,T_i$ is the $\frac{1}{2}$-homothety of T_i with respect to its center.) Then, for $i = 1, 2, \dots$, we set

$$f(x) := \frac{1}{i}\,(1 - \varphi_i(x)) + \frac{1}{i+1}\,\varphi_i(x), \quad x \in T_i.$$

This defines f on G as a C^∞ function.

To estimate the modulus of continuity

$$\omega_1(t\,; f)_G := \sup_{|x-y|\le t} \big\{|f(x) - f(y)|\,;\, [x, y] \subset G\big\},$$

we note that for every $i \geq 1$,

$$\sup_{x,y\in G_i} |f(x) - f(y)| = \frac{1}{i} - \frac{1}{i+1} < \frac{1}{i^2}$$

and the supremum is attained at points x, y lying at the opposite sides of T_i. But if $[x, y] \subset G_i$, then $|y - x| \approx 2^{-i}$ and therefore

$$\omega_1(t\,; f)_G \leq O(1)\Big(\log\frac{1}{t}\Big)^{-2} \text{ for } 0 < t \leq \frac{1}{2}\,.$$

Hence, $f \in \dot{\Lambda}^{1,\omega}(G)$ with $\omega(t) \approx \big(\log\frac{1}{t}\big)^{-2}$ near $t = 0$.

On the other hand, any δ-neighborhood G_δ, contains all of T_i with sufficiently large i, say $i > i_\delta$. If $\tilde{f} : G_\delta \to \mathbb{R}$ is a continuous extension of f, then $\tilde{f}$ must be 0 at the origin. Therefore for $i > i_\delta$ and $t \approx 2^{-i}$,

$$\omega_1(t\,;\tilde{f})_{G_\delta} \geq \big|\tilde{f}(x) - \tilde{f}(0)\big| = \frac{1}{i}\,,$$

provided that x belongs to the lower side of T_i. Hence,

$$\omega_1(t\,;\tilde{f})_{G_\delta} \geq \gamma\Big(\log\frac{1}{t}\Big)^{-1}$$

for some $\gamma > 0$ and all sufficiently small t.

Hence, any such extension does not belong to $\dot{\Lambda}^{1,\omega}(G_\delta)$ with $\omega(t) \approx \big(\log\frac{1}{t}\big)^{-2}$.

We leave it to the reader to consider the case of the space $\dot{\Lambda}^{k,\omega}(G)$ with $k > 1$.

9.3.3 Uniform domains and Whitney's cubes

We begin with a special property of Whitney's cubes contained in a domain G; this family is recalled to be denoted by $\mathcal{W}(G)$ ($:= \mathcal{W}_{G^c}$ in the notation of Lemma 2.14 of Volume I, see also Proposition 9.1).

Lemma 9.59. *Let Q be a cube centered at the closure $\overline{G}$ and intersecting ∂G (i.e., $Q \in \mathcal{K}_{\overline{G}}$). Then every $K \in \mathcal{W}(G)$ intersecting Q satisfies*

$$r_K \leq 2r_Q. \tag{9.149}$$

Proof. First, let the center $c_Q \in \partial G$. Since by the definition of Whitney cubes $2K \subset G$, see (9.6) for $S := G^c$, and $Q \cap K \neq \emptyset$, it is true that

$$2r_K \leq \operatorname{dist}(c_K, G^c) \leq \|c_K - c_Q\|_\infty \leq r_K + r_Q$$

and (9.149) follows.

Now, let $c_Q \in G$ and a_Q be the closest to c_Q point of ∂G (in the ℓ_∞-metric). Then $\|c_Q - a_Q\|_\infty \leq r_Q$ as $Q \cap \partial G \neq \emptyset$. Further, by $\widetilde{Q}$ we denote the cube of length side $4r_Q$ (radius $2r_Q$) centered at a_Q. This clearly contains Q and therefore $K \cap \widetilde{Q} \neq \emptyset$. According to the previous case $r_K \leq r_{\widetilde{Q}} = 2r_Q$, and the result is done. $\square$

Stipulation.

(a) Hereafter G denotes a (c_0, c_1)-uniform domain in $\mathbb{R}^n$ subject to Definition 9.49 but with the ℓ_∞-norm of $\mathbb{R}^n$ instead of the ℓ_2-norm. In particular, all distances are measured in this norm.

(b) All quantities appearing in the proofs under the name "constant" depend only on inessential parameters c_0, c_1, n. They will be denoted by a, b, c, d or $O(1)$. If such a constant depends also on an additional parameter, say $\lambda > 0$, we will write $O_\lambda(1)$.

Lemma 9.60. *Let the cube $Q \in \mathcal{K}_{\overline{G}}$ intersect ∂G and $r_Q \le c_0$. Then there are a constant $\ell > 0$ and a cube $K \in W(G)$ such that*[1]

$$K \subset \ell Q \quad \textit{and} \quad r_K \approx r_Q. \tag{9.150}$$

Proof. By Proposition 9.50, for some constant $c > 0$, we have

$$\operatorname{diam}(Q \cap G) \ge c r_Q.$$

Therefore there exist points x, y in $Q \cap G$ such that, for some constant $b > 0$,

$$b r_Q \le \|x - y\| \le 2 r_Q.$$

From this and Definition 9.49 it follows that for some curve $\gamma : [0, 1] \to G$ joining x and y,

$$\ell(\gamma) \le c_1 \|x - y\|_\infty \le 2 c_1 r_Q.$$

Now we choose a point $z \in \gamma([0, 1])$ such that

$$\|x - z\|_\infty = \|y - z\|_\infty \ge \frac{1}{2} \|x - y\|_\infty.$$

Applying the (equivalent) definition of (c_0, c_1)-uniformity which exploits (9.125) instead of (9.124) we then have for this point

$$\operatorname{dist}(z, G^c) \ge c_1 \min\big(\|x - z\|_\infty, \|y - z\|_\infty\big) \ge \frac{c_1 b}{2} r_Q.$$

On the other hand, by the choice of γ, we get

$$\|x - z\|_\infty \le \ell(\gamma) \le c_1 \|x - y\|_\infty \le 2 c_1 r_Q.$$

Now, let K be a cube from $\mathcal{W}(G)$ containing z. Taking into account inequalities (9.7) and (9.9) with $S := G^c$ we can write for this cube

$$\frac{c_1 b}{2} r_Q \le \operatorname{dist}(z, G^c) \le \|c_K - z\|_\infty + \operatorname{dist}(c_K, G^c) \le r_K + 6 r_K,$$

[1] Recall that $\approx$ means a two-sided inequality with positive constants of equivalence.

whence

$$r_K \geq \frac{c_1 b}{14} r_Q.$$

Also, by the previous estimate and the belonging of x to $Q \cap G$, we have

$$\|z - c_Q\|_\infty \leq \|x - z\|_\infty + \|x - c_Q\|_\infty \leq (2c_1 + 1)r_Q.$$

Denoting the right-hand side by t we then have

$$tQ \cap \partial G \neq \emptyset \quad \text{and} \quad tQ \cap K \neq \emptyset.$$

Therefore Lemma 9.59 implies that

$$r_K \leq 2t := (4c_1 + 2)r_Q.$$

Set $\ell := (10c_1 + 5)$; then $\ell \geq 5t$ and therefore $K \subset \ell Q$. Together with the proved inequality

$$\frac{c_1 b}{14} r_Q \leq r_K \leq (4c_1 + 2)r_Q;$$

this proves the lemma. □

Lemma 9.61. *Let $Q_i \in \mathcal{W}(G)$ be Whitney cubes of radius r_i and center c^i, $i = 1, 2$. Assume that*

$$\operatorname{dist}(Q_1, Q_2) \leq c_0 \tag{9.151}$$

and for some parameter $\lambda > 0$,

$$\lambda^{-1} r_1 \leq r_2 \leq \lambda r_1 \quad \text{and} \quad \|c^1 - c^2\|_\infty \leq \lambda(r_1 + r_2). \tag{9.152}$$

Then there exists a "chain" $\{K_i\}_{1 \leq i \leq m+1} \subset \mathcal{W}(G)$ joining Q_1 and Q_2, meaning that:

(a) *K_i and K_{i+1} touch, $1 \leq i \leq m$;*

(b) *$K_1 = Q_1$ and $K_{m+1} = Q_2$;*

(c) *$K_i \neq K_j$ if $i \leq j$.*

Moreover, this chain meets the following conditions:

(d) *For some $c = O_\lambda(1)$ and all i, j,*

$$K_j \subset cK_i; \tag{9.153}$$

(e) *$m \leq O_\lambda(1)$.*

Proof. Excluding the trivial case, we assume that $Q_1 \neq Q_2$. Using (9.151) we fix the points $z^i \in Q_i$, $i = 1, 2$, so that

$$\|z^1 - z^2\|_\infty \leq c_0.$$

Since G is (c_0, c_1)-uniform, this implies the existence of a curve $\gamma : [0,1] \to G$ joining z^1 and z^2 and satisfying the condition of Definition 9.49, i.e.,

$$\ell(\gamma) \leq c_1 \|z^1 - z^2\|_\infty \quad \text{and} \quad Kn(\gamma\,; c_1) \subset G.$$

Now we introduce the family of Whitney cubes intersecting γ:

$$\mathcal{W}_\gamma := \big\{K \in \mathcal{W}(G)\,; K \cap \gamma[0,1] \neq \emptyset\big\}.$$

The family possesses the following two properties:

(i) For some constants $a_i(\lambda)$, $i = 1, 2$, and all cubes $K \in \mathcal{W}_\gamma$

$$Q_1 \subset a_1(\lambda)K \quad \text{and} \quad K \in a_2(\lambda)Q_2. \tag{9.154}$$

(ii) The number of cubes in $\mathcal{W}_\gamma$ is estimated by

$$\operatorname{card} \mathcal{W}_\gamma \leq O_\lambda(1). \tag{9.155}$$

We postpone the proof of (i), while now we derive from it the second assertion.

The order of the Whitney cover $\mathcal{W}(G)$ is bounded by $c(n)$, see Proposition 9.1, and therefore Lemma 9.53 asserts that the family $\mathcal{W}_\gamma$ is the union of at most $c_1(n) := 2^n\big(c(n) - 1\big) + 1$ disjoint subfamilies. Due to (9.154), every such subfamily is contained in the cube $a_2(\lambda)Q_2$. The comparison of volumes then estimates their cardinality by $\frac{a_2(\lambda)^n |Q_2|}{\min\{|K|\,; K \in \mathcal{W}_\gamma\}}$. On the other hand, every $K \in \mathcal{W}_\gamma$ contains the cube $a_1(\lambda)^{-1}Q_1$ and therefore the denominator of this ratio is at least $a_1(\lambda)^{-n}|Q_1|$.

Hence, the number of cubes in $\mathcal{W}_\gamma$ is at most $c_1(n)\big(a_1(\lambda)a_2(\lambda)\big)^n \frac{|Q_2|}{|Q_1|}$.

Finally, due to condition (9.152), for some constants $b_1(\lambda), b_2(\lambda)$,

$$Q_1 \subset b_1(\lambda)Q_2 \quad \text{and} \quad Q_2 \subset b_2(\lambda)Q_1; \tag{9.156}$$

together with the previous inequality this leads to the required result:

$$\operatorname{card} \mathcal{W}_\gamma \leq c_1(n)\big(a_1(\lambda)a_2(\lambda)b_2(\lambda)\big)^n = O_\lambda(1).$$

Now we construct the desired chain $\{K_i\}_{1 \leq i \leq m}$. To this end we first define inductively an increasing sequence of families $\mathcal{C}_i \subset \mathcal{W}_\gamma$ starting from $\mathcal{C}_1 := \{Q_2\}$. If then $\mathcal{C}_i$ has been defined for $i \geq 1$, we set

$$\mathcal{C}_{i+1} := \mathcal{C}_i \cup \big\{K \in \mathcal{W}_\gamma \backslash \mathcal{C}_i\,; K \cap Q \neq \emptyset \text{ for some } Q \in \mathcal{C}_i\big\}.$$

In other words, we add to $\mathcal{C}_i$ every cube of $\mathcal{W}_\gamma$ which does not belong to any $\mathcal{C}_j$, $j \le i$, but touches some cube from $\mathcal{C}_i$. After a finite number of steps, say J, this procedure terminates, see (9.155). The initial cube Q_1 of the constructed chain belongs to $\mathcal{C}_J$, since otherwise the connected set $\gamma([0,1])$ would be covered by two disjoint nonempty closed subsets, the finite unions of (closed) cubes from $\mathcal{W}\backslash\mathcal{C}_J$ and $\mathcal{C}_J$, respectively.

Let m be the smallest of indices i for which $Q_1 \in \mathcal{C}_i$. By the definition of $\mathcal{C}_m$, there exists a cube in $\mathcal{C}_{m-1}$ touching Q_1. Denoting this cube by K_2 and renaming Q_1 by K_1, we then find a cube K_3 from $\mathcal{C}_{m-2}$ touching K_2 and so on. Since $\mathcal{C}_1 := \{Q_2\}$, the last cube K_{m+1} of this collection coincides with Q_2.

The family $\{K_i\}_{1\le i\le m+1}$ clearly satisfies conditions (a)–(c) of Lemma 9.61. It also satisfies (e) due to (9.155) and (d) due to (9.154); this implies that for every pair of cubes K', K'' from $\mathcal{W}_\gamma$,

$$K' \subset O_\lambda(1)K''.$$

To complete the proof of the lemma it remains to establish the embeddings in (9.154).

First, let the cube $K \in \mathcal{W}_\gamma$ intersect one of the cubes Q_i^*, $i = 1, 2$, say Q_1^*. Since Q_i and K are Whitney cubes, Proposition 9.1 implies that

$$Q_1 \subset 9K \quad \text{and} \quad K \subset 9Q_1.$$

The required embeddings of (9.154) follow from these and the embeddings in (9.156).

Now let $K \in \mathcal{W}_\gamma$ and

$$K \cap \{Q_1^* \cup Q_2^*\} = \emptyset. \tag{9.157}$$

Let w be a point of the set $K\cap\gamma([0,1])$. Due to Definition 9.49 there exists a curve $\gamma : [0,1] \to G$ joining some fixed points $z^i \in Q_i$, $i = 1, 2$, such that

$$\operatorname{dist}(Q_2, K) \le \|c^2 - w\|_\infty + \|w - c^1\|_\infty \le \ell(\gamma) \le c_1\|z^1 - z^2\|_\infty. \tag{9.158}$$

By (9.152), the norm on the right-hand side is bounded by

$$c_2\big(r_1 + r_2 + \|c^1 - c^2\|_\infty\big) \le c_1(1+\lambda)^2 r_2,$$

whence

$$\operatorname{dist}(Q_2, K) \le c_1(1+\lambda)^2 r_2. \tag{9.159}$$

This obviously implies that the required embeddings in (9.154) will follow from the equivalence

$$r_K \approx r_2 \tag{9.160}$$

with constants depending on λ (and inessential parameters).

To establish this, we first note that by the definition of Whitney cubes

$$6r_K \geq \operatorname{dist}(c_K\,;\,G^c), \tag{9.161}$$

see (9.7). Moreover, $\|c_K - w\| \leq r_K$ and therefore

$$\operatorname{dist}(w\,;\,G^c) \leq \operatorname{dist}(c_K\,;\,G^c) + \|c_K - w\|_\infty \leq 7r_K. \tag{9.162}$$

Furthermore, applying inequality (9.125) equivalent to condition (b) in Definition 9.49 to the above introduced points w, z^1, z^2 and the curve γ we obtain

$$\operatorname{dist}(w, G^c) \geq c_1 \min\big\{\|w - z^1\|_\infty,\ \|w - z^2\|_\infty\big\}.$$

Due to (9.157) and (9.152) the right-hand side here is at least

$$\min \operatorname{dist}(K, Q_i) \geq \min r_i \geq c_1 \min\{1, \lambda^{-1}\} r_2.$$

Combining this and (9.162) we get

$$r_K \geq O_\lambda(1) r_1.$$

The converse inequality follows from Proposition 9.1 and inequality (9.159) which yield

$$r_K \leq 5 \operatorname{dist}(K, G^c) \leq 5\big(\operatorname{dist}(K, Q_2) + \operatorname{dist}(Q_2, G^c) + 2r_2\big) \leq 5\big((1+\lambda)^2 c_1 + 7\big) r_2.$$

The result is proved. □

Lemma 9.62. *Let $Q_i \in \mathcal{W}(G)$ be a cube of radius r_i and center c^i, $i = 1, 2$. Assume that*[2]

$$\|c^1 - c^2\|_\infty \leq \frac{1}{2} \cdot \frac{c_0}{c_1} \tag{9.163}$$

and for some parameter $\lambda > 0$,

$$r_1 + \|c^1 - c^2\|_\infty \leq \lambda r_2. \tag{9.164}$$

Then there is a chain $\{K_i\}_{1 \leq i \leq m} \subset \mathcal{W}(G)$ satisfying conditions (a) *and* (b) *of Lemma* 9.61 *and such that every cube occurs in the chain at most $O_\lambda(1)$ times. Moreover, for each pair $1 \leq j \leq i \leq m$,*

$$K_j \subset O_\lambda(1) K_i. \tag{9.165}$$

[2] Recall that c_0, c_1 are the constants of uniformity for G.

Proof. Since for the case $Q_1 \cap Q_2 \neq \emptyset$ the result immediately follows from Lemma 9.61, we may and will assume that $Q_1 \cap Q_2 = \emptyset$. We define a sequence of cubes $\{L_j\}_{0 \le i \le J}$ such that $L_1 = Q_2, L_J = Q_1$ and for every $j \le J$ the pair L_j, L_{j+1} satisfies the assumption of Lemma 9.61. Using this, we find for every pair L_j, L_{j+1} the chain of cubes denoted by $[L_j, L_{j+1}]$ satisfying conditions (a), (b), (c) and (d) of Lemma 9.61. Joining all the chains $[L_j, L_{j+1}]$ we define the required chain.

To realize this program, we first define the index J by the condition

$$\frac{1}{2} r_1 \le 2^{-J} \|c^1 - c^2\| \le r_1.$$

Further, because of the inequality $\|c^1 - c^2\|_\infty \le c_0$ there is a curve $\gamma : [0,1] \to G$ joining the points c^1 and c^2. Since the functions $y \mapsto \|x-y\|_\infty$ and γ are continuous, for every $j \le J$ there exists a point $w^j \in \gamma\big([0,1]\big)$ such that

$$\|c^1 - w^j\|_\infty = 2^{-j}\, \|c^1 - c^2\|_\infty. \tag{9.166}$$

Now let L_j denote a (unique) cube of $\mathcal{W}(G)$ containing w^j, $1 \le j < J-1$; set also $L_0 := Q_2$ and $L_J := Q_1$. We simplify the notation by setting

$$\rho_j := r_{L_j}, \quad \sigma^j := c_{L_j};$$

in particular, $\rho_0 = r_2$, $\rho_J = r_1$, and $\sigma^0 = c^2$.

Later we will prove that

$$\rho_j \approx 2^{-j} \|c^1 - c^2\|_\infty, \quad j = 0, 1, \dots, J, \tag{9.167}$$

with constants depending only on an inessential parameter and λ from (9.164). From here we derive for $j < i$,

$$\|\sigma^j - \sigma^i\|_\infty \le O_\lambda(1)\rho_j. \tag{9.168}$$

In fact, inserting points w^j and w^i and using the triangle inequality we bound the left-hand side of (9.168) by

$$\|\sigma^j - w^j\|_\infty + \|w^j - c^1\|_\infty + \|c^1 - w^i\|_\infty + \|w^i - \sigma^i\|_\infty \le \rho_j + \rho_i + (2^{-i} + 2^{-j})\|c^1 - c^2\|_\infty.$$

This and (9.167) estimate the left-hand side in (9.168) by

$$O_\lambda(1)\big(2^{-i} + 2^{-j}\big)\|c^1 - c^2\| \le O_\lambda(1)\rho_j.$$

Applying again (9.167) we get $\rho_i \le O_\lambda(1)\rho_j$ for $j < i$; this and (9.168) imply that

$$L_i \subset O_\lambda(1) L_j \text{ for } j < i. \tag{9.169}$$

Further, every cube L_j contains the point w^j from $\gamma\big([0,1]\big)$ and therefore Definition 9.49 and (9.163) yield

$$\begin{aligned} \operatorname{dist}(L_j, L_{j+1}) &\le \|w^j - w^{j+1}\|_\infty \le \|w^j - c^1\|_\infty + \|c^1 - w^{j+1}\|_\infty \\ &\le 2\ell(\gamma) \le 2c_1\|c^1 - c^2\|_\infty \le c_1 \cdot \frac{c_0}{c_1} = c_0. \end{aligned} \tag{9.170}$$

Hence, the pair of cubes L_j, L_{j+1} satisfies the first assumption of Lemma 9.61. Inequalities (9.167) and (9.168) show that it satisfies the second assumption with λ replaced by some constant $O_\lambda(1)$. Applying this lemma we find for all $j < J$ a chain $[L_j, L_{j+1}] := \{L_{ji}\}_{1\le i\le m_j}$ such that the adjacent cubes touch, $m_j \le O_\lambda(1)$ and

$$L_{ji_1} \subset O_\lambda(1)L_{ji_2} \text{ for } 1 \le i_1, i_2 \le m_j,\ 0 \le j < J. \tag{9.171}$$

Now we define the desired chain joining Q_1 and Q_2 by setting

$$[Q_1, Q_2] := \bigcup_{j=0}^{J-1} [L_j, L_{j+1}].$$

That is to say, the chain $[Q_1, Q_2]$ is defined by joining the chains $[L_j, L_{j+1}]$ and $[L_{j+1}, L_{j+2}]$ at L_{j+1}, $0 \le j < J$. This clearly satisfies conditions (a) and (b) which are common for Lemmas 9.61 and 9.62.

It remains to check two other assertions of the latter lemma, see (9.165). To this end we enumerate the cubes of the chain $[Q_1, Q_2]$ as $\{K_i\}_{1\le i\le m}$ where $m := \sum_{j=0}^{J-1} m_j - (J-1)$ so that $K_m := Q_2 (:= L_{01})$ and adjacent cubes are numbered by adjacent indices.

Let $K_r, K_s \in [Q_1, Q_2]$ where $r \le s$. By definition there are $j \le i$ such that

$$K_r \in [L_i, L_{i+1}] \quad \text{and} \quad K_s \in [L_j, L_{j+1}].$$

If $j = i$, then $K_r \subset O_\lambda(1)K_s$ by the construction of $[L_i, L_{i+1}]$. Otherwise, $j < i$ and by (9.169) and (9.171) the previous embedding is also true.

Now let us estimate the multiplicity of each cube K_r in $[Q_1, Q_2]$. Assume that for some $i < j$ simultaneously

$$K_r \in [L_i, L_{i+1}] \quad \text{and} \quad K_r \in [L_j, L_{j+1}].$$

By the constructions of the chains we then have

$$L_i \subset O_\lambda(1)K_r \quad \text{and} \quad L_j \subset O_\lambda(1)K_r.$$

Hence, the radii ρ_i and ρ_j of these cubes satisfy $\rho_i \le O_\lambda(1)\rho_j$. Together with (9.167) this then implies that

$$j - i \le O_\lambda(1),$$

meaning that each cube K_r occurs in $[Q_1, Q_2]$ at most $O_\lambda(1)$ times. This fact and conditions (c) and (d) of Lemma 9.61, which hold for every $[L_i, L_{i+1}]$, imply that the remaining two assertions of Lemma 9.62 are true.

It remains to prove (9.167). Since $Q_1 \cap Q_2 = \emptyset$, the set $\gamma\big([0,1]\big)\setminus(Q_1 \cup Q_2) \ne \emptyset$. Let w be a point of this set and K be a cube of $\mathcal{W}(G)$ containing w. Show that

$$r_K \approx \|c^1 - w\|_\infty. \tag{9.172}$$

By condition (9.125) and Definition 9.49 (b),

$$\operatorname{dist}(w,G^c)\ge c_1\min\{\|w-c^1\|_\infty,\|w-c^2\|_\infty\},\tag{9.173}$$

while the first condition of this definition, together with (9.164), yields

$$\|w-c^1\|_\infty\le\ell(\gamma)\le c_1\|c^1-c^2\|_\infty\le O_\lambda(1)r_2.$$

In addition, $w\notin Q_2$ and therefore

$$\|w-c^2\|_\infty\ge r_2.$$

Combining these inequalities with (9.173) we obtain

$$\|w-c^1\|_\infty\le O_\lambda(1)\operatorname{dist}(w,G^c).$$

In turn, $w\notin Q_1$, so that $\|w-c^1\|_\infty\ge r_1$, and therefore

$$\operatorname{dist}(w,G^c)\le\operatorname{dist}(c^1,G^c)+\|c^1-w\|_\infty\le 5r_1+\|w-c^1\|_\infty\le 6\|w-c^1\|_\infty.$$

Together with the previous inequality this leads to the equivalence

$$\operatorname{dist}(w,G^c)\approx\|w-c^1\|_\infty.$$

Now we use Lemma 2.14 of Volume I asserting that for every $K\in\mathcal{W}(G)$,

$$\operatorname{dist}(c_K,G^c)\approx r_K.$$

The last two equivalences imply that

$$\begin{aligned}r_K\approx\operatorname{dist}(c_K,G^c)&\le\operatorname{dist}(w,G^c)+\|w-c_K\|_\infty\\&\le 2\operatorname{dist}(w,G^c)\le 2\{\operatorname{dist}(c_K,G^c)+\|c_K-w\|_\infty\}\le O(1)r_K,\end{aligned}$$

whence

$$\|w-c^1\|_\infty\approx\operatorname{dist}(w,G^c)\approx r_K.$$

Along with the definitions of w^j and L_j, see (9.166), this gives

$$\rho_j\approx 2^{-j}\|c^1-c^2\|_\infty,\quad 1\le j\le J,$$

as required.

The proof of Lemma 9.62 is complete. □

Comments

Theorems 9.3 and 9.8 were due to Yu. Brudnyi and Shvartsman in [BSh-1982] and [BSh-1999], respectively. Another characteristic of the space $\Lambda^{2,\omega}(\mathbb{R}^n)|_S$ was given by Hanin in [Han-1987]. The extension method (9.15) exploited in this chapter was introduced by Yu. Brudnyi [Br-1970b] along with the chain condition of Theorem 9.3 which, however, was used only as a technical tool. The role of this condition was emphasized by the results of Jonsson and Wallin [JW-1984, Thm. III.2 and Cor. III.3] giving the next local approximation characteristic of the trace space $\dot{B}^\sigma_\infty(\mathbb{R}^n)\big|_S$ where $S \subset \mathbb{R}^n$ is an arbitrary closed set. (For the definition of the Besov space $\dot{B}^\sigma_\infty(\mathbb{R}^n)$, see, e.g., (9.174) below with $p = \theta = \infty$.)

The function f belongs to $\dot{B}^\sigma(\mathbb{R}^n)\big|_S$ if and only if for every cube $Q \in \mathcal{K}_S$ there exists a polynomial P_Q of degree $\ell := [\sigma]$ such that

(a) $E_\ell(f\,; Q \cap S) \le c|Q|^{\frac{\sigma}{n}}$;

(b) *for every pair of intersecting cubes Q, Q',*

$$\max_{Q\cap Q'} |p_Q - p_{Q'}| \le c \max\big\{|Q|, |Q'|\big\}^{\frac{\sigma}{n}};$$

here c is independent of f and Q, Q'.
Moreover, there exists a linear continuous extension operator from the trace space into $\dot{B}^\sigma_\infty(\mathbb{R}^n)$.

The proof begins with the preliminary quantitative approximation of a trace $f \in \dot{B}^\sigma_\infty(\mathbb{R}^n)\big|_S$ by a sequence of traces $\{f_j\} \subset \dot{B}^{\tilde\sigma}_\infty(\mathbb{R}^n)$ where $\tilde\sigma > \sigma$ and is *noninteger*. Since for $\dot{B}^{\tilde\sigma}_\infty(\mathbb{R}^n) = \dot{C}^{k,\omega}(\mathbb{R}^n)$ for some k and ω, the Whitney–Glaeser Theorem 2.13 of Volume I may be applied to extend every f_j to the function $F_j \in \dot{B}^{\tilde\sigma}_\infty(\mathbb{R}^n)$ such that $|F_j - F_{j'}|_{B^{\tilde\sigma}_\infty(\mathbb{R}^n)}$ is controlled by the trace seminorm of $f_j - f_{j'}$. This provides convergence of $\{F_j\}$ in $\dot{B}^{\tilde\sigma}_\infty(\mathbb{R}^n)$ to some function F together with a quantitative estimate of $|F_j - F_j|_{B^{\tilde\sigma}_\infty(\mathbb{R}^n)}$. The latter implies that F belongs to $\dot{B}^\sigma_\infty(\mathbb{R}^n)$ and therefore gives the required extension of the trace f.

Markov sets were introduced and studied by Jonsson and Wallin [JW-1984, Ch. II]; they, in particular, proved Theorem 9.21 and Proposition 9.24 describing the basic characteristics of Markov sets.

The extension results of Section 9.2 were proved by A. and Yu. Brudnyi by developing the methods of the papers [Br-1970b] and [BB-2007a].

The chains formed by the Whitney cover for a uniform domain G were first used by P. Jones [Jon-1981] to construct a map from $\mathcal{W}(G)$ into $\mathcal{W}_{G^c}$ modeling the Ahlfors *reflection property*[3] of a quasidisk $D \subset \mathbb{R}^2$. The latter property is the key point in the Gol'dstein–Vodop'yanov proof of the extension theorem for $W^1_2(D)$, see [GV-1980] and Theorem 2.83 of Volume I. Similarly, the Jones "reflection"

[3] meaning that there exists a bi-Lipschitz bijection of $\mathbb{R}^2$ which maps D onto its open exterior and is the identity on ∂D, see, e.g., [Ge-1982, p. 28].

map is the main geometric tool in the proof of his extension theorem for Sobolev spaces on uniform domains.

However, the Whitney cube chains used in Section 9.3 are much more complicated and are aimed at the proof of the local approximation estimate of Proposition 9.52. This result (in a much more general form) and the geometric lemmas of subsection 9.3.3 were proved in Shvartsman's Ph.D. thesis (1984).

The local approximation methods for the study of multivariate real functions were developed by Yu. Brudnyi and summarized in [Br-1971] and in several manuscripts circulated in the Yaroslavl function theory seminar; some results of these manuscripts are described in Appendices I–III to the paper [Br-1994] and in the survey [Br-2008]. In the aforementioned P. Shvartsman's thesis these methods were applied to the study of extension problems for a wide range of smoothness spaces of integrable functions including Lipschitz functions of higher order over $L_p(\mathbb{R}^n)$, $0 < p \le \infty$. We present several important results of this thesis using its extended version [Shv-1986] (the text can be found in the Russian Academy of Science archive of manuscripts).

The space of integrable Lipschitz functions in question is introduced similarly to that of continuous ones, see subsection 9.1.2, with the space $L_p^{\mathrm{loc}}(G)$ substituted for $L_\infty^{\mathrm{loc}}(\mathbb{R}^n)$; here $G \subset \mathbb{R}^n$ is a domain and $1 \le p \le \infty$. In fact, a refinement of this definition is in use. Its definition is based on the seminorm

$$|f|_{\Lambda_{p\theta}^{k,\omega}(G)} := \left\{ \int_0^\infty \left(\frac{\omega_k(t\,; f)_{L_p(G)}}{\omega(t)} \right)^\theta \frac{dt}{t} \right\}^{\frac{1}{\theta}}; \tag{9.174}$$

here $1 \le \theta \le \infty$ and ω is a k-majorant.

This space is denoted by $\dot{\Lambda}_{p\theta}^{k,\omega}(G)$ while the corresponding nonhomogeneous space (obtained by adding $\|f\|_{L_p(G)}$) is denoted by the same symbol without the upper point.

Since the trace and extension problems for these objects are far from being solved in the continuous case, we consider only the simplest problems of this kind concerning subsets of positive Lebesgue measure. Below, $S \subset \mathbb{R}^n$ stands for such a set, and $G \subset \mathbb{R}^n$ denotes a domain.

Trace Problem. *Characterize functions from $L_p^{\mathrm{loc}}(S)$ belonging to the trace space $\dot{\Lambda}_{p\theta}^{k,\omega}(\mathbb{R}^n)\big|_S$.*

Simultaneous Extension Problem. *Under what conditions on S and ω does there exist a linear continuous extension operator from $\dot{\Lambda}_{p\theta}^{k,\omega}(\mathbb{R}^n)\big|_S$ into $\dot{\Lambda}_{p\theta}^{k,\omega}(\mathbb{R}^n)$?*

The same question arises for the nonhomogeneous case.

It is easily seen that the answer to the second question is positive for $p = \theta = 2$ and any S and ω. Probably, the same may be proved for $1 < p < \infty$.

Restricted Extension Problem. *Under what conditions on G and ω does there exist a simultaneous extension from $\dot{\Lambda}_{p\theta}^{k,\omega}(G)$ into $\dot{\Lambda}_{p\theta}^{k,\omega}(\mathbb{R}^n)$?*

The same question arises for the nonhomogeneous case.

In general, these problems remain unsolved, but Shvartsman's results give solutions for a wide class of sets and majorants. The key point to his approach is an "atomic" decomposition of k-modulus of continuity due to Yu. Brudnyi [Br-1971].

Theorem A. *There exist constants of equivalence depending only on n and k such that for every $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and $t > 0$,*

$$\omega_k(t\,; f)_{L_p(\mathbb{R}^n)} \approx \sup_{\pi}\left\{ \sum_{Q\in\pi} E_k(Q\,; f)^p_{L_p} \right\}^{\frac{1}{p}}, \tag{9.175}$$

where π runs over all disjoint families of cubes of volume t^n.

Let us recall that the local approximation involved here is a set function given, for $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ and bounded measurable $S \subset \mathbb{R}^n$, by

$$E_k(S\,; f)_{L_p} := \inf_{m\in\mathcal{P}_{k-1,n}} \|f - m\|_{L_p(S)}.$$

The theorem motivates an extension of the concept of k-modulus of continuity to the case of functions defined on measurable subsets distinct from domains.

Specifically, let $f \in L_p^{\mathrm{loc}}(S)$ where $S \subset \mathbb{R}^n$ is a set of positive measure. *Approximation k-modulus of continuity for f* is a function in $t \in (0, 2\,\mathrm{diam}\,S)$ given by

$$\Omega_k(t\,; f)_{L_p(S)} := \sup_{\pi}\left\{ \sum_{Q\in\pi} E_k(Q\cap S\,; f)^p_{L_p} \right\}^{\frac{1}{p}}$$

where π runs over all disjoint families of cubes Q of volume t^n centered at S (i.e., $Q \in \mathcal{K}_S$).

Sometimes an integral form of this notion is also exploited. For its definition, we introduce *normalized* local approximation

$$\mathcal{E}_k(S\,; f)_{L_p} := |S|^{-\frac{1}{p}} E_k(S\,; f)_{L_p}$$

and then set, for $t \in (0, 2\,\mathrm{diam}\,S)$,

$$\widehat{\Omega}_k(t\,; f)_{L_p(S)} := \left(\int_S \mathcal{E}_k\big(S\cap Q_t(x)\,; f\big)^p_{L_p} dx \right)^{\frac{1}{p}}.$$

Using the Besicovitch covering lemma [Bes-1946], it can be easily shown that, for some constants $c_1, c_2 > 0$ depending only on n and k, and all t,

$$c_1\Omega_k\left(\frac{t}{2}\,; f\right)_{L_p(S)} \le \widehat{\Omega}_k(t\,; f)_{L_p(S)} \le c_2\Omega_k(t\,; f)_{L_p(S)}.$$

Now, replacing in (9.174) ω_k by Ω_k (or, equivalently, by $\widehat{\Omega}_k$), we define the Lipschitz space of integrable functions on an arbitrary measurable set S preserving the same notation for these objects, e.g., $\dot{\Lambda}^{k,\omega}_{p\theta}(S)$ stands for the homogeneous space.

Replacing the ℓ_p-norm in (9.174) by the ℓ_q-norm, $1 \le q \le \infty$, we then define a more general concept of (k,q)-*modulus of continuity*, a functional useful in some applications, see Yu. Brudnyi [Br-1971].

The first Shvartsman result [Shv-1978, Shv-1986] concerns a linear extension of this functional controlling its order of growth in t. We present only two consequences of his result related to θ equal to p or ∞.

Let $S \subset \mathbb{R}^n$ be an (Ahlfors) n-regular set, i.e., for some $\mu > 0$ and all cubes $Q \in \mathcal{K}_S$,

$$|Q \cap S| \ge \mu |Q|. \tag{9.176}$$

Theorem B. *Assume that a k-majorant ω satisfies, for some $c \ge 1$ and all $t > 0$, the condition*

$$\left\{ \int_t^\infty \left(\frac{\omega(u)}{u^k} \right)^p \frac{du}{u} \right\}^{\frac{1}{p}} \le c\, \frac{\omega(t)}{t^k}\,. \tag{9.177}$$

Then up to equivalence of the norms

$$\Lambda^{k,\omega}_{p\theta}(\mathbb{R}^n)\big|_S = \Lambda^{k,\omega}_{p\theta}(S),$$

and there exists a simultaneous extension from the trace space into $\Lambda^{k,\omega}_{p\theta}(\mathbb{R}^n)$.

If S is bounded, the same holds for the homogeneous Lipschitz space.

Remark. (a) The first assertion is true under a local version of (9.176) with $Q = Q_r(x)$ satisfying $x \in S$ and $0 < r \le r_0$ for some constant r_0.

(b) It was shown in [Shv-1978] that condition (9.177) cannot be weakened.

In particular, the theorem does not hold for the majorant $\omega_k : t \mapsto t^k$ when the corresponding Lipschitz space coincides with the Sobolev space $W^k_p(\mathbb{R}^n)$ for $1 < p \le \infty$, see, e.g., Theorem 2.32 and Remark 2.33 of Volume I; we present the result for this case separately. For the majorant $\omega_\sigma : t \mapsto t^\sigma$, $0 < \sigma < k$, the theorem is true and concerns the Besov space $B^\sigma_{p\theta}(\mathbb{R}^n) = \Lambda^{k,\omega_\sigma}_{p\theta}(\mathbb{R}^n)$. (A similar equality holds for the homogeneous case with k being the smallest integer greater than σ, see, e.g., Triebel [Tri-1992].)

The second consequence of Shvartsman's theorem concerns the trace of the Morrey–Campanato space $\dot{M}^{k,\omega}_p(\mathbb{R}^n)$ to n-regular sets. Let us recall, see (9.98), that this is defined by the seminorm

$$|f|_{M^{k,\omega}_p(\mathbb{R}^n)} := \sup_Q \frac{\mathcal{E}_k(Q\,; f)_{L_p}}{\omega(r_Q)}, \tag{9.178}$$

where ω is a monotone function on $(0, +\infty)$ satisfying (9.178) with $p = \infty$, i.e., for some $c \geq 1$ and all $t > 0$,

$$\sup_{u \geq t} \frac{\omega(u)}{u^k} \leq c \, \frac{\omega(t)}{t^k} \,. \tag{9.179}$$

By definition, a k-majorant satisfies this condition with $c = 1$, but it also holds for decreasing functions. If, e.g., $\omega : t \mapsto t^{-\sigma}$, $\sigma \geq 0$, then the corresponding Morrey–Campanato space, up to factorization by $\mathcal{P}_{k-1,n}$, coincides, respectively, with the Morrey space $\mathcal{M}_p^\sigma(\mathbb{R}^n)$ for $\sigma < \frac{k}{p}$, and $\mathrm{BMO}(\mathbb{R}^n)$ for $\sigma = 0$, and $L_p(\mathbb{R}^n)$ for $\sigma = \frac{n}{p}$.

In order to formulate the result, we extend the definition (9.178) to the measurable subsets $S \subset \mathbb{R}^n$ of positive measure, cf. (9.100), defining a space denoted by $\dot{M}_p^{k,\omega}(S)$ by the seminorm

$$|f|_{M_p^{k,\omega}(S)} := \operatorname*{ess\,sup}_Q \frac{\mathcal{E}_k(Q \cap S\,; f)_{L_p}}{\omega(r_Q)},$$

where Q runs over all cubes centered at S of length sides at most $2 \operatorname{diam} S$.

Theorem C. *Assume that $S \subset \mathbb{R}^n$ is n-regular and ω satisfies* (9.179)*. Then up to equivalence of the seminorms*

$$\dot{M}_p^{k,\omega}(\mathbb{R}^n)\big|_S = \dot{M}_p^{k,\omega}(S),$$

and there exists a simultaneous extension from the trace space into $\dot{M}_p^{k,\omega}(\mathbb{R}^n)$.

For ω being a k-majorant, the theorem is a consequence of Theorem 9.36 which asserts much stronger results. However, for slowly increasing or decreasing ω the functions of $\dot{M}_p^{k,\omega}(S)$ have no traces to sets of measure zero, and Theorem 9.36 is not valid for these cases.

The second Shvartsman result concerns uniform domains. Let $G \subset \mathbb{R}^n$ be a (c_0, c_1)-uniform domain, see Definition 9.49. It can be easily shown that G satisfies the local version of condition (9.176) and therefore Theorem B holds for $\Lambda_{p\theta}^{k,\omega}(\mathbb{R}^n)\big|_G$; moreover, it was proved by Shvartsman that in this case the result is true with *arbitrary* k-majorants, in particular, for Sobolev spaces $W_p^k(\mathbb{R}^n)$. But for quasipower majorants Shvartsman obtained a stronger result giving a partial solution to the Restricted Extension Problem.

Theorem D. *Assume that $G \subset \mathbb{R}^n$ is a (c_0, c_1)-uniform domain and ω is a quasipower majorant. Then there exists a linear continuous extension operator from $\Lambda_{p\theta}^{k,\omega}(G)$ into $\Lambda_{p\theta}^{k,\omega}(\mathbb{R}^n)$.*

Since $\omega : t \mapsto t^\sigma$ is quasipower for $0 < \sigma \leq k$, the theorem yields, as a special case, the P. Jones theorem for Sobolev spaces [Jon-1981] and a similar result for Besov spaces.

Let us note that Theorem 9.55 is a slight refinement for the special case of Theorem D with $p = \theta = \infty$. Using an L_p-version of the criterion of Theorem 9.3 it is possible to prove the same refinement for Theorem D.

Since Theorem B excludes Sobolev spaces we briefly discuss the situation for this case and for Sobolev spaces of fractional smoothness $L_p^\sigma(\mathbb{R}^n)$, $\sigma > 0$ (also known as potential spaces). Let us recall that $L_p^\sigma(\mathbb{R}^n)$ consists of functions $f \in L_p(\mathbb{R}^n)$ for which the norm

$$\|f\|_{L_p^\sigma(\mathbb{R}^n)} := \|f * \mathcal{G}_\sigma\|_{L_p(\mathbb{R}^n)} < \infty;$$

here the convolution kernel is the *Bessel potential*, see, e.g., Stein [Ste-1970, Sec. V.3]. The relation between these and Sobolev spaces is given by the classical Calderón theorem, see, e.g., [Ste-1970, Thm V.3], asserting that for $1 < p < \infty$ and $\sigma \in \mathbb{N}$,

$$L_p^\sigma(\mathbb{R}^n) = W_p^\sigma(\mathbb{R}^n).$$

The key point for the proof of analogs of Theorem B for a Sobolev space is its local polynomial representation. For $\dot{W}_p^k(\mathbb{R}^n)$ this was due to Yu. Brudnyi [Br-1971, Sec. 4.3] who generalized the classical F. Riesz lemma characterizing $\dot{W}_p^1(\mathbb{R})$ for $1 < p < \infty$, see [R-1910, § 5], as follows.

Let us define a variation of the set function $Q \mapsto \mathcal{E}(Q\,; f)_{L_q}$ by setting

$$V_{k,p}(f\,; L_q) := \sup_\pi \left\{ \sum_{Q\in\pi} |Q| \left(\frac{(\mathcal{E}_k(Q\,; f)_{L_q}}{|Q|^{\frac{k}{n}}} \right)^p \right\}^{\frac{1}{p}}. \tag{9.180}$$

Assuming that $1 < p \le q \le \infty$, and

$$\frac{k}{n} \ge \frac{1}{p} - \frac{1}{q} \text{ for } q < \infty \text{ and } \frac{k}{n} > \frac{1}{p} - \frac{1}{q} \text{ for } q = \infty,$$

we have, for every $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$ the equivalence

$$|f|_{W_p^k(\mathbb{R}^n)} \approx V_{k,p}(f\,; L_q)$$

with constants independent of f.

For fractional Sobolev spaces, a local polynomial characteristic using Taylor polynomials was due to Strichartz [Str-1967] for $0 < \sigma < 1$ and Dahlberg [Da-1979] for arbitrary noninteger $\sigma > 0$. The result, going back to the Marcinkiewicz characterization of $\dot{W}_p^1(\mathbb{R})$ for $1 < p < \infty$ [Ma-1938], exploits a Taylor analog of the normalized best local approximation given for $f \in L_p^\sigma(\mathbb{R}^n)$ and $Q = Q_r(x)$ by

$$R_\sigma(x, r\,; f) := \frac{1}{|Q|} \int_Q \left|f - T_x^\ell f\right| dy \tag{9.181}$$

where ℓ is the integer part of σ and

$$T_x^\ell f(y) := \sum_{|\alpha|\le\ell} \frac{D^\alpha f(x)}{\alpha!}\,(y-x)^\alpha.$$

It is known that these derivatives exist for almost all x, see, e.g., [Ste-1970, Sec. V.3.4].

The result in question asserts that for $1 < p < \infty$ and noninteger $\sigma > 0$,

$$\|f\|_{L_p^\sigma(\mathbb{R}^n)} \approx \|f\|_{L_p(\mathbb{R}^n)} + \left\| \left\{ \int_0^\infty \left(\frac{R_\sigma(\cdot, r\,; f)}{r^\sigma} \right)^2 \frac{dr}{r} \right\}^{\frac{1}{2}} \right\|_{L_p(\mathbb{R}^n)}. \tag{9.182}$$

The entity $R_\sigma(x, r\,; f)$ can be replaced by a smaller one, $\mathcal{E}_{\ell+1}\big(Q_r(x)\,; f\big)_{L_1}$, by estimating the Taylor remainder via local best approximation, see [Br-1971, Sec. 2.1]. In fact, equivalence (9.182) with $\mathcal{E}_{\ell+1}\big(Q_r(x)\,; f\big)_{L_1}$ substituted for $R_\sigma(x, r\,; f)$ was due to Dorronsoro [Dor-1986] who used another approach based on the complex interpolation of Sobolev spaces.

From these representations the analog of Theorem B for Sobolev spaces can be derived, namely, if $S \subset \mathbb{R}^n$ is n-regular, then the trace space $\dot{W}_p^k(\mathbb{R}^n)\big|_S$ for $1 < p \le \infty$, is characterized by variation (9.180) but with $\mathcal{E}_k(Q\,; f)_{L_q}$ replaced by $\mathcal{E}_k(Q \cap S\,; f)_{L_q}$ where Q are cubes from the family $\mathcal{K}_S$. Similarly, the norm of $L_p^\sigma(\mathbb{R}^n)\big|_S$ for $1 < p < \infty$ and $\sigma > 0$ is equivalent to

$$\|f\|_{L_p(S)} + \left\| \left\{ \int_0^\infty \left(\frac{\mathcal{E}_{\ell+1}(S \cap Q_r(\cdot))_{L_1}}{r^\sigma} \right)^2 \frac{dr}{r} \right\}^{\frac{1}{2}} \right\|_{L_p(S)}.$$

In conclusion, we single out the Calderón representation of $\dot{W}_p^k(\mathbb{R}^n)$ for $1 < p \le \infty$ via a maximal function using the Taylor remainder (9.181) with $\ell = k-1$. Again, the Taylor remainder can be replaced by the local best approximation of order k, and the (equivalent) seminorm of $\dot{W}_p^k(\mathbb{R}^n)$, $p > 1$, is then given by

$$\left\| \sup_{r>0} \frac{\mathcal{E}_k(Q_r(\cdot)\,; f)_{L_1}}{r^k} \right\|_{L_p(\mathbb{R}^n)},$$

see DeVore and Sharpley [DeVSh-1984, Thm 6.2].

A similar result for $\dot{W}_p^k(\mathbb{R}^n)\big|_S$ with the same S as above may be formulated by substituting $Q_r(x) \cap S$ where $Q_r(x) \in \mathcal{K}_S$ for $Q_r(x)$ and $L_p(S)$ for $L_p(\mathbb{R}^n)$.

Chapter 10

Whitney Problems

This chapter contains results concerning two basic conjectures on the extension properties of the smoothness spaces $C^\ell\Lambda^{k,\omega}(\mathbb{R}^n)$, $C_b^k(\mathbb{R}^n)$ and the related jet-spaces. The first conjecture claims that the *finiteness constant* $\mathcal{F}(X)$ of every space X from this family is bounded by a constant depending only on ℓ, k and n (this and other concepts appearing in this chapter will be introduced in Section 10.1, see also Volume I, subsection 2.4.3). Finiteness of $\mathcal{F}(X)$ allows us essentially to simplify the solution to the *first Whitney problem* on distinguishing the traces $f|_S \in X|_S$ from other functions defined on S. Actually, in presence of the *finiteness property* (meaning finiteness of $\mathcal{F}(X)$), we need only study the extension properties of functions defined on subsets of cardinality at most $\mathcal{F}(X)$. For instance, due to Whitney's Theorems 2.13 and 2.47 of Volume I, we need only examine two-point subsets for the jet space $J_b^\ell(\mathbb{R}^n)$ and $(\ell+2)$-point subsets for the space $C_b^\ell(\mathbb{R})$.

Several results justifying the above formulated *finiteness conjecture* are given in subsection 10.3.1 (the sharp result for $C^{1,\omega}(\mathbb{R}^n)$), subsections 10.3.2 and 10.4.1 (upper bounds for $C_b^\ell(\mathbb{R}^n)$ and $C^{\ell,\omega}(\mathbb{R}^n)$, $\ell \geq 2$) and subsection 10.5.1 (an upper bound for $J^\ell\Lambda^{2,\omega}(\mathbb{R}^n)$).

All these bounds are of exponential growth in n and ℓ. In subsections 10.2.2 and 10.2.3 we will show that for a wide class of subsets (Markov and weak Markov) the finiteness constant $\mathcal{F}(\Lambda^{k,\omega}(\mathbb{R}^n))$ has an essentially smaller upper bound (of polynomial growth in k and n).

The final step to the solution of Whitney's first problem is the next extremally difficult question.

Estimate the trace norm of $f \in X|_S$ *for* $\operatorname{card} S \leq \mathcal{F}(X)$ *via the set* $\{f(x)\}_{x\in S}$ *with the constants of equivalence depending only on* ℓ, k, n.

For the univariate function spaces of the family in question the solution was presented in subsection 2.4.3 of Volume I. For instance, if $X := C^{\ell,\omega}(\mathbb{R})$, the

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Whitney Theorem 2.52 of Volume I gives for $S \subset \mathbb{R}$ of cardinality $\mathcal{F}(X) = \ell + 2$,

$$|f|_{X|_S} \approx \frac{|f[S]|}{\omega^*(\operatorname{diam} S)},$$

where $\omega^*(t) := \frac{t}{\omega(t)}$, $t > 0$.

In the multivariate case, the result of this kind is known only for Markov sets, see subsection 10.2.2 where the notion of divided difference is defined for these settings. For the general case, only a few results are known (namely, for $J_b^\ell(\mathbb{R}^n)$, $J^{\ell,\omega}(\mathbb{R}^n)$ and $C^{1,\omega}(\mathbb{R}^2)$, $\Lambda^{2,\omega}(\mathbb{R}^2)$, see Section 10.1).

The *second Whitney problem* asks for the proof of existence of a simultaneous (= linear continuous) extension from $X|_S$ into X, where X is a space in question. The problem is, in general, open, and solved only for some special cases. These solutions are presented in subsection 10.4.2 (for $C_b^\ell(\mathbb{R}^n)$ and $C_b^{\ell,\omega}(\mathbb{R}^n)$), in subsection 10.5.2 (for $J^\ell\Lambda^{2,\omega}(\mathbb{R}^n)$) and in subsection 10.2.4 (for $\Lambda^{k,\omega}(\mathbb{R}^n)$ and Markov subsets).

The proofs of the finiteness and linearity results for the spaces $C^{1,\omega}(\mathbb{R}^n)$, $\Lambda^{k,\omega}(\mathbb{R}^n)$ and $J^\ell\Lambda^{2,\omega}(\mathbb{R}^n)$ are carried out in accordance with the geometric analysis line of this book, combining the Lipschitz selections theorems of Sections 5.3 and 5.4 of Volume I with the local approximation extension criteria of Section 9.1.

A detailed account of Ch. Fefferman's ground-breaking results for the spaces $C^\ell(\mathbb{R}^n)$ and $C^{\ell,\omega}(\mathbb{R}^n)$ cannot be presented here, since the series of papers devoted to these results consists of hundreds of pages of a highly complicated text. We present instead a rather complete survey of Fefferman's results and a very sketchy description of his methods. A deeper insight into the Fefferman methods may, as we hope, lead to essential further progress.

Challenging problems of the area whose solution is inaccessible by known methods are the finiteness and linearity conjectures for the space of C^1 functions on $\mathbb{R}^n$ whose derivatives satisfy the Zygmund condition, i.e., for the space $B^2_\infty(\mathbb{R}^n)$ (for $B^s_\infty(\mathbb{R}^n)$ with noninteger s and for $s = 1$ solutions have been given, respectively, by Ch. Fefferman and Shvartsman).

10.1 Formulation of the problems

We introduce several basic concepts (partially discussed in subsection 2.4.3 of Volume I) that are used in formulations of problems and conjectures. They concern the smoothness spaces of multivariate functions or jets introduced earlier, i.e., $C^\ell\Lambda^{k,\omega}(\mathbb{R}^n)$, $\ell, k \geq 0$, and $J^\ell\Lambda^{k,\omega}(\mathbb{R}^n)$ and their homogeneous counterparts, identified by a point over Λ or C if $k = 0$. For instance, $\dot{C}_b^\ell(\mathbb{R}^n)$ stands for the space of ℓ-times continuously differentiable functions equipped with the seminorm

$$|f|_{C_b^\ell(\mathbb{R}^n)} := \max_{|\alpha|=\ell} \sup_{\mathbb{R}^n} |D^\alpha f|$$

while $C_b^\ell(\mathbb{R}^n)$ is a subspace of $\dot{C}_b^\ell(\mathbb{R}^n)$ determined by the norm

$$\|f\|_{C_b^\ell(\mathbb{R}^n)} := \sup_{\mathbb{R}^n} |f| + |f|_{C_b^\ell(\mathbb{R}^n)}.$$

10.1.1 Trace spaces

In this section, X denotes one of the aforementioned normed or seminormed spaces of functions or jets. All of them are continuously embedded into the space $C(\mathbb{R}^n)$ (or into the direct sum of $\binom{n+\ell}{n}$ copies of $C(\mathbb{R}^n)$ for a space of ℓ-jets) and therefore the *trace space* $X|_S$ is well defined; hereafter S denotes a *closed* subset of $\mathbb{R}^n$ unless explicitly stated otherwise.

We discuss the properties of this object in the basic normed case; the facts of this kind for the remaining smoothness spaces may be obtained analogously.

In the formulation of the first result, we use the null-space

$$N_S(X) := \{f \in X\, ;\, f|_S = 0\}.$$

Since S is closed, this is a closed subspace of X.

Proposition 10.1. *The factor-space $X/N_S(X)$ is isometric to $X|_S$.*

Proof. Denoting the norm of the factor-space by $\|\cdot\|^*$ we get

$$\|f\|_{X|_S} := \inf\{\|g\|\, ;\, g|_S = f\} = \inf\{\|f - h\|_X\, ;\, h|_S = 0\} =: \|f\|^*. \qquad \square$$

Corollary 10.2. *The trace space $X|_S$ admits a simultaneous extension into X if and only if $N_S(X)$ is complemented in X.*

Proof. A linear continuous extension operator $E : X|_S \to X$ gives rise to the projection $P := Id_X - E$ of X onto $N_S(X)$ and vice versa. $\square$

As it will be explained later, most of the smoothness spaces in question have a property introduced by

Definition 10.3. A Banach space $Y \subset C(\mathbb{R}^n)$ is of dual type if:

(a) there exists a Banach space Z such that its dual Z^* is isometric to Y;

(b) the linear hull of the set of evaluations $\delta_{\mathbb{R}^n} := \{\delta_x\, ;\, x \in \mathbb{R}^n\}$ is dense in Z.

Remark 10.4. Since $Y \subset C(\mathbb{R}^n)$, every evaluation $\delta_x : x \mapsto f(x)$ is a linear continuous functional, i.e., an element of Y^*. Using the canonical isometric embedding $Z \subset Z^{**}$ $(= X^*)$ we may and will identify $\delta_{\mathbb{R}^n}$ with a subset of Z. Then (b) asserts that $\text{hull}(\delta_{\mathbb{R}^n})$ after this identification becomes a dense subset in Z.

Now we check that every trace space of X inherits this property.

Proposition 10.5. *If X is of dual type, then for every S the trace space $X|_S$ is also dual.*

Proof. Let $Y^* = X$ and let $Y(S)$ denote the closure in Y of the hull(δ_S), where $\delta_S := \{\delta_x\,;\, x \in S\}$, which by assumption (b) belongs to Y. We show that after the canonical identification we have

$$X|_S = Y(S)^*$$

isometrically.

Let $\ell \in Y(S)^*$. By the Hahn-Banach theorem there exists an extension $L \in Y^*$ of ℓ such that

$$\|\ell\|_{Y(S)^*} = \|L\|_{Y^*}.$$

These functionals give rise to functions $f_L : \mathbb{R}^n \to \mathbb{R}$ and $f_\ell : S \to \mathbb{R}$ given by

$$f_L(x) := L(\delta_x) \quad \textit{and} \quad f_\ell(x) := \ell(\delta_x).$$

Because of density of hull$(\delta_{\mathbb{R}^n})$ in Y we get

$$\|f_L\|_X = \|L\|_{Y^*} = \|\ell\|_{Y(S)^*}.$$

Moreover, $f_L|_S = f_\ell$ by definition.

This clearly implies that

$$\|f_\ell\|_{X|_S} \le \|f_L\|_X = \|\ell\|_{Y(S)^*},$$

i.e., the map $\ell \mapsto f_\ell$ is an injection of $Y(S)^*$ into $X|_S$ of norm at most 1. Performing this argument in the opposite direction starting with a function $f \in X|_S$ and the associated functional $\ell_f \in Y(S)^*$ defined on hull(δ_S) by $\ell_f(\delta_x) := f(x)$ we prove the converse inequality and the proposition. □

Remark 10.6. Every space X defined over the Lipschitz space $\Lambda^{k,\omega}(\mathbb{R}^n)$ is subject to Definition 10.3. Since this fact will not be used later, we only outline the main steps of its (lengthy) proof for $X := C^\ell\Lambda^{k,\omega}(\mathbb{R}^n)$, $k \ge 1$, $\ell \ge 0$, and the corresponding ℓ-jet spaces.

(a) Let $\Sigma(s,n)$ be a *minimal interpolation subset* in $\mathbb{R}^n$ for polynomials of degree $s-1$. In other words, its cardinality equals $\dim \mathcal{P}_{s-1,n}$ and every polynomial of degree $s-1$ is uniquely determined by its values at the points of $\Sigma(s,n)$.

Using the inequality

$$E_{k+\ell}(Q; f) \le O(1) r^\ell \max_{|\alpha|=\ell} \omega_k(Q; D^\alpha f)$$

which straightforwardly follows from inequality (9.3) we conclude that uniformly in $f \in X$,

$$\|f\|_X \approx \sum_{|\alpha| \le \ell-1} |D^\alpha f(0)| + \max_{|\alpha|=\ell} \sup_{t>0} \frac{\omega_k(t; D^\alpha f)}{\omega(t)}.$$

The second term here is the seminorm of f in the homogeneous space $\dot{X} := C^\ell \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ denoted below by $|f|_X$.

In turn, the functional $f \mapsto |f|_X$ is a norm on the null-space

$$N_{\Sigma(s,n)}(X) := \{f \in X\, ;\, f|_{\Sigma(s,n)} = 0\},$$

where $s := \ell + k - 1$, which is equivalent to its norm induced from X. Hence, it suffices to establish the required result for the null-space equipped with the norm $f \mapsto |f|_X$.

(b) First let $\ell = 0$, i.e., $X := \Lambda^{k,\omega}(\mathbb{R}^n)$. To prove that X is of dual type we use an analog of the free-Lipschitz space construction of Theorem 4.89 of Volume I. Let B denote the closed unit ball of the null-space $N := N_{\Sigma(k,n)}(\Lambda^{k,\omega}(\mathbb{R}^n))$. We define the desired predual space by

$$Y := \overline{\operatorname{hull}(\delta_{\mathbb{R}^n})} \quad \text{(closure in } \ell_\infty(B)).$$

It was explained after the proof of Theorem 4.91 of Volume I why Y^* is linearly isomorphic to X. This implies that X is of dual type.

(c) The same argument may be applied to the space $X = C^\ell \Lambda^{k,\omega}(\mathbb{R}^n)$ with $\ell, k \geq 1$. The reason is a possibility to equip this space with an equivalent norm using a kind of a k-module of continuity. Specifically, we have uniformly in $f \in X$,

$$\|f\|_X \approx \sup_{\mathbb{R}^n} |f| + \sup_{h,g \in \mathbb{R}^n} \frac{\|\Delta_g^\ell \Delta_h^k f\|_{\ell_\infty(\mathbb{R}^n)}}{\|g\|^\ell \omega(\|h\|)},$$

see the proof of this in Theorem 2.31 of Volume I for $k = 1$; the general case may be proved similarly. Therefore the corresponding null-space is now determined by $\Sigma(\ell + k, n)$ and the proof follows with small changes to that for $\ell = 0$.

Finally, the corresponding results for jet-spaces over $\Lambda^{k,\omega}(\mathbb{R}^n)$ may be straightforwardly derived from the previous ones.

One more family of dual type spaces is given by

Example 10.7. (a) Let the parameters of the Sobolev space $W_p^\ell(\mathbb{R}^n)$ satisfy $\frac{\ell}{n} > \frac{1}{p}$. Then $W_p^\ell(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ modulo measure zero and therefore the trace operator and the trace space $W_p^\ell(\mathbb{R}^n)|_S$ are well defined for *every* closed subset $S \subset \mathbb{R}^n$.

The image of $W_p^\ell(\mathbb{R}^n)$ with $1 < p < \infty$ in $C(\mathbb{R}^n)$ is the space of functions of bounded (ℓ, p)-variations, see (9.180) with $q = \infty$. Moreover, in the Yu. Brudnyi paper [Br-2008], a predual to this image is presented by an atomic decomposition approach. It may be easily verified that $\operatorname{hull}(\delta_{\mathbb{R}^n})$ is dense in the so-introduced predual space. Hence, the space $W_p^\ell(\mathbb{R}^n)$ is of dual type if $1 < p < \infty$ and $\frac{\ell}{n} > \frac{1}{p}$. Due to Proposition 10.5 every trace space $W_p^\ell(\mathbb{R}^n)|_S$ is also of dual type.

(b) Since $W_2^\ell(\mathbb{R}^n)$ is Hilbert, the image of this space in $C(\mathbb{R}^n)$ (for $\frac{\ell}{n} > \frac{1}{2}$) is also Hilbert up to equivalent renorming. Then every closed subspace of this image is complemented and Corollary 10.2 implies that every trace space of $W_2^\ell(\mathbb{R}^n)$ to an arbitrary closed subset of $\mathbb{R}^n$ admits a simultaneous extension to $W_2^\ell(\mathbb{R}^n)$.

10.1.2 Finiteness Property

The Whitney Trace Problem for the spaces under consideration is divided into two subproblems. The solution to the first one, the *Finiteness Problem*, allows us to reduce the initial problem for a closed set to that for N-points subsets, where N is the *finiteness constant* introduced for some special cases in subsection 2.4.2 of Volume I. Now we discuss in detail the italized notions for spaces of functions; the case of ℓ-jets is considered similarly.

In the next formulation, given a continuous function $g : S \to \mathbb{R}$ and a finite subset $F \subset S$, we put, for brevity,

$$\|g\|_F := \|g|_F\|_{X|_F} \tag{10.1}$$

and then for $N \in \mathbb{N}$ denote by $\delta_N : C(S) \to \mathbb{R}$ the function

$$\delta_N(g; S; X) := \sup\{\|g\|_F\, ;\, F \subset S \text{ and } \operatorname{card} F \le N\}. \tag{10.2}$$

Definition 10.8. A space X has the Finiteness Property with respect to a class Σ of closed subsets of $\mathbb{R}^n$ if, for some integer $N > 1$ and every $S \in \Sigma$, the trace space $X|_S$ coincides with the linear space $\{g \in C(S)\, ;\, \delta_N(g; S; X) < \infty\}$.

The minimal N here is said to be the *finiteness constant* and is denoted by $\mathcal{F}_\Sigma(X)$ while the class of spaces subject to this definition is denoted by $\mathcal{FP}(\Sigma)$ or simply $\mathcal{FP}$ if Σ consists of all nonempty closed subsets of $\mathbb{R}^n$.

Definition 10.8 may be applied to the case of smoothness spaces distinct from those studied in this chapter. The only restriction is the existence of a well-defined trace operator to the subsets of a given class Σ. This is the case of the Sobolev space $W_p^\ell(\mathbb{R}^n)$, where $\frac{\ell}{n} > \frac{1}{p}$, see Example 10.7(a) and Proposition 10.9 below.

We now present a result that may be regarded as a qualitative form of Definition 10.8.

Proposition 10.9. *A space X belongs to $\mathcal{FP}(\Sigma)$ with $\mathcal{F}_\Sigma(X) \le N$ if and only if for every $S \in \Sigma$ and $g \in C(S)$,*

$$\|g\|_S \le c\delta_N(g; S; X), \tag{10.3}$$

where $c > 0$ is a constant independent of g.

Proof. Since finiteness of $\delta_N(g; S; X)$ and (10.3) imply that $g \in X|_S$, the space $X \in \mathcal{FP}(\Sigma)$ and $\mathcal{F}_\Sigma(X) \le N$.

To prove the converse we introduce the normed linear space

$$X_N(S) := \{g \in C(S)\,;\, \delta_N(g; S; X) < \infty\}.$$

To show that δ_N is a norm, it suffices to check that the equality $\delta_N(g; S; X) = 0$ implies $g = 0$. But $X|_F$ is continuously embedded into the space $C(F)$ and therefore for every $x \in S$ and some $F \subset S$ of cardinality at most N containing x we get

$$|g(x)| \le \max_F |g| \le c\|g\|_F \le c\delta_N(g; S; X).$$

Therefore, if $\delta_N(g; S; X) = 0$, then $g = 0$.

Further, it is the matter of definition to check that $X_N(S)$ is complete, i.e., is a Banach space.

Now, if $g \in X_N(S)$, then by Definition 10.8 $g \in X|_S$. Hence, the linear embedding $X_N(S) \subset X|_S$ holds. Since both spaces are Banach and are continuously embedded into the Fréchet space $C(S)$, the Open Mapping Theorem, see, e.g., [DS-1958], implies that this embedding is continuous and therefore (10.3) is true. □

In this chapter, we establish the Finiteness Property for several families of smoothness spaces with constants in inequality (10.3) *independent* of S. We single out this property by

Definition 10.10. A space X has the Uniform Finiteness Property with respect to a class Σ if X belongs to $\mathcal{FP}(\Sigma)$ and (10.3) holds for $N = \mathcal{F}_\Sigma(X)$ with a constant independent of S and g. The infimum of these constants is denoted by $\gamma_\Sigma(X)$.

Geometrically, this property means that for every $S \in \Sigma$ the intersection $\bigcap_F \{g \in C(S)\,;\, \|g\|_F \le 1\}$, where F runs over all subsets of S of cardinality at most N, is contained in a closed ball of $X|_S$ of radius independent of S. Note that every (convex) set of this intersection is unbounded in the space $X|_S$ and contains its closed unit ball, if $\operatorname{card} S = \infty$.

Example 10.11. (a) The space $\Lambda^{1,\omega}(\mathbb{R}^n)$ has the Uniform Finiteness Property and its finiteness constant equals 2.

The norm of this space is equivalent to that given by $f \mapsto |f(0)| + \sup_{t>0} \frac{\omega_1(t;f)}{\omega(t)}$. Therefore it suffices to prove the claim for its subspace consisting of functions vanishing at 0. This, in turn, is isometric to the Lipschitz space $\mathrm{Lip}_0(\mathbb{R}^n, d_\omega)$ on the pointed metric space $(\mathbb{R}^n, d_\omega, 0)$, where the metric d_ω is given on $\mathbb{R}^n$ by

$$d_\omega(x, y) := \omega(\|x - y\|)$$

(recall that the 1-majorant ω is subadditive).

To prove the required property for $\mathrm{Lip}_0(\mathbb{R}^n, d_\omega)$ we consider a closed metric subspace $(S, d_\omega, 0) \subset (\mathbb{R}^n, d_\omega, 0)$ and assume that for a function $g \in C(S)$ and every set $F := \{x, y\} \subset S$ there exists an extension $g_F \in$

$\mathrm{Lip}_0(\mathbb{R}^n, d_\omega)$ of $g|_F$ of norm bounded by a constant, say γ, independent of F. Then we get

$$|g(x) - g(y)| = |g_F(x) - g_F(y)| \le \gamma\omega(\|x - y\|),$$

i.e., g belongs to $\mathrm{Lip}(S, d_\omega)$ and its Lipschitz constant is bounded by γ.

According to McShane's Theorem 1.27 of Volume I there exists an extension of g to $\mathbb{R}^n$ with the Lipschitz constant at most γ. Returning to the initial space we conclude that every function $g \in C(S)$ has the trace norm bounded by $\delta_2(g; S; \Lambda^{1,\omega}(\mathbb{R}^n))$, i.e., inequality (10.3) holds for $c = 1$ and $N = 2$.

Thus, $\Lambda^{1,\omega}(\mathbb{R}^n)$ has the Uniform Finiteness Property and its finiteness constant is at most 2.

On the other hand, the restriction of every $g : S \to \mathbb{R}$ to a one-point set $\{x_0\} \subset S$ extends by a constant to $\mathbb{R}^n$ as a function of $\Lambda^{1,\omega}(\mathbb{R}^n)$ with norm $|g(x_0)|$. Since g may not be from $\Lambda^{1,\omega}(\mathbb{R}^n)|_S$, the finiteness constant is greater than 1. Consequently,

$$\mathcal{F}(\Lambda^{1,\omega}(\mathbb{R}^n)) = 2.$$

It is worth noting that the McShane theorem is a straightforward consequence of Helly's theorem regarded in subsection 1.9.2 of Volume I as a Lipschitz selection result. Hence, the above derivation may be seen as a simple illustration of the main line of our research in this chapter, where Lipschitz selection results are basic geometric tools.

(b) The version of the Whitney Theorem 2.55 presented in subsection 2.4.2 of Volume I asserts that the homogeneous space $\dot{C}^{\ell,\omega}(\mathbb{R})$ has the finiteness constant $\ell + 2$. The proof of this theorem shows that this space has, in fact, the Uniform Finiteness Property.

(c) The space $\dot{J}^{\ell,\omega}(\mathbb{R}^n)$ has the Uniform Finiteness Property and its finiteness constant is 2.

This claim may be proved by an argument similar to that used in part (a) but with the Whitney-Glaeser Theorem 2.19 of Volume I in place of the McShane theorem.

(d) The image of the Sobolev space $W_p^\ell(\mathbb{R}^n)$ with $\frac{\ell}{n} > \frac{1}{p}$ into $C(\mathbb{R}^n)$ does not have the Finiteness Property. To fix an idea we consider the simplest case of the homogeneous space $\dot{W}_1^1(\mathbb{R})$ whose image in $C(\mathbb{R})$ is known to be the space AC of absolutely continuous functions on $\mathbb{R}$ equipped with the seminorm $f \mapsto \int_{\mathbb{R}} |f'|\, dx$, see, e.g., [Ru-1987, Thm. 7.18].

To show that AC does not have the Finiteness Property, we define a "sawtooth" continuous function $g : [0, 1] \to \mathbb{R}$ whose graph over the segment

$I_n := [2^{-n}, 2^{-n+1}]$ is the equilateral triangle with the base I_n and of height $\frac{1}{n \cdot 2^n}$, $n \in \mathbb{N}$. Then

$$\int_0^1 |g'|\,dx = 2\sum_{i\in\mathbb{N}} \frac{1}{n} = \infty,$$

i.e., g has no extension to $\mathbb{R}$ as an AC function. On the other hand, for every $N > 1$ it is true that $\delta_N(g; [0,1]; AC) \le 2$, i.e., the space AC does not have the Finiteness Property.

Actually, let $F \subset [0,1]$ be a set of cardinality at most N. We extend $g|_F$ continuously by constants outside the segment $\operatorname{conv}(F)$ and linearly between adjacent points of F. This extension, say g_F, is clearly absolutely continuous and $|g'_F| \le 2$ over every segment determined by the adjacent points of F. Hence, $\|g'_F\|_{L_1(\mathbb{R})} \le 2$, as required.

10.1.3 Finiteness Problem

The problem concerns the smoothness spaces of C^ℓ functions or ℓ-jets over spaces of Lipschitz functions of higher order. As above, X denotes the common object of this family.

Finiteness Problem 10.12. *Prove that every X has the Uniform Finiteness Property.*

The results presented in Example 10.11 and in Sections 10.2–10.5 suggest that the solution to the problem is positive. However, it might require (and surely does require) completely new ideas and tools.

10.1.4 Finiteness constants

Let $\mathcal{FP}_u(\Sigma)$ be the class of spaces having the Uniform Finiteness Property with respect to a class of closed nonempty subspaces Σ. Every smoothness space $X \in \mathcal{FP}_u(\Sigma)$ determines two finiteness constants, $\mathcal{F}_\Sigma(X)$ and the optimal constant in inequality (10.3) $\gamma_\Sigma(X)$. Hence, for every $c > \gamma_\Sigma(X)$ and $N := \mathcal{F}_\Sigma(X)$,

$$\delta_N(f; S; X) \le c\|f\|_{X|_S} \tag{10.4}$$

provided that $f \in C(S)$ and $S \in \Sigma$.

Clearly, $\mathcal{F}_\Sigma(X)$ and $\gamma_\Sigma(X)$ are nondecreasing whenever Σ is increasing from the class of all finite subsets Φ to the class of all bounded subsets (in the latter case the constants are denoted by $\mathcal{F}(X)$ and $\gamma(X)$). We will show that for spaces of dual type these constants are independent of Σ.

Proposition 10.13. *Assume that a space X is of dual type and has the uniform Finiteness Property. Then it is true that*

$$\mathcal{F}_\Phi(X) = \mathcal{F}(X) \quad \textit{and} \quad \gamma_\Phi(X) = \gamma(X). \tag{10.5}$$

Proof. By definition

$$\gamma_\Phi(X) \le \gamma(X) \quad \text{and} \quad \mathcal{F}_\Phi(X) \le \mathcal{F}(X) \tag{10.6}$$

and we must prove that these are equalities.

Let $S \subset \mathbb{R}^n$ be an arbitrary closed set and $f \in C(S)$ be such that

$$\delta_N(f; S; X) < 1 \quad \text{for} \quad N := \mathcal{F}_\Phi(X). \tag{10.7}$$

We should show that the f belongs to $X|_S$ and its trace norm is bounded by $\gamma_\Phi(X)$.

Let $\{S_j\}_{j\in\mathbb{N}}$ be an increasing sequence of finite subsets of S such that their union is dense in S. In view of (10.7) we have, for every j,

$$\delta_N(f; S_j; X) \le \delta_N(f; S; X) < 1,$$

while by the assumption on X the trace $f|_{S_j}$ belongs to the trace space $X|_{S_j}$ and its trace norm there satisfies, in notation (10.1), the inequality

$$\|f\|_{S_j} \le \gamma_\Phi(X).$$

Then there exists a function $f_j \in X$ such that

$$f_j|_{S_j} = f|_{S_j} \quad \text{and} \quad \|f_j\|_X \le \gamma_\Phi(X) + \frac{1}{j}.$$

In particular, the sequence $\{f_j\}_{j\in\mathbb{N}}$ is contained in a closed ball of X. But X is of dual type, in particular, $X = Y^*$ for some Banach space Y, and the Banach-Alaoglu theorem, see, e.g., Dunford-Schwartz [DS-1958], asserts that a (bounded) closed ball of X is compact in the weak* topology.

Let $\hat{f}$ be a limit point of the set $\{f_j\}_{j\in\mathbb{N}}$ in this topology. Because of weak* semicontinuity of the norm in X,

$$\|\hat{f}\|_X \le \varliminf_{j\to\infty} \|f_j\|_X \le \gamma_\Phi(X)$$

and it remains to show that $\hat{f}$ agrees with f on S. Since f and $\hat{f}$ are continuous, it suffices to check this claim at a point x of the dense subset $\bigcup_{j\in\mathbb{N}} S_j$. By monotonicity of $\{S_j\}$ the point x belongs to all S_j with $j \ge j_0(x)$. Since $\delta_x \in Y$ and $\hat{f} \in X = Y^*$, we then get

$$\hat{f}(x) := \hat{f}(\delta_x) = \varliminf_{j\to\infty} f_j(\delta_x) = f_{j_0}(x) = f(x).$$

Hence, (10.7) implies that f belongs to $X|_S$ and its trace norm is bounded by $\gamma_\Phi(X)$. By Definition 10.10 this means that

$$\mathcal{F}(X) \le \mathcal{F}_\Phi(X) \quad \text{and} \quad \gamma(X) \le \gamma_\Phi(X).$$

Together with (10.7) this proves the result. □

Remark 10.14. Let $\mathcal{K}$ denote the class of all compact subsets of $\mathbb{R}^n$. Since $\Phi \subset \mathcal{K}$, the proposition, in particular, implies that

$$\mathcal{F}_{\mathcal{K}}(X) = \mathcal{F}(X) \quad \text{and} \quad \gamma_{\mathcal{K}}(X) = \gamma(X). \tag{10.8}$$

If, however, X is such that for some $c > 0$ and $k \in \mathbb{N}$ and every compactly supported C^∞ function ϕ,

$$\|\phi f\|_X \leq c\|\phi\|_{C^k(\mathbb{R}^n)} \cdot \|f\|_X \text{ for all } f \in X,$$

then the first equality in (10.8) is valid without the duality assumption on X.

Now we formulate the basic problem concerning the growth of the finiteness constants. In its formulation, we set

$$d_n(s) := \dim \mathcal{P}_{s-1,n} = \binom{n+s-1}{n}.$$

Problem 10.15. (a) *Prove that for $k, \ell \in \mathbb{Z}_+$,*

$$\mathcal{F}(C^\ell \Lambda^{k,\omega}(\mathbb{R}^n)) \leq 2^{d_n(k+\ell)}. \tag{10.9}$$

(b) *Prove that for $\ell \in \mathbb{N}$ and $k \in \mathbb{Z}_+$,*

$$\mathcal{F}(J^\ell \Lambda^{k,\omega}(\mathbb{R}^n)) \leq 2^{d_n(k)}. \tag{10.10}$$

(c) *Prove similar estimates for the homogeneous counterparts of these spaces.*

The first estimate looks to be too rough, since for $n = 1$ the right-hand side in (10.9) is $2^{k+\ell}$ while the corresponding finiteness constant equals $k + \ell + 2$. However, the exponential growth for $n > 1$ is a typical situation, as the inequality

$$\mathcal{F}(\Lambda^{2,\omega}(\mathbb{R}^n)) \geq 3 \cdot 2^{n-1} \tag{10.11}$$

following from Shvartsman's theorem [Shv-1987] shows. Since the Shvartsman proof requires an extension theorem which will be proved only in Section 10.5, we present below a similar result for the space $C^{1,\omega}(\mathbb{R}^n)$ whose proof is based on his argument, see Yu. Brudnyi and Shvartsman [BSh-2001b, sec. 4]. Specifically, we prove that

$$\mathcal{F}(C^{1,\omega}(\mathbb{R}^n)) \geq 3 \cdot 2^{n-1}. \tag{10.12}$$

Equivalently, the following is true.

Theorem 10.16. *There exists a compact set $S := S_\omega \subset \mathbb{R}^n$ and a function $F := F_\omega : S \to \mathbb{R}$ such that its restriction to every subset $T \subset S$ of cardinality $3 \cdot 2^{n-1} - 1$ extends to a function F_T satisfying*

$$\|F_T\|_{C^{1,\omega}(\mathbb{R}^n)} \leq 1$$

while F itself does not belong to $C^{1,\omega}(\mathbb{R}^n)|_S$.

Proof. We first describe the basic facts used in the proof and derive from them the result.

Claim I. There exists a sequence $\{S_j\}_{j\in\mathbb{Z}_+}$ of subsets in $\mathbb{R}^n$ such that

(a_1) $S_0 := \{0\}$ and $\operatorname{card} S_j = 3\cdot 2^{n-1}$ for $j \geq 1$.

(b_1) These subsets are mutually disjoint.

(c_1) $\lim_{j\to\infty} S_j = \{0\}$ (convergence in the Hausdorff metric).

From (c_1) it follows that the set

$$S := \bigcup_{j\in\mathbb{Z}_+} S_j$$

is compact.

Claim II. There exists a sequence of functions $f_j : S_j \to \mathbb{R}$, $j \geq 1$, such that:

(a_2) The trace of f_j to a *proper* nonempty subset of S_j, $j \geq 1$, admits an extension to a function from $C^{1,\omega}(\mathbb{R}^n)$ whose norm in this space is bounded by $c(n)$.

(b_2) The supports of these extensions for distinct j do not intersect.

(c_2) The trace norms of the functions f_j in $C^{1,\omega}(\mathbb{R}^n)|_{S_j}$ (denoted by $\|f_j\|_{S_j}$) satisfy

$$\sup_j \|f_j\|_{S_j} = \infty.$$

Now we define the required function $F : S \to \mathbb{R}$ by setting

$$F := f_j \text{ on } S_j,\ j \in \mathbb{N}, \text{ and } F(0) := 0.$$

Due to (b_1) F is well defined.

The function F does not belong to the trace space $C^{1,\omega}(\mathbb{R}^n)|_S$, since each of its extensions to $\mathbb{R}^n$, say $\tilde{F}$, satisfies the inequality

$$\|\tilde{F}\|_{C^{1,\omega}(\mathbb{R}^n)} \geq \sup_j \|f_j\|_{S_j} = \infty$$

by (c_2).
However, the trace of F to every subset $T \subset \mathbb{R}^n$ with $\operatorname{card} T \leq 3\cdot 2^{n-1} - 1$ admits an extension to a function from $C^{1,\omega}(\mathbb{R}^n)$, say F_T, such that

$$\|F_T\|_{C^{1,\omega}(\mathbb{R}^n)} \leq c_1(n).$$

Actually, let $T_j := S_j \cap T \neq \emptyset$ for some $j \geq 1$. Then T_j is a proper subset of S_j by (a_1). Therefore (a_2) implies that there exists an extension of $f_j|_{T_j}$ ($= F|_{T_j}$) to $\mathbb{R}^n$, say $\tilde{f}_j$, such that

$$\|\tilde{f}_j\|_{C^{1,\omega}(\mathbb{R}^n)} \leq c(n) \quad \text{and} \quad (\operatorname{supp} \tilde{f}_j) \cap (\operatorname{supp} \tilde{f}_{j'}) = \emptyset \text{ if } j \neq j'.$$

Setting now

$$F_T := \sum_{S_j \cap T \neq \emptyset} \tilde{f}_j$$

we obtain a function that coincides with F on T and satisfies the inequality

$$\|F_T\|_{C^{1,\omega}(\mathbb{R}^n)} \leq (3 \cdot 2^{n-1} - 1) \cdot c(n).$$

Hence, $C^{1,\omega}(\mathbb{R}^n)$ does not possess the Uniform Finiteness Property with $N := 3 \cdot 2^{n-1} - 1$, i.e., (10.12) is true.

Now we realize this program beginning with the sequence $\{S_j\}_{j \geq 1}$. Its definition is based on a certain inductive procedure determined by sequences of numbers $\{t_i\}_{i=1}^n$ and $\{r_i\}_{i=0}^n$ satisfying the conditions:

$$0 < t_{i+1} < t_i \text{ for } 1 \leq i \leq n-1 \text{ and } t_1 = r_0; \tag{10.13}$$

$$0 < 8r_i \leq r_{i+1} \leq 1 \text{ for } 0 \leq i \leq n-1. \tag{10.14}$$

Using these we first construct a sequence of finite sets $\{T_i\}_{i=1}^n$ whose final term T_n gives rise to S_j by an appropriate choice of the sequences $\{t_i\}$ and $\{r_i\}$. The inductive procedure giving the sets T_i is presented in

Lemma 10.17. *There exist points $x^{(i)}, y^{(i)}$ and sets T_i in $\mathbb{R}^n$, $1 \leq i \leq n$, such that the following is true.*

(a) *For every i,*

$$x^{(i)}, y^{(i)} \in T_i \subset \mathbb{R}^i := \{x \in \mathbb{R}^n \,;\, x_k = 0 \text{ for } i+1 \leq k \leq n\}. \tag{10.15}$$

Moreover, for $x \in T_i$,

$$0 \leq x_i \leq 2r_{i-1}. \tag{10.16}$$

(b) *The number of points in T_i satisfies*

$$\operatorname{card} T_i = 3 \cdot 2^{i-1}.$$

(c) *The size of T_i in the ℓ_∞-metric is estimated by*

$$\operatorname{diam} T_i \leq 2 \sum_{k=0}^{i-1} r_i. \tag{10.17}$$

Proof. At the first step we set

$$x^{(1)} = y^{(1)} := t_1 e_1 \quad \text{and} \quad T_1 := \{0, t_1 e_1, 2t_1 e_1\}, \tag{10.18}$$

where $\{e_i\}_{1 \leq i \leq n}$ is the standard basis of $\mathbb{R}^n$:

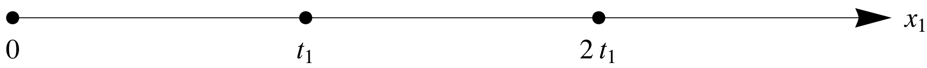

Figure 10.1: $T_1 := \{0, t_1e_1, 2t_1e_1\}$.

Clearly, (10.15)–(10.17) hold for $i = 1$. Then we proceed inductively assuming that the points and sets satisfying (10.15)–(10.17) have been defined for $i < n$. Then we set

$$y^{(i+1)} := x^{(i)} + t_{i+1}e_{i+1} \quad \text{and} \quad x^{(i+1)} := 2r_ie_1 - y^{(i+1)} \tag{10.19}$$

and define the set $\widetilde{T}_{i+1}$ by

$$\widetilde{T}_{i+1} := (T_i \setminus \{x^{(i)}\}) \cup \{y^{(i+1)}\}. \tag{10.20}$$

Using this we define the desired $(i + 1)$-th set, see Figures 10.2, 10.3 below, by

$$T_{i+1} := \widetilde{T}_{i+1} \cup \{2r_ie_1 - \widetilde{T}_{i+1}\}. \tag{10.21}$$

Now (10.15) holds for T_{i+1} by definition. Further, by (10.16) and (10.20) we have for $x \in \widetilde{T}_{i+1}$,

$$0 \le x_1 \le 2r_{i-1}$$

and for $x \in 2r_ie_1 - \widetilde{T}_{i+1}$,

$$2(r_i - r_{i-1}) \le x_1 \le 2r_i. \tag{10.22}$$

These and (10.21) imply (10.16) for $i + 1$.

Finally, by the induction hypothesis and (10.20), (10.21) we get

$$\operatorname{card} T_{i+1} = 2 \operatorname{card} T_i = 3 \cdot 2^i$$

and, moreover,

$$\begin{aligned}\operatorname{diam} T_{i+1} &\le 2r_i + 2 \operatorname{diam} \widetilde{T}_{i+1} = 2r_i + 2 \max\{\operatorname{diam} T_i, t_{i+1}\}\\ &\le 2r_i + 2\sum_{k=0}^{i-1} r_k = 2\sum_{k=0}^{i} r_k,\end{aligned}$$

as required in (b) and (c). □

Now we specify the sequences $\{t_i\}, \{r_i\}$ in this construction to define the desired sets S_j, $j \in \mathbb{N}$. Namely, we set

$$r_n^{(j)} := 4^{-j} \tag{10.23}$$

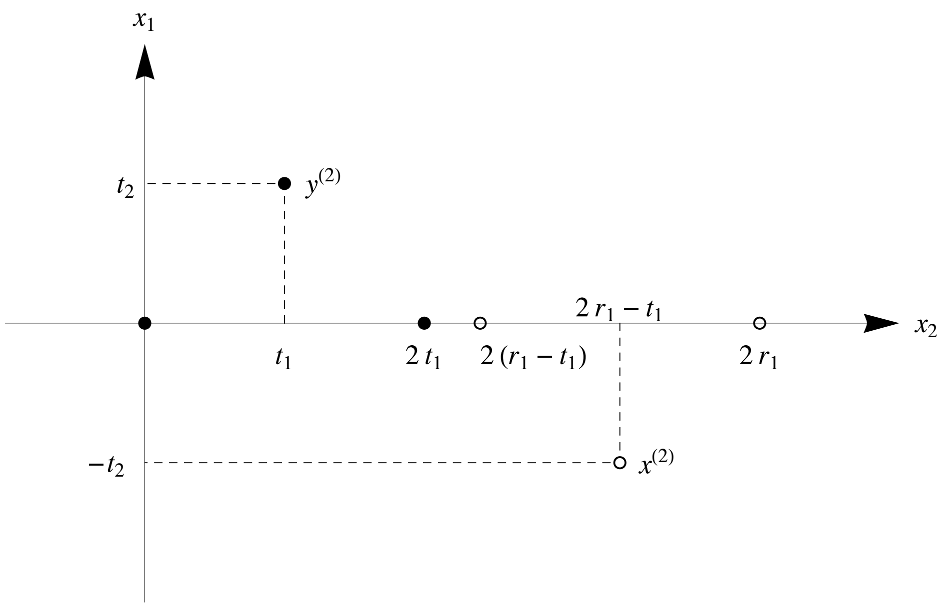

Figure 10.2: $\widetilde{T}_2$ is the set of black dots; $2r_1e_1 - \widetilde{T}_2$ is the set of circles.

and then define the remaining numbers $r_i^{(j)}$ as solutions of the equations

$$\omega(r) = \frac{\omega(4^{-j})}{2^{(j+3)(n-i)}}, \quad 0 \le i < n.$$

Since ω is continuous and nondecreasing and $\omega(0+) = 0$, the equation has a solution for every i. The sequence $\{t_i^{(j)}\}_{1\le i\le n}$ is then defined by

$$t_1^{(j)} := r_0^{(j)} \text{ and } t_i^{(j)} := r_0^{(j)} \cdot \frac{\omega(r_1^{(j)})}{\omega(r_i^{(j)})} \text{ for } i \ge 1. \tag{10.24}$$

Lemma 10.18. *The just introduced sequences satisfy conditions* (10.13) *and* (10.14).

Proof. We must only prove (10.14). Since ω is a 1-majorant, the function $t \mapsto \frac{\omega(t)}{t}$ is nonincreasing. Therefore

$$\frac{r_{i+1}^{(j)}}{r_i^{(j)}} \ge \frac{\omega(r_{i+1}^{(j)})}{\omega(r_i^{(j)})} := 2^{j+3} \ge 8,$$

as required. □

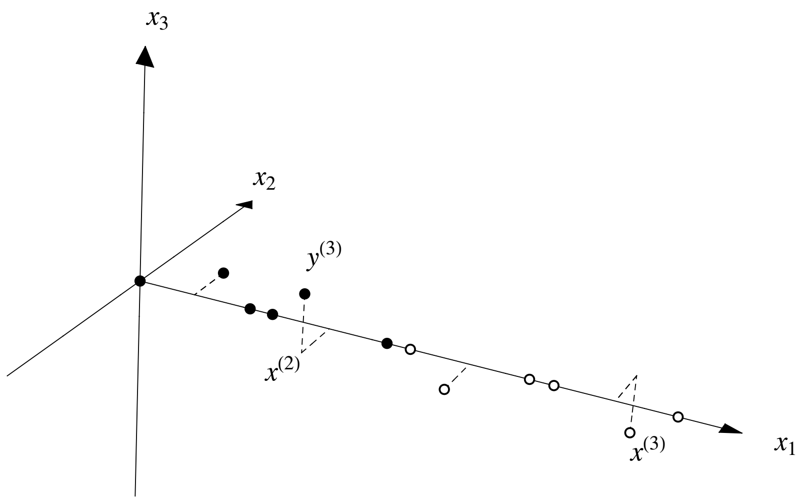

Figure 10.3: $\widetilde{T}_3$ is the set of black dots; $2r_2e_1 - \widetilde{T}_3$ is the set of circles.

Now we define the sets S_j by setting

$$S_j := T_n^{(j)} + r_n^{(j)} e_1 \text{ for } j \geq 1 \text{ and } S_0 := \{0\}. \tag{10.25}$$

Here the sets $T_i^{(j)}$ are given by Lemma 10.17 with sequences $\{t_i^{(j)}\}$ and $\{r_i^{(j)}\}$ in place of $\{t_i\}$ and $\{r_i\}$. The corresponding distinguished points of $T_i^{(j)}$ are denoted by $x^{(ij)}, y^{(ij)}$. For $i = n$ we omit here the first index so that $x^{(j)} := x^{(nj)}$, $y^{(j)} := y^{(nj)}$; due to (10.19) we have in this notation

$$y^{(j)} := x^{(n-1\,j-1)} + t_n^{(j)} e_1, \quad x^{(j)} = 2r_{n-1}^{(j-1)} e_1 - y^{(j)}. \tag{10.26}$$

The following result proves assertions (a_1)–(c_1) of Claim I.

Lemma 10.19. (a) *For every $j \in \mathbb{N}$ it is true that*

$$S_j \subset L_j := \{x \in \mathbb{R}^n \,;\, r_n^{(j)} \leq x_1 \leq 2r_n^{(j)}\}. \tag{10.27}$$

(b) *The sets S_j, $j \geq 0$, are mutually disjoint.*

(c) *The set $S := \bigcup_{j \in \mathbb{Z}_+} S_j$ is compact.*

(d) card $S_j = 3 \cdot 2^{n-1}$, $j \geq 1$.

Proof. Assertion (a) follows from (10.16) (with $i = n$ and $r_n := r_n^{(j)}$) and (10.25).

Further, $2r_n^{(j)} < r_n^{(j+1)}$, see (10.23), and therefore the layers L_j are mutually disjoint. This and (a) imply (b).

Next, by (10.17) and (10.14),

$$\operatorname{diam} S_j \le 2 \sum_{j=0}^{n-1} r_i^{(j)} \le 2 \left(\sum_{i=0}^{n-1} 8^i \right) r_n^{(j)} \to 0 \text{ as } j \to \infty.$$

This and (10.27) imply that 0 is a unique limit point of the set $\cup_{j\ge 1} S_j$. Therefore the bounded set $S := \{0\} \bigcup (\cup_{j\ge 1} S_j)$ is compact.

Finally assertion (d) follows from Lemma 10.17(b). □

At the next stage we should introduce the sequence of functions $f_j : S_j \to \mathbb{R}$, $j \in \mathbb{Z}_+$, satisfying Claim II. The main step in this direction is the next proposition dealing with the main objects of Lemma 10.17.

Proposition 10.20. *Given a point $y \in T_i$, $1 \le i \le n$, there exists a function $f_{i,y} : \mathbb{R}^n \to \mathbb{R}$ such that*

(a$_3$) *$f_{i,y} = \mu \delta_{y^{(i)}}$ on $T_i \backslash \{y\}$ where and hereafter*

$$\mu := \min_{1 \le i \le n} t_i \omega(r_{i-1});$$

in particular, $f_{i,y} = 0$ on this set if $y = y^{(i)}$.

(b$_3$) *The support of $f_{i,y}$ is contained in the layer*

$$\widehat{L}_i := \left\{ x \in \mathbb{R}^n \,;\, |x_1| \le \frac{1}{2} r_i \right\}.$$

(c$_3$) *For some constant $\gamma(n) > 0$,*

$$\|f\|_{C^{1,\omega}(\mathbb{R}^n)} \le \gamma(n).$$

Proof. We derive the result by induction on i. To this end we need

Lemma 10.21. *Given an index $i \in \{1, \dots, n\}$ and a point $y = (y_1, \dots, y_n) \in Q_r(0)$, where $0 < r \le 1$ and $y_i \ne 0$, there exists a real function $\phi := \phi_{i,r,y}$ on $\mathbb{R}^n$ such that:*

(a) $\phi(y) = 1$;

(b) *$\phi(-x) = -\phi(x)$ if $x \in Q_r(0)$;*

(c) *ϕ equals zero on the sets $\mathbb{R}^{i-1} \cap Q_r(0)$ and $Q_{2r}(0)^c := \mathbb{R}^n \setminus Q_{2r}(0)$;*

(d) *for some constant $c = c(n) > 0$,*

$$\|\phi\|_{C^{1,\omega}(\mathbb{R}^n)} \le \frac{c}{|y_i| \omega(r)}.$$

Proof. We first define a function $\widetilde{\phi}$ on the set $Q_r(0) \cup Q_{2r}(0)^c$ and then extend it to $\mathbb{R}^n$ to obtain the desired function ϕ. Namely, we set

$$\widetilde{\phi}(x) := \begin{cases} \psi(x_i), & \text{if } x \in Q_r(0), \\ 0, & \text{if } x \in Q_{2r}(0)^c, \end{cases}$$

where the function $\psi : \mathbb{R} \to \mathbb{R}$ is given by

$$\psi(t) := \begin{cases} \dfrac{t(2r-|t|)^2}{y_i(2r-|y_i|)^2}, & \text{if } |t| \le 2r, \\ 0 & \text{otherwise.} \end{cases}$$

A straightforward computation gives for the seminorm $|\psi|_{C^{1,\omega}([-r,r])}$ the upper bound

$$\sup_{0<t\le r} \frac{10t}{|y_i|(2r-|y_i|)^2\omega(t)} \le \frac{10r}{|y_i|r^2} \cdot \frac{r}{\omega(r)} = \frac{10}{|y_i|\omega(r)}.$$

Now let $\{\widetilde{\phi}_\alpha\}_{|\alpha|\le 1}$ be a 1-jet generated by the function $\widetilde{\phi}$, i.e., $\widetilde{\phi}_\alpha := D^\alpha \widetilde{\phi}$, $|\alpha| \le 1$. Then $\widetilde{\phi}_\alpha(x)$ equals 0 identically if $\alpha = e_j$ and $j \ne i$, and equals $\psi'(x_i)$ if $\alpha = e_i$. Due to the previous estimate this 1-jet satisfies the Taylor chain condition of the Whitney-Glaeser Theorem 2.19 of Volume I with $k = 1$ and the constant $\frac{10}{|y_i|\omega(r)}$. By this theorem $\widetilde{\phi}$ admits an extension to a function, say ϕ, from $C^{1,\omega}(\mathbb{R}^n)$ whose norm in this space is bounded by $\frac{c(n)}{|y_i|\omega(r)}$.

Hence, condition (d) holds for this ϕ while those of (a)–(c) are true for ϕ due to the definition of $\widetilde{\phi}$. □

Now we begin with the inductive derivation of Proposition 10.20. For $i = 1$ we have, see (10.18),

$$T_1 := \{0, t_1e_1, 2t_1e_1\}, \quad x^{(1)} = y^{(1)} = t_1e_1$$

and for $y := y^{(1)}$ we may simply set $f_{1,y} = 0$. Clearly, assertions (a_3)–(c_3) of Proposition 10.20 are valid for this $f_{1,y}$.

In the remaining case, $y \in \{0, 2t_1e_1\}$ and the required function is given at $x \in \mathbb{R}^n$ by

$$f_{1,y}(x) := \mu\phi_{1,r,\widetilde{y}}(x), \quad x \in \mathbb{R}^n,$$

where $r := \frac{1}{8}r_0$ and $\widetilde{y} := t_1e_1 - y$. Then by Lemma 10.21 (d),

$$\|f_{1,y}\|_{C^{1,\omega}(\mathbb{R}^n)} \le c(n)\,\frac{\mu}{|\widetilde{y}_1|\omega(r)} := c(n)\,\frac{\min\limits_{1\le i\le n} t_1\omega(r_{i-1})}{t_1\omega\big(\frac{1}{8}r_0\big)} \le c(n)\,\frac{t_1\omega(r_0)}{t_1\omega\big(\frac{1}{8}r_0\big)} \le 8c(n),$$

and assertion (c_3) holds.

Further, by the same lemma,

$$f_{1,y}(t_1y_1) := \mu\phi_{1,r,\tilde{y}}(\tilde{y}) = \mu$$

and, moreover,

$$f_{1,y}(0) = f_{1,y}(2t_1e_1) = 0.$$

The latter computation takes into account the vanishing of $f_{1,y}$ on the sets $\mathbb{R}^0 \cap Q_r(0) := \{0\}$ and $\widetilde{L}^c$ where $\widetilde{L} := \left\{x \in \mathbb{R}^n\,; |x_1 - y_1| \le 2r := \frac{r_0}{4}\right\}$ which do not contain $2t_1e_1 := 2r_0e_1$ (see Lemma 10.21 (c)).

Hence, assertion (a_3) is also true.

Finally, due to (10.14), for $x \in \widetilde{L}$,

$$|x_1| \le |x_1 - y_1| + |y_1| \le \frac{r_0}{4} + 2t_1 := \frac{r_0}{4} + 2r_0 \le \frac{r_0}{4} + \frac{r_1}{4} < \frac{r_1}{2}$$

and therefore

$$\operatorname{supp} f_{1,y} \subset \widetilde{L} \subset L_1 := \left\{x \in \mathbb{R}^n\,; |x_1| \le \frac{1}{2}\,r_1\right\},$$

as required by (b_3).

Now assume that the proposition holds for $i < n$. To show that it is true for $i+1$ we consider separately two cases:

$$y \in \widetilde{T}_{i+1} \quad \text{and} \quad y \in 2r_ie_1 - \widetilde{T}_{i+1},$$

where the waved set is defined by (10.21).

Lemma 10.22. *Proposition* 10.20 *holds for $y \in \widetilde{T}_{i+1}$. Moreover, the function $f_{i+1,y}$ for this y equals zero whenever $|x_1| > r_i$.*

Proof. For $y = y^{(i+1)}$ ($\in \widetilde{T}_{i+1}$) we simply set $f_{i+1,y} = 0$; then the assumptions of the proposition trivially hold.

Now let $y \in \widetilde{T}_{i+1}$ but $y \ne y^{(i+1)}$. By the induction hypothesis for the point $y^* := 2r_{i-1}e_1 - y$ there exists a function $\widetilde{f} := f_{i,y^*} \in C^{1,\omega}(\mathbb{R}^n)$ satisfying the conditions:

$$\|\widetilde{f}\|_{C^{1,\omega}(\mathbb{R}^n)} \le c(n); \tag{10.28}$$

$$\widetilde{f} = 0 \text{ on } T_i \setminus \{y^*, y^{(i)}\} \text{ and on } \left\{x \in \mathbb{R}^n\,;\, |x_1| > \frac{1}{2}r_i\right\}, \tag{10.29}$$

and, moreover,

$$\widetilde{f}(y^{(i)}) = \mu := \min_{1\le i\le n} t_i\omega(r_{i-1}). \tag{10.30}$$

We then define the required function $f_{i+1,y}$ by setting

$$f_{i+1,y}(x) = \widetilde{f}(2r_{i-1}e_1 - Pr_i(x)), \quad x \in \mathbb{R}^n, \tag{10.31}$$

where $Pr_i(x) := (x_1, \dots, x_i, 0, \dots, 0) \in \mathbb{R}^i$.

Then condition (c_3) for $f_{i+1,y}$ straightforwardly follows from (10.28).

We prove (a_3) and (b_3) for this function. Due to (10.19) we have

$$Pr_i(y^{(i+1)}) := Pr_i(x^{(i)} + t_{i+1}e_{i+1}) = x^{(i)}$$

and therefore (10.30) implies that

$$f_{i+1,y}(y^{(i+1)}) = \widetilde{f}(2r_{i-1}e_1 - x^{(i)}) = \widetilde{f}(y^{(i)}) = \mu.$$

On the other hand, (10.19), (10.20) and the equality $y^* := 2r_{i-1}e_1 - y$ imply

$$\widetilde{T}_{i+1} \setminus \{y^{(i+1)}, y\} = T_i \setminus \{x^{(i)}, y\} \quad \text{and} \quad 2r_{i-1}e_1 - (T_i \setminus \{x^{(i)}, y\}) = T_i \setminus \{y^{(i)}, y^*\}.$$

Hence, by (10.29) and (10.31), $f_{i+1,y}$ vanishes on the set $\widetilde{T}_{i+1} \setminus \{y^{(i+1)}, y\}$.

Moreover, $\widetilde{f}$ vanishes whenever $|x_1| > \frac{1}{2}r_i$ and therefore $f_{i+1,y}$ is zero on the set $\{x \in \mathbb{R}^n\,;\, |x_1| > r_{i-1} + \frac{1}{2}r_i\}$, see (10.31). Since $r_{i-1} \le \frac{1}{8}r_i$, we, in particular, get

$$f_{i+1,y}(x) = 0 \text{ if } |x_1| > r_i. \tag{10.32}$$

Further, the first coordinate of $x \in 2r_ie_1 - \widetilde{T}_{i+1}$ satisfies the last inequality by (10.22), hence, $f_{i+1,y}$ vanishes also on $2r_ie_1 - \widetilde{T}_{i+1}$ and therefore is zero on the set

$$T_{i+1} \setminus \{y^{(i+1)}, y\} := \Big(\widetilde{T}_{i+1} \cup (2r_ie_1 - \widetilde{T}_{i+1})\Big) \setminus \{y^{(i+1)}, y\}.$$

Combining these results we get $f_{i+1,y} = \mu \cdot \delta_{y^{(i+1)}}$ and, hence, prove (a_3). In turn, (10.32) implies (b_3). □

Lemma 10.23. *Proposition* 10.20 *holds for* $y \in 2r_ie_1 - \widetilde{T}_{i+1}$.

Proof. The required function $f_{i+1,y}$ is defined using two auxiliary functions; the first one is given by

$$\widetilde{\phi} := \mu\phi_{i+1,r,y^*}, \tag{10.33}$$

where we set $r := \frac{1}{8}\, r_{i+1}$ and $y^* := x^{(i+1)} - r_ie_1$.
The second function is defined by

$$\widetilde{f} := f_{i+1,\widetilde{y}}, \text{ where } \widetilde{y} := 2r_ie_1 - y; \tag{10.34}$$

since $\tilde{y}$ belongs to $\widetilde{T}_{i+1}$ by the assumption, $\tilde{f}$ has already been introduced in Lemma 10.22.

Now we set for $x \in \mathbb{R}^n$,

$$f_{i+1,y}(x) := \widetilde{f}(2r_ie_1 - x) + \widetilde{\phi}(r_ie_1 - x) \tag{10.35}$$

and prove (a_3)–(c_3) for the function so defined and the point $y \in 2r_ie_1 - \widetilde{T}_{i+1}$.

By virtue of Lemma 10.22, $\widetilde{f}$ satisfies the conditions:

$$\|\widetilde{f}\|_{C^{1,\omega}(\mathbb{R}^n)} \le c_1(n); \tag{10.36}$$

$$\widetilde{f}(y^{(i+1)}) = \mu; \tag{10.37}$$

$$\widetilde{f}(x) = 0 \text{ if } x \in T_{i+1} \setminus \{y^{(i+1)}, \widetilde{y}\} \text{ and if } |x_1| \ge r_i. \tag{10.38}$$

Since $2r_ie_1 - y^{(i+1)}$ belongs to the first set in (10.38) and, moreover,

$$\widetilde{\phi}(y^{(i+1)}) = \mu\phi_{i+1,r,y^*}(y^*) = \mu$$

by (10.33) and Lemma 10.21, we get

$$f_{i+1,y}(y^{(i+1)}) = \widetilde{\phi}(y^{(i+1)}) = \mu.$$

Hence, to complete the proof of assertion (a_3) for the function $f_{i+1,y}$, we should show that it vanishes on the set $T_{i+1} \setminus \{y^{(i+1)}, y\}$.

We first check this for $x \in T_{i+1} \setminus \big\{y^{(i+1)}, x^{(i+1)}, y\big\}$. In this case, we get from (10.16) and (10.19)–(10.21),

$$|x_1 - r_i| \le r_i \quad \text{and} \quad |x_j| \le t_j < r_j, \; j > 1,$$

i.e., $x \in Q_{r_i}(re_1)$.

Moreover, by the definition of T_{i+1}, we also have $T_{i+1} \setminus \big\{y^{(i+1)}, x^{(i+1)}\big\} \subset \mathbb{R}^i$. Hence, $x - r_ie_1 \in \mathbb{R}^i \cap Q_{r_i}(0) \subset \mathbb{R}^i \cap Q_{\frac{1}{8}r_{i+1}}(0)$, and by the choice of $\widetilde{\phi}$, see (10.33), and Lemma 10.21,

$$\widetilde{\phi}(-x + r_ie_1) = -\widetilde{\phi}(x - r_ie_1) = 0.$$

Further, by (10.34) and Lemma 10.22, $\tilde{f} = 0$ on $T_{i+1} \setminus \{x^{(i+1)}, \tilde{y}\}$, and therefore $\tilde{f}(2r_ie_1 - x) = 0$, since $2r_ie_1 - x$ belongs to $T_{i+1} \setminus \{x^{(i+1)}, 2r_ie_1 - y\} := T_{i+1} \setminus \{x_{i+1}, \tilde{y}\}$.

Combining these facts with (10.35) we get for $x \in T_{i+1} \setminus \{x^{(i+1)}, y^{(i+1)}, y\}$,

$$f_{i+1,y}(x) = \widetilde{f}(2r_ie_1 - x) + \widetilde{\phi}(r_ie_1 - x) = 0.$$

The same is given for $x = x^{(i+1)}$ by the next evaluation:

$$f_{i+1,y}(x^{(i+1)}) = \widetilde{f}(y^{(i+1)}) + \widetilde{\phi}(r_ie_1 - x^{(i+1)}) = \mu + \mu\phi_{i+1,r,y^*}(-y^*) = 0,$$

see (10.35), (10.37) and Lemma 10.21(b). Hence, condition (a_3) holds.

Next, condition (b_3) straightforwardly follows from (10.33), (10.35), Lemma 10.21(c) and equality $\widetilde{f}(x) = 0$ if $|x_1| \ge r_i$.

Finally, due to (10.33), (10.36) and Lemma 10.21(d),

$$\begin{aligned} &\|f_{i+1,y}\|_{C^{1,\omega}(\mathbb{R}^n)} \le \|\widetilde{f}\|_{C^{1,\omega}(\mathbb{R}^n)} + \mu\|\phi_{i+1,r,y^*}\|_{C^{1,\omega}(\mathbb{R}^n)} \\ &\le c_1(n) + c(n)\frac{t_{i+1}\omega(r_i)}{t_{i+1}\omega(\frac{1}{8}r_{i+1})} \le c_1(n) + c(n)\frac{t_{i+1}\omega\big(\frac{1}{8}\, r_{i+1}\big)}{t_{i+1}\omega\big(\frac{1}{8}\, r_{i+1}\big)} = c_1(n) + c(n). \end{aligned}$$

This proves assertion (c_3) and the lemma. □

Since $T_{i+1} := \widetilde{T}_{i+1} \bigsqcup \left(2r_i e_1 - \widetilde{T}_{i+1}\right)$, Lemmas 10.22 and 10.23 imply Proposition 10.20. □

At the final step, we introduce the required functions $f_j : S_j \to \mathbb{R}$, $j \geq 1$, by setting

$$f_j(x) := \begin{cases} \min\limits_{1\le i\le n} t_i^{(j)} \omega\left(r_{i-1}^{(j)}\right), & \text{if } x = y^{(j)} + r_n^{(j)} e_1, \\ 0 & \text{otherwise.} \end{cases} \tag{10.39}$$

Let us recall that $S_j := T_n^{(j)} + r_n^{(j)} e_1$ and $y^{(j)}$ are given by (10.25) and (10.26), respectively.

Proposition 10.24. *The family* $\{f_j\}_{j\geq 1}$ *satisfies conditions* (a_2), (b_2) *of Claim* II.

Proof. Let T be a nonempty proper subset of S_j. To prove (a_2) we should find an appropriate $C^{1,\omega}$ extension of the trace $f_j|_T$. We use for this aim the function $f_{n,y}$ of Proposition 10.20 defined for the set $T_n^{(j)}$ and the sequences $\{t_i^{(j)}\}_{1\le i\le n}$ and $\{r_i^{(j)}\}_{0\le i\le n}$. Specifically, we set for $x \in \mathbb{R}^n$ and $y \in S_j \setminus T$,

$$f_{T,j}(x) := f_{n,y}(x - r_j^{(n)}).$$

Due to Proposition 10.20 and (10.29),

$$f_{T,j}(x) = f_j(x) \text{ for } x \in S_j \setminus \{y\},$$

i.e., $f_{T,j}$ is an extension of $f_j|_T$ to $\mathbb{R}^n$. By the same proposition the $C^{1,\omega}$-norm of this extension is bounded by $c(n)$, and, moreover,

$$\operatorname{supp} f_{T,j} \subset L_j := \{x \in \mathbb{R}^n\,;\, |x - r_n^{(j)}| \le \frac{1}{2} r_n^{(j)}\}.$$

Since $r_n^{(j)} := 4^{-j}$, the family $\{L_j\}_{j\geq 1}$ consists of mutually disjoint sets and therefore $\operatorname{supp} f_{T,j}$ for distinct j do not intersect. This proves assertion (a_2).

To prove (b_2) we need the next fact concerning the sets T_k and points $y^{(k)}$, see (10.21) and (10.19).

Lemma 10.25. *If a function* $f \in C^{1,\omega}(\mathbb{R}^n)$ *vanishes on the set* $T_k \setminus \{y^{(k)}\}$, $1 \le k \le n$, *then for some* $c = c(k) > 0$,

$$|f(y^{(k)})| \le c\, |f|_{C^{1,\omega}(\mathbb{R}^n)} \sum_{i=1}^{k} t_i \omega(r_{i-1}). \tag{10.40}$$

Proof. We prove the result by induction on k. Since $T_1 := \{0, t_1e_1, 2t_1e_1\}$ and $y^{(1)} := t_1e_1$, we have

$$|f(y^{(1)})| \leq \int_0^{t_1} |(D^1f)((t+t_1)e_1) - (D^1f)(te_1)|\,dt$$
$$\leq t_1\omega(t_1) \sup_{x\neq y} \max_{|\alpha|=1} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\omega(\|x-y\|)} := t_1\omega(r_0)|f|_{C^{1,\omega}(\mathbb{R}^n)}.$$

This proves (10.40) for $k = 1$.

Now let (10.40) hold for $k < n$ and let $f \in C^{1,\omega}(\mathbb{R}^n)$ be such that $f = 0$ on $T_{k+1} \setminus \{y^{(k+1)}\}$. We define a function $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(x) := f(2r_{k-1}e_1 - x).$$

Due to (10.20) and (10.21) $2r_{k-1}e_1 - (T_k \setminus \{y^{(k)}\}) \subset T_{k+1} \setminus \{y^{(k+1)}\}$ and so g vanishes on $T_k \setminus \{y^{(k)}\}$. Then by the induction hypothesis

$$|f(x^{(k)})| =: |g(y^{(k)})| \leq c(k)|g|_{C^{1,\omega}(\mathbb{R}^n)} \sum_{i=1}^{k} t_i\omega(r_{i-1}) = c(k)|f|_{C^{1,\omega}(\mathbb{R}^n)} \sum_{i=1}^{k} t_i\omega(r_{i-1}).$$

Now we define a function $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x) := f(2(r_k - r_{k-1})e_1 + x).$$

By a similar argument h vanishes on $T_k \setminus \{y^{(k)}\}$ and therefore (10.20) along with the induction hypothesis imply that

$$|f(2r_ke_1 - x^{(k)})| =: |h(y^{(k)})| \leq c(k)|f|_{C^{1,\omega}(\mathbb{R}^n)} \sum_{i=1}^{k} t_i\omega(r_{i-1}).$$

Now we present $f(y^{(k+1)})$ as

$$f(y^{(k+1)}) = \Big(\big[f(y^{(k+1)}) - f(x^{(k)})\big] - \big[f(2r_ke_1 - x^{(k)}) - f(x^{(k+1)})\big]\Big)$$
$$+ f(x^{(k+1)}) + f(x^{(k)}) + f(2r_ke_1 - x^{(k)}).$$

The last two terms have been already estimated while the first one is bounded by

$$\int_0^{t_{k+1}} |(D^1f)(x^{(k)} + te_1) - (D^1f)(x^{(k+1)} + te_1)|\,dt$$
$$\leq |f|_{C^{1,\omega}(\mathbb{R}^n)} \cdot t_{k+1}\omega(\|x^{(k)} - x^{(k+1)}\|).$$

Here we use equalities (10.19):

$$y^{(k+1)} = x^{(k)} + t_{k+1}e_1, \quad x^{(k+1)} + t_{k+1}e_1 = 2r_ke_1 - x^{(k)}.$$

Since $f(x^{(k+1)}) = 0$ for $x^{(k+1)} \in T_{k+1} \setminus \{y^{(k+1)}\}$, this and the previous inequalities yield

$$|f(y^{(k+1)})| \le |f|_{C^{1,\omega}(\mathbb{R}^n)} \left(t_{k+1}\omega(\|x^{(k)} - x^{(k+1)}\|) + 2c(k) \sum_{i=1}^{k} t_i \omega(r_{i-1}) \right).$$

It remains to note that by (10.19) $\|x^{(k)}\| \le 2r_{k-1}$, hence,

$$\omega(\|x^{(k)} - x^{(k+1)}\|) \le 4\omega(r_k).$$

This proves (10.37) for $k+1$ and therefore completes the proof of the lemma. $\square$

Now let $f \in C^{1,\omega}(\mathbb{R}^n)$ be an almost optimal extension of f_j, that is, for a fixed $0 < q < 1$,

$$\|f_j\|_{S_j} := \inf\{\|g\|_X \, ; \, g|_{S_j} = f_j\} \ge q\|f\|_{C^{1,\omega}(\mathbb{R}^n)}.$$

By the definition of f_j, see (10.39), the function $f(\cdot + r_n^{(j)} e_1)$ vanishes on the set $T_n^{(j)} \setminus \{y_n^{(j)}\}$ and equals $\min_{1\le i\le n} t_i \omega(r_{i-1})$ at $y_n^{(j)}$. Then (10.40) yields

$$\|f\|_{C^{1,\omega}(\mathbb{R}^n)} \ge |f|_{C^{1,\omega}(\mathbb{R}^n)} \ge \frac{1}{c(n)} \frac{|f(y_n^{(j)})|}{\sum_{i=1}^n t_i^{(j)} \omega(r_{i-1}^{(j)})} = \frac{1}{c(n)} \frac{\min_{1\le i\le n} t_{i-1}^{(j)} \omega(r_{i-1}^{(j)})}{\sum_{i=1}^n t_i^{(j)} \omega(r_{i-1}^{(j)})}.$$

By the choice of the sequences $\{r_i^{(j)}\}$ and $\{t_i^{(j)}\}$, see (10.23) and (10.24), we then have

$$\min_{1\le i\le n} t_i^{(j)} \omega(r_{i-1}^{(j)}) = t_0^{(j)} \omega(r_0^{(j)}), \quad \sum_{i=1}^{n} t_i^{(j)} \omega(r_{i-1}^{(j)}) = 2^{-j-3} n t_1^{(j)} \omega(r_0^{(j)}).$$

Together with the previous inequalities this leads to the required estimate

$$\|f_j\|_{S_j} \ge \frac{q}{c(n)} \frac{2^{j+3}}{n}.$$

Hence,

$$\sup_{j\ge 1} \|f_j\|_{S_j} = \infty$$

and condition (a_2) of Claim II is also true. $\square$

Since the assertion of Theorem 10.16 has already been derived from Claims I, II, Proposition 10.20 and Lemma 10.19 complete the proof of this theorem. $\square$

10.1.5 The trace problem for finite sets

Assuming the Uniform Finiteness Property to be fulfilled for the space $X := C^\ell\Lambda^{k,\omega}(\mathbb{R}^n)$, $k \geq 1$, we may characterize its traces to a closed subset $S \subset \mathbb{R}^n$ by the following (conditional)

Criterion 10.26. *A continuous function $g : S \to \mathbb{R}$ belongs to the trace space $X|_S$ if and only if for $N := \mathcal{F}(X)$,*

$$\sup\big\{\|g\|_F\,; F \subset S \quad \textit{and} \quad \operatorname{card} S \leq N\big\} < \infty. \tag{10.41}$$

Moreover, the supremum is equivalent to the trace norm of g in $X|_S$ with constants of equivalence depending only on n, k, ℓ.

As above, by $\|g\|_F$ we denote the trace norm of $g|_F$ in $X|_F$. Moreover, g_F stands in the sequel for the optimal extension of $g|_F$, i.e., $\|g_F\|_X = \|g\|_F$.

The analogous description of the space $C_b^\ell(\mathbb{R}^n)$ is more subtle, since in this case the trace of the family $\{g_F\}$ to every closed cube Q should be not only uniformly bounded but also precompact in $C_b^\ell(Q)$, see, e.g., Whitney's extension Theorem 2.47 of Volume I for $n = 1$. Therefore the desired conventional criterion for this space may be formulated either by using the family of extensions $\{g_F\}$ or by exploiting the compactness criterion for $C_b^\ell(\mathbb{R}^n)$. It asserts, cf. Proposition 10.54 below, that a bounded set $H \subset C_b^\ell(\mathbb{R}^k)$ is precompact if and only if its trace to every closed cube Q is contained in $C^{\ell,\omega}(Q)$ for some 1-majorant $\omega = \omega(Q)$. This allows us to derive the conditional finiteness criterion and the Finiteness Property for $C_b^\ell(\mathbb{R}^n)$ from that for $C^{\ell,\omega}(\mathbb{R}^n)$, see Theorem 10.102.

The conditional criterion for the jet-space $J^\ell\Lambda^{k.\omega}(\mathbb{R}^n)$ with $k \geq 1$ may be formulated similarly (the case of $k = 1$ is already covered by the unconditional criterion of Theorem 2.13 of Volume I).

The above discussion explains importance of the next problem.

Problem 10.27. *Given a finite set $F \subset \mathbb{R}^n$ and a function $g : F \to X$, find a function $\psi_F(\cdot; X) : \ell_\infty(F) \to \mathbb{R}$ such that*

$$\|g\|_F \approx \psi_F(g\,; X)$$

with the constants of equivalence independent of g and F.

Several results presented below demonstrate the difficulties encountered in its solution. We begin with two relatively simple cases.

Example 10.28. (a) Let $X := \Lambda^{1,\omega}(\mathbb{R}^n)$, so that $\mathcal{F}(X) = 2$. Extending $g : \{x, y\} \to \mathbb{R}$ by an interpolating at $\{x, y\}$ affine function multiplied by a suitable C^1 test-function, we easily get

$$\psi_{\{x,y\}}(g\,; X) = |g(x)| + \frac{|g(x) - g(y)|}{\omega(\|x - y\|)}\,.$$

(b) Let $X := J^{\ell,\omega}(\mathbb{R}^n)$. Using the geometric approach that will be presented in full generality in Section 10.5 one may show that $\mathcal{F}(X) = 2$. Then the solution to Problem 10.27 would imply the Whitney–Glaeser characterization of the trace space $J^{\ell,\omega}(\mathbb{R}^n)\big|_S$ given by Theorem 2.19 of Volume I. The solution may be reached by the following simple argument.

Let $F := \{u, v\} \subset \mathbb{R}^n$ and let $\vec{g} := (g_\alpha)_{|\alpha|\le\ell}$ be an ℓ-jet defined on F. To estimate the trace norm $\|\vec{g}\|_F$, we use the $C^{\ell,\omega}$ function $g : \mathbb{R}^n \to \mathbb{R}$ given by

$$g(x) := \varphi(x) \sum_{|\alpha|\le\ell} \frac{g_\alpha(u)}{\alpha!}(x-u)^\alpha + \big(1-\varphi(x)\big) \cdot \sum_{|\alpha|\le\ell} \frac{g_\alpha(v)}{\alpha!}(x-v)^\alpha,$$

where $\varphi : \mathbb{R}^n \to [0,1]$ is a C^∞ function equivalent to the distance function $x \mapsto d(x\,; F) := \min\big(\|x-u\|_2,\ \|x-v\|_2\big)$. E.g., we may set $\varphi(x) := \|x-v\|_2 \cdot h\Big(\frac{\|x-u\|_2}{\|x-v\|_2}\Big)$, where $h : \mathbb{R}_+ \to [0,1]$ is a univariate C^∞ function that agrees with the function $t \mapsto \min(1,t)$, $t \ge 0$, outside $\big(\frac{1}{2}, 1\big)$.

A straightforward computation shows that $\vec{g} = \big(D^\alpha g|_F\big)_{|\alpha|\le\ell}$ and the $C^{\ell,\omega}$ norm of g is bounded by

$$W(\vec{g}; u, v) := c(\ell, n) \max_{|\alpha|\le\ell} \big(|g_\alpha(u)| + |g_\alpha(v)| + |g_\alpha(u,v)| + |g_\alpha(v,u)|\big),$$

where we set

$$g_\alpha(u,v) := \Bigg(g_\alpha(u) - \sum_{|\beta|\le\ell-|\alpha|} \frac{g_\beta(u)}{\beta!}(v-u)^\beta\Bigg) \Bigg/ \|u-v\|_2^{\ell-|\alpha|}\omega\big(\|u-v\|_2\big).$$

Hence, the continuous ℓ-jet $\vec{f} : S \to \mathbb{R}^n$ belongs to the trace space $C^{\ell,\omega}(\mathbb{R}^n)|_S$ (identified with $J^{\ell,\omega}(\mathbb{R}^n)$) if

$$\sup_{\{u,v\}\subset S} W(\vec{f}|_{\{u,v\}}; u, v) < \infty$$

and this supremum bounds the trace norm of $\vec{f}$.

Since the converse assertion is evident, we obtain a rather simple proof of the Whitney–Glaeser theorem. However, the original proof contains an important additional information asserting that the required extension is attained by a *linear continuous operator*.

Now we present a much more involved result concerning the homogeneous space $\dot{C}^{1,\omega}(\mathbb{R}^n)$ for $n = 1, 2$. As it has already been mentioned and will be proved in Theorem 10.69,

$$\mathcal{F}\big(\dot{C}^{1,\omega}(\mathbb{R}^n)\big) = 3 \cdot 2^{n-1}. \tag{10.42}$$

For $n = 1$ the constant equals 3 and the corresponding function ψ_F is presented in Theorem 2.52 of Volume I by the formula

$$\psi_F\big(g\,;\dot{C}^{1,\omega}(\mathbb{R})\big) = \frac{|f[F]|}{\rho_1(F)},$$

where $g[F]$ is the divided difference of g over the 3-point set F and

$$\rho_1(F) := \frac{\omega(\operatorname{diam} F)}{\operatorname{diam} F}\,.$$

For $n = 2$ the set F is of cardinality 6, and the function ψ_F hardly depends on the combinatorial structure of F. To introduce it, we define an analog of a divided difference for a three-point set (triangle) $\Delta \subset \mathbb{R}^2$ denoted, as before, by $f[\Delta]$. Specifically, if Δ is contained in a straight line L, we define $f[\Delta]$ to be equal to the divided difference of the univariate function $f|_L$ over the node set Δ .

Otherwise, we use a uniquely defined affine polynomial $P_\Delta(f)$ interpolating f at the vertices of a nondegenerate triangle Δ and then set

$$f[\Delta] := \nabla P_\Delta(f)\ (\in \mathbb{R}^2).$$

Now the function ψ_F for a six-point set F is the maximum of two functions, ψ_F^1 and ψ_F^2, where the former is given by

$$\psi_F^1\big(f\,;\dot{C}^{1,\omega}(\mathbb{R}^2)\big) = \max_{\Delta\subset\Sigma} \frac{|f[\Delta]|}{\rho_1(\Delta)}\,; \tag{10.43}$$

here Δ runs over the set of all degenerate triangles of F (the maximum here and below equals zero being taken over the empty set).

The second function is given by

$$\psi_F^2\big(f\,;\dot{C}^{1,\omega}(\mathbb{R}^2)\big) := \max_{\Delta,\Delta'} \frac{\big\|f[\Delta]-f[\Delta']\big\|_2}{\rho_2(\Delta,\Delta')}, \tag{10.44}$$

where (Δ, Δ') runs over the set of all pairs of nondegenerate triangles in F and ρ_2 is defined by

$$\rho_2(\Delta,\Delta') := \frac{\omega(\operatorname{diam}\Delta)}{|\sin\theta_\Delta|} + \frac{\omega(\operatorname{diam}\Delta')}{|\sin\theta_{\Delta'}|} + \omega\big(\operatorname{diam}(\Delta\cup\Delta')\big).$$

Here θ_Δ is the maximal angle of the triangle Δ.

Now we set

$$\psi_F := \max\{\psi_F^1, \psi_F^2\}$$

and formulate the following result due to Shvartsman [Shv-2002].

Theorem 10.29. *The function f defined on a closed subset $S \subset \mathbb{R}^2$ belongs to the trace space $\dot{C}^{1,\omega}(\mathbb{R}^2)\big|_S$ if and only if*

$$\sup\Big\{\psi_F\big(f\,;\dot{C}^{1,\omega}(\mathbb{R}^2)\big)\,; F \subset S \quad \text{and} \quad \operatorname{card} F \le 6\Big\} < \infty.$$

Moreover, this supremum is equivalent to the trace norm of f with the constants of equivalence independent of f and S.

The miraculous looking formula for ψ_F is, in fact, a by-product of the proof for (10.42). Likewise, we may handle the cases of $n = 3, 4, \dots$. Specifically, using induction on n we first present $\psi_F\big(\,\cdot\,; \dot{C}^{1,\omega}(\mathbb{R}^n)\big)$ as

$$\psi_F = \max\{\psi_F^1, \dots, \psi_F^n\},$$

where ψ_F^k for $k < n$ is determined by the set of all k-*degenerate* subsets of F; a subset is k-degenerate if it is contained in a k-dimensional affine subspace of $\mathbb{R}^n$ and its cardinality is more than $k+1$.

For instance, if $n = 3$, then the finiteness number is 12 and the functions ψ_F^1 and ψ_F^2 are evaluated by (10.43) and (10.44), respectively. Removing from F the k-degenerate parts with $k = 1, 2$, we then obtain a subset $F' \subset F$ with the following property:

Every triple $\Delta := \{x^1, x^2, x^3\} \subset F'$ uniquely determines a two-dimensional affine subspace of $\mathbb{R}^3$ denoted by $L(\Delta)$ that does not contain other points of F'. Further, every such triple determines a unique affine polynomial P_Δ in two variables interpolating the trace $f|_{L(\Delta)}$ at the points of Δ. Finally, for every pair $\Delta \ne \Delta'$ of such triples we denote by $\theta_{\Delta,\Delta'}$ the smallest angle between the subspaces $L(\Delta)$ and $L(\Delta')$. Using this data we define the desired function ψ_F^3 as in (10.44) and then introduce the required function $\psi_F\big(\,\cdot\,; \dot{C}^{1,\omega}(\mathbb{R}^3)\big)$.

In view of the discussion presented, a constructive solution to Problem 10.27 seems to be unattainable even for the space $C^{k,\omega}(\mathbb{R}^n)$ with small $k, n \ge 2$.

The following version of the problem might be more comprehensible.

Problem 10.30. *Given the finiteness constants for $C^{k,\omega}(\mathbb{R}^n)$ and the corresponding function ψ_F, find effective upper bounds of the analogous quantities for $C^{k+1,\omega}(\mathbb{R}^n)$ and $C^{k,\omega}(\mathbb{R}^{n+1})$.*

The starting point here is, of course, the space $C^{2,\omega}(\mathbb{R}^2)$. The result discussed later gives the number 64 as an upper bound for $\mathcal{F}\big(C^{2,\omega}(\mathbb{R}^2)\big)$ (conjecturally, it equals 24). Moreover, the basic configurations are now not triangles but the minimal interpolation sets $\Sigma(3,2)$ of cardinality 7, see Remark 10.6 (a) for their definition, that determine interpolation polynomials of degree 2 involved in the final step. This makes the corresponding formula for $\psi_F\big(C^{2,\omega}(\mathbb{R}^2)\big)$ essentially more complicated.

10.1.6 Linear extension problems

To formulate the main problem we, as in Chapter 7, denote by $\mathrm{Ext}(S\,;X)$ the Banach space of all linear extension operators from $X|_S$ into X and then set

$$\lambda(S\,;X) := \big\{\|T\|\,;T \in \mathrm{Ext}(S\,;X)\big\}. \tag{10.45}$$

By stipulation, this is infinity if $\mathrm{Ext}(S\,;X) = \phi$.

Problem 10.31. *Prove that*

$$\lambda(X) := \sup\big\{\lambda(S\,;X)\,;\ S \neq \phi\big\}$$

is finite.

Here X stands for one of the smoothness spaces considered in this section.

Positive solutions to this problem for the function spaces $C_b^\ell(\mathbb{R})$ and $C^{\ell,\omega}(\mathbb{R})$ as well as for the spaces of ℓ-jets $J_b^\ell(\mathbb{R}^n)$ and $J^{\ell,\omega}(\mathbb{R}^n)$ have already been presented in Chapter 2 of Volume I. The situation for the spaces of multivariate functions or ℓ-jets over $\Lambda^{k,\omega}(\mathbb{R}^n)$ with $k \geq 2$ is much more complicated. Although several recent results (with very involved proofs) that will be discussed later give quite a bit of evidence of solvability of the problem, the general approach still has not been found. The difficulty of their proofs may be explained by closeness of their assertions to the borderline separating true from wrong results.

The next two facts illustrate the last statement. Let $C_u^1(\mathbb{R}^n)$ be the space of bounded functions with uniformly continuous derivatives. Then the trace space $C_u^1(\mathbb{R}^n)\big|_S$ admits a simultaneous extension to $C_u^1(\mathbb{R}^n)$ if S is compact and, in general, does not admit otherwise. The first result is a direct consequence of the Fefferman Theorem 10.104 [F-2007a] that will be discussed in Section 10.4, while the second one presented now is due to Yu. Brudnyi and Shvartsman [BSh-1999, Sec. 9].

Theorem 10.32. *There exists a non-empty closed subset $S_0 \subset \mathbb{R}^2$ such that*

$$\mathrm{Ext}\big(S_0\,;C_u^1(\mathbb{R}^2)\big) = \emptyset.$$

Proof. The desired set S_0 is given by

$$S_0 := \bigcup_{i\in\mathbb{Z}_+} \Delta_i := \bigcup_{i\in\mathbb{Z}_+} \{x^{(i)}, y^{(i)}, z^{(i)}\}$$

where the points are given by

$$x^{(i)} := (2i, 0),\ y^{(i)} := (2i+1, 0),\ z^{(i)} := (2i+1, w_i)$$

and the weight $w := (w_i)_{i\in\mathbb{Z}_+}$ satisfies $0 < w \leq 1$.

We denote by $w^{\lim}$ the set of limit points for the set $\{w_i\}$, and assume that the following holds:

(*A*) $\qquad$ 0 is a nonisolated point of $w^{\lim}$.

E.g., $\{w_i\}$ may be chosen to be dense in $(0,1)$.

Let the result not hold under this choice of w. Then according to Corollary 10.2 the condition

$$(B) \qquad \operatorname{Ext}\big(S_0\,;C_u^1(\mathbb{R}^2)\big) \neq \emptyset$$

holds if and only if the null-space $\mathcal{N} := \big\{f \in C_u^1(\mathbb{R}^2)\,; f\big|_{S_0} = 0\big\}$ is *complemented* in $C_u^1(\mathbb{R}^2)$. To get a contradiction it will be more preferable to deal instead with a closed subspace of the initial trace space given by

$$\mathcal{L} := \big\{f \in C_u^1(\mathbb{R}^n)\big|_{S_0}\,; f(x^{(i)}) = f(y^{(i)}) = 0, \quad i \in \mathbb{Z}_+\big\}$$

and with the related complementedness problem.

Specifically, using a linear map that transforms $\mathcal{L}$ into a sequence space, we will show that condition (A) implies existence in $\mathcal{L}$ of an embedded pair $X_0 \subset X_1$ of closed subspaces linearly isomorphic to the pair $c_0 \subset \ell_\infty$. In turn, condition (B) allows us to construct a linear continuous operator $T : X_1 \to X_0$ whose restriction to X_0 is invertible. Then $\big(T\big|_{X_0}\big)^{-1}T$ would be a linear continuous projection of X_1 onto X_0 and therefore c_0 is complemented in ℓ_∞. The latter contradicts the Phillips theorem [Ph-1940], and therefore (B) is not true, as required in Theorem 10.32.

The main tool in fulfilling this program is a variant of the Whitney–Glaeser Theorem 2.19 of Volume I formulated for this case. Since the space of bounded functions whose derivatives are uniformly continuous on $\mathbb{R}^n$ is the union of the spaces $C^{1,\omega}(\mathbb{R}^n)$ where ω runs over the set of all 1-majorants, the required result may be formulated as follows.

Proposition 10.33. *A continuous* 1*-jet* $(f,\vec{g}) := (f, g_1, g_2) : S \to \mathbb{R}^3$ *belongs to* $C_u^1(\mathbb{R}^2)\big|_S$ *if and only if, for some* $\lambda > 0$, *the following conditions hold:*

(a)
$$\|\vec{g}\|_{\ell_\infty(S)} := \sup_S \max\big\{|g_1(x)|, |g_2(x)|\big\} \leq \lambda.$$

(b) *The Taylor remainder*
$$R(f\,;x,y) := f(y) - f(x) - \langle \vec{g}(x), y - x\rangle$$
satisfies for $x, y \in S$,
$$\big|R(f\,;x,y)\big| \leq \lambda\|x-y\|.$$

(c) *If* $\|x-y\| \to 0$, *then uniformly in* $x, y \in S$,
$$R(f\,;x,y) = o\big(\|x-y\|\big).$$

(d) *The trace norm of* $(f,\vec{g})$ *in* $J_b^1(\mathbb{R}^2)\big|_S$ *is equivalent with some numerical constants to* $\inf\lambda$.

Hereafter $\|x\|$ stands for $\max_{1\le i\le n} |x_i|$.

Now we establish a relation between the subspaces of the null-space $\mathcal{L}$ and those of the sequence space $\Sigma(w) := c_0 + \ell_\infty(w)$ equipped with the norm

$$\|s\|_{\Sigma(w)} := \inf_{s=s^1+s^2} \big\{ \|s^1\|_{\ell_\infty} + \|s^2\|_{\ell_\infty(w)} \big\}.$$

Here the norm of the second term in the right-hand side is defined by the formula $\|s\|_{\ell^\infty(w)} := \sup_{i\in\mathbb{Z}_+} \frac{|s_i|}{w_i}$.

For this aim we use a linear operator $U : \Sigma(w) \to \ell_\infty(S_0)$ given on the triangle Δ_i by

$$(Us)(x^{(i)}) = (Us)(y^{(i)}) := 0, \ (Us)(z^{(i)}) := s_i, \quad i \in \mathbb{Z}_+. \tag{10.46}$$

Lemma 10.34. *The image of U is a subspace of the null-space $\mathcal{L}$ and its norm is bounded by a numerical constant.*

Proof. Given $q > 1$, we choose a decomposition $s = s^1 + s^2$ for $s \in \Sigma(w)$ such that

$$\|s^1\|_{\ell_\infty} \le q\|s\|_{\Sigma(w)}, \ \|s^2\|_{\ell_\infty(w)} \le q\|s\|_{\Sigma(w)}.$$

Then we define a 1-jet $(f, \vec{g})$ given on the set S_0 by

$$f(x) := (Us)(x), \ g_1(x) := 0 \text{ and } g_2|_{\Delta_i} := \frac{s^{(2)}}{w_i}, \ i \in \mathbb{Z}_+, \tag{10.47}$$

and show that the assumptions of Proposition 10.33 hold for this 1-jet with $\lambda = 6q\|s\|_{\Sigma(w)}$. Because of (10.46) this would imply the assertion of the lemma.

First, by (10.47) and the choice of s^2, we have

$$\|\vec{g}\,\|_{\ell_\infty(S_0)} = \|s^2\|_{\ell_\infty(w)} \le q\|s\|_{\Sigma(w)},$$

i.e., condition (a) of the proposition holds.

To check (b) we first consider the case of $x \in \Delta_i$, $y \in \Delta_j$ where $i \ne j$. Then $\|x - y\| \ge 1$ by the definition of S_0 and therefore

$$\begin{aligned}\big|R(f\,; x, y)\big| &\le |f(x)| + |f(y)| + q\|x - y\|\, \|\vec{g}\,\|_{\ell_\infty(S)} \\ &\le \big(|s_i| + |s_j| + 2q\|s\|_{\Sigma(w)}\big)\|x - y\|,\end{aligned}$$

see (10.47) and (10.46). Since $s = s^1 + s^2$ is a q-optimal decomposition and $0 < w_i \le 1$, we get

$$|s_i| \le |s_i^1| + |s_i^2| \le \|s^1\|_{\ell_\infty} + \|s^2\|_{\ell_\infty(w)} \le 2q\|s\|_{\Sigma(w)}.$$

Combining these inequalities, we obtain

$$\big|R(f\,; x, y)\big| \le 6q\|s\|_{\Sigma(w)}\|x - y\|,$$

i.e., condition (b) holds in this case.

In the remaining case, $x, y \in \Delta_i$ for some i. It suffices to consider the case $x = x^{(i)}$, $y = z^{(i)}$, since $R(f\,; x, y) = 0$ if $x = x^{(i)}$, $y = y^{(i)}$. Due to (10.47) and (10.46), we now have for these x, y,

$$\begin{aligned}\big|R(f\,;x,y)\big| &:= \big|f(z^{(i)}) - f(x^{(i)}) - \langle \vec{g}\,(x^{(i)}), z^{(i)} - x^{(i)}\rangle\big| \\ &:= \left|s_i - \frac{s_i^2}{w_i}\,w_i\right| = |s_i^1|\,\|x^{(i)} - z^{(i)}\| := |s_i^1| \cdot \|x-y\|,\end{aligned}$$

(note that $\|x^{(i)} - z^{(i)}\| := \max\{1, w_i\} = 1$ by the definition of S_0) and we conclude that

$$\big|R(f\,;x,y)\big| \le \|s^1\|_{\ell_\infty}\|x-y\| \le q\|s\|_{\Sigma(w)}\|x-y\|.$$

Hence, condition (b) has been verified.

It remains to check condition (c). Since now $\|x-y\| \to 0$, we may assume that $\|x-y\| < 1$. This, in turn, implies that $x, y \in \Delta_i$ for some i. As above, we then have for this case

$$\big|R(f\,;x,y)\big| \le |s_i^1|\,\|x-y\|.$$

Since $s_i^1 \in c_0$ and $i \to \infty$ as $\|x-y\| \to 0$, this bounds the right-hand side by $o\big(\|x-y\|\big)$, i.e., condition (c) holds as well.

Now applying Proposition 10.33 to 1-jet (10.47) we conclude that the function $f := Us$ is the trace to S_0 of the C_u^1 function, and its trace norm is bounded by $cq\|s\|_{\Sigma(w)}$, where c is a numerical constant. Moreover, by (10.46), Us belongs to the null-space $\mathcal{L}$. □

At the next step we relate the subspaces of $\Sigma(w)$ to those of $\ell_\infty(w)$ using a supposedly existing simultaneous extension operator.

Lemma 10.35. *An operator $E \in \operatorname{Ext}\big(S_0\,; C_u^1(\mathbb{R}^2)\big)$ gives rise to a linear operator $T : \Sigma(w) \to \ell_\infty(w)$ such that $Id_{\Sigma(w)} - T$ maps $\Sigma(w)$ into the space c_0, and, moreover, for some constant $c = c(n) > 0$,*

$$\|T\| + \|Id_{\Sigma(w)} - T\| \le c\|E\|.$$

Proof. If f belongs to the null-space $\mathcal{L} \subset C_u^1(\mathbb{R}^2)\big|_{S_0}$, then $\tilde{f} := Ef$ belongs to $C_u^1(\mathbb{R}^2)$ and extends f from S_0 to $\mathbb{R}^n$. Using this we define a linear operator $V : \mathcal{L} \to \ell_\infty(w)$ by setting

$$Vf := \left(w_i\Big(\frac{\partial \tilde{f}}{\partial x_2}\Big)(x^{(i)})\right)_{i\in\mathbb{Z}_+}$$

and introduce the desired operator T by $T := VU$.

Due to the inequality

$$\|Vf\|_{\ell_\infty(w)} := \sup_{i\in\mathbb{Z}_+}\left|\Big(\frac{\partial \tilde{f}}{\partial x_2}\Big)(x)\right| \le \|\tilde{f}\|_{C_b^1(\mathbb{R}^2)} \le \|E\| \cdot \|f\|_{C_u^1(\mathbb{R}^2)|_{S_0}},$$

we get $T : \mathcal{L} \to \ell_\infty(w)$ and, moreover,

$$\|T\| \le \|E\|\,\|U\| \le c\|E\|.$$

To show that $\widetilde{T} := Id_{\Sigma(w)} - T$ maps $\Sigma(w)$ into c_0, we fix $s \in \Sigma(w)$ and write

$$s^2 := Ts \quad \text{and} \quad s^1 := s - s^2 =: \widetilde{T}s.$$

We must show that $s^1 \to 0$ as $i \to \infty$. To this end we set $f := Us\,(\in \mathcal{L})$ and use its C_u^1 extension $\tilde{f} := Ef$ to define a function $\vec{g} = (g_1, g_2) : S_0 \to \mathbb{R}^2$ by setting $\vec{g} := (\nabla \tilde{f})|_{S_0}$.

Assertion (c) of Proposition 10.33 clearly holds for the 1-jet $(f, \vec{g})$ and therefore

$$\big|R(f\,; x^{(i)}, y^{(i)})\big| := 2\big|g_1(x^{(i)})\big| = o\big(\|x^{(i)} - y^{(i)}\|\big) = o(1) \text{ as } i \to \infty.$$

Applying the same argument to the points $x^{(i)}, z^{(i)}$ we get

$$\begin{aligned}\big|R(\tilde{f}\,; x^{(i)}, z^{(i)})\big| &:= \big|f(z^{(i)}) - f(x^{(i)}) - g_2(x^{(i)})w_i\big| \\ &= o\big(\|x^{(i)} - z^{(i)}\|\big) = o(1) \text{ as } i \to \infty.\end{aligned}$$

Further, $f(z^{(i)}) := (Us)(z^{(i)}) := s_i$ and $(Ts)_i := w_i g_2(x^{(i)})$ and we have

$$\begin{aligned}|s_i^1| &:= |s_i - (Ts)_i| := \big|f(z^{(i)}) - w_i g_2(x^{(i)})\big| \\ &\le \big|R(\tilde{f}\,; x^{(i)}, z^{(i)})\big| + \big|g_1(x^{(i)})\big|.\end{aligned} \tag{10.48}$$

Due to two previous estimates, the right-hand side here tends to zero as $i \to \infty$, i.e., $s^1 \in c_0$, as required.

It remains to estimate the norm of $\widetilde{T} : \Sigma(w) \to c_0$. To this end we first note that

$$\big|g_1(x^{(i)})\big| := \Big|\Big(\frac{\partial \tilde{f}}{\partial x_2}\Big)(x^{(i)})\Big| \le \|\tilde{f}\|_{C_u^1(\mathbb{R}^2)} \le \|E\| \cdot \|f\|^*$$

where we set, for brevity, $\|f\|^* := \|f\|_{C_u^1(\mathbb{R})|_{S_0}}$.

Further, by Proposition 10.33 (b),

$$\frac{|R(\tilde{f}\,; x^{(i)}, z^{(i)})|}{\|x^{(i)} - z^{(i)}\|} = \big|R(\tilde{f}\,; x^{(i)}, z^{(i)})\big| \le c\|E\|\,\|f\|^*.$$

Using this and (10.48) we then get

$$|s_i^1| \le c\|x^{(i)} - z^{(i)}\|\,\|E\|\,\|f\|^* + \|E\|\,\|f\|^* \le (2c+1)\|E\|\,\|f\|^*.$$

This implies

$$\|\widetilde{T}s\|_{\ell_\infty} := \|s^1\|_{\ell_\infty} \le (2c+1)\|E\|\,\|f\|^*,$$

i.e., $\|\widetilde{T}\|_{\ell_\infty} := \|Id_{\Sigma(w)} - T\|_{\ell_\infty} \le (2c+1)\|E\|$, as required. □

Now we use condition (A) which implies, for every $\varepsilon > 0$, existence of a subsequence $\widetilde{w} := \{\widetilde{w}_{n_i}\}_{i\in\mathbb{Z}_+} \subset w$ such that $\varepsilon \le \widetilde{w}_{n_i} \le 2\varepsilon$ for $i \in \mathbb{Z}_+$. This implies that for every $s \in \ell_\infty(\widetilde{w})$,

$$(2\varepsilon)^{-1}\|s\|_{\ell_\infty} \le \|s\|_{\ell_\infty(\widetilde{w})} \le \|s\|_{\ell_\infty} \tag{10.49}$$

and that the space $\Sigma(\widetilde{w}) := c_0 + \ell_\infty(\widetilde{w})$ coincides with ℓ_∞; moreover,

$$\|s\|_{\Sigma(\widetilde{w})} \le \|s\|_{\ell_\infty(\widetilde{w})} \le \varepsilon^{-1}\|s\|_{\ell_\infty}.$$

Now let ρ be the canonical trace operator given for $s \in \Sigma(w)$ by

$$\rho(s) := s\big|_{\{n_i\}_{i\in\mathbb{Z}_+}}$$

and $\nu(s)$ be the extension of $s : \{n_i\}_{i\in\mathbb{Z}_+} \to \mathbb{R}$ to $\mathbb{Z}_+$ by zero. Then the operators

$$\widehat{T} := \rho T \nu \quad \text{and} \quad Id_{\Sigma(\widetilde{w})} - \widehat{T}$$

satisfy the inequalities

$$\|\widehat{T}\| \le \|T\| \quad \text{and} \quad \|Id_{\Sigma(\widetilde{w})} - \widehat{T}\| \le \|Id_{\Sigma(w)} - T\|, \tag{10.50}$$

since $\|\rho\| = \|\nu\| = 1$.

Then by Lemma 10.35 $\widehat{T}$ maps $\Sigma(\widetilde{w}) = \ell_\infty$ into ℓ_∞ and

$$\|\widehat{T}\| \le c\varepsilon^{-1}\|E\|.$$

Moreover, $Id_{\Sigma(\widetilde{w})} - \widehat{T}$ maps ℓ_∞ into c_0, and therefore $\widehat{T}(c_0) \subset c_0$.

At the final step we should show that the restriction $\widehat{T}|_{c_0} : c_0 \to c_0$ is invertible. To this end we estimate the norm of $(Id_{\ell_\infty} - \widehat{T})\big|_{c_0}$ and show that it becomes less than 1 for sufficiently small ε. Specifically, given $s \in c_0$, we have, by (10.49), (10.50) and Lemma 10.35,

$$\|s - \widehat{T}s\|_{\ell_\infty} \le 2\varepsilon\|\widehat{T}s\|_{\ell_\infty(\widetilde{w})} \le 2\varepsilon\|T\|\,\|s\|_{\Sigma(\widetilde{w})} \le 2\varepsilon c\|E\|\,\|s\|_{\Sigma(\widetilde{w})}.$$

However, by the definition of $\|\cdot\|_{\Sigma(\widetilde{w})}$ we also have

$$\|s\|_{\Sigma(\widetilde{w})} := \inf_{s=s^1+s^2}\left\{\|s^1\|_{c_0} + \|s^2\|_{\ell_\infty(\widetilde{w})}\right\} \le \|s\|_{\ell_\infty}.$$

Together with the previous inequality this yields, for $s \in c_0$,

$$\|s - \widehat{T}s\|_{\ell_\infty} \le 2c\varepsilon\|E\|\,\|s\|_{\ell_\infty},$$

i.e., $\left\|Id_{c_0} - \widehat{T}|_{c_0}\right\| \le 2c\varepsilon\|E\| < 1$ for a suitable $\varepsilon > 0$. This clearly implies that $\widehat{T}|_{c_0}$ is invertible, see, e.g., [DS-1958].

Now we set $P := \left(\widehat{T}|_{c_0}\right)^{-1}\widehat{T}$. This operator acts from ℓ_∞ into c_0, and for $s \in c_0$,

$$Ps = s.$$

Hence, P is a linear continuous projection of ℓ_∞ onto c_0, whose existence contradicts the Phillips theorem [Ph-1940], and therefore the extension operator $E \in \mathrm{Ext}\big(S_0\,; C_u^1(\mathbb{R}^2)\big)$ does not exist.

Theorem 10.32 has been proved. □

10.1.7 Remarks on structure of extension algorithms

The algorithms developed in Chapters 2 (of Volume I) and 10 for study of Whitney's problems are nonlinear but admit (mostly highly involved) linearization. It is natural to ask whether the nonlinear part may be eliminated in these algorithms.

The following discusses two basic problems concerning such linear algorithms. To formulate the first problem, we introduce a linear analog of the function δ_N, see (10.2), by setting

$$\lambda_N(\Sigma\,; X) := \sup_{S\in\Sigma} \sup\big\{\lambda(F\,; X)\,; F \subset S \text{ and } \operatorname{card} F \le N\big\}$$

where we recall, see Definition 7.1,

$$\lambda(S\,; X) := \inf\big\{\|T\|\,; T \in \mathrm{Ext}(S\,; X)\big\}.$$

This conventionally equals ∞ if $\mathrm{Ext}(S\,; X) = \phi$.

Definition 10.36. A smoothness space X possesses the Linear Finiteness Property with respect to the family of sets Σ (briefly, $X \in \mathcal{LFP}(\Sigma)$) if there exists an integer $N > 1$ and a constant $c = c(\Sigma, X) > 0$ such that $\lambda_N(\Sigma\,; X)$ is finite and for every $S \in \Sigma$,

$$\lambda(S\,; X) \le c\lambda(\Sigma\,; X). \tag{10.51}$$

The minimal N is called the *linear finiteness constant* and is denoted by $\mathcal{L}_\Sigma(X)$; the subscript is omitted if Σ consists of all nonempty closed subsets.

The main question concerning relations between linear and nonlinear extension properties of the distinguished family of functions and jet spaces is as follows.

Problem 10.37. (a) *Prove that for the spaces from the class in question,*

$$\mathcal{LFP} = \mathcal{FP}.$$

(b) *Prove that for the space X of this class,*

$$\mathcal{F}(X) = \mathcal{L}(X).$$

The results already proved confirm the first claim, while the validity of the stronger second claim is rather questionable.

We conclude with several facts illustrating the situation.

Example 10.38. (a) Let $X := \Lambda^{1,\omega}(\mathbb{R}^n)$, so that $\mathcal{F}(X)=2$, see Example 10.11 (a). A simple modification of Whitney's extension method gives a linear extension operator $T_S \in \mathrm{Ext}(S\,; X)$ constructed from the family of two-point extension operators $T_F \in \mathrm{Ext}(F\,; X)$. Hence, $\mathcal{L}(X) = \mathcal{F}(X)$ in this case.

However, the optimal constant c in (10.51) is much larger than the corresponding nonlinear extension constant $\gamma(X)$. In view of the McShane Theorem 1.27 of Volume I, the latter is 1, while the former is at least $c_0 n^{\frac{1}{8}}$ where $c_0 > 0$ is numerical, see Corollary 8.12.

(b) Theorems from Sections 10.2–10.5 confirm the first claim of Problem 10.37 although the linearization procedures in most cases are very involved. A detailed consideration of this procedure for the spaces $C_b^1(\mathbb{R}^n)$ and $C^\ell\Lambda^{k,\omega}(\mathbb{R}^n)$, where $\ell + k = 2$ and $\ell < 2$, shows that the second claim of the problem is also true for this case.

The second problem concerns the *localization property* of an extension operator. We explain this concept beginning with the simplest case of $\Lambda^{1,\omega}(\mathbb{R}^n)$.

The Whitney extension operator $T_S \in \mathrm{Ext}\big(S\,; \Lambda^{1,\omega}(\mathbb{R}^n)\big)$ is given by

$$T_S f := \sum_{Q \in \mathcal{W}_S} f(s_Q)\varphi_Q$$

where s_Q is the closest to Q point of S. Since the order of the cover $\{\operatorname{supp}\varphi_Q\}_{Q\in\mathcal{W}_S}$ of the complement S^c is bounded by some integer $d = d(n) > 1$, for every $f \in \Lambda^{1,\omega}(\mathbb{R}^n)\big|_S$ and $x \in \mathbb{R}^n$ there exist points $x^i \in S$ and numbers λ_i, $1 \le i \le d$, such that

$$T_S f(x) = \sum_{i=1}^{d} \lambda_i f(x_i). \tag{10.52}$$

We will say that T_S is of *depth d*.

The general definition introduced by Fefferman in [F-2007b] says that an extension operator $T_S : X|_S \to X$ is of depth d if (10.52) holds. Here λ_i are linear or nonlinear functionals depending on linearity or nonlinearity of T_S.

The proofs of the known extension results presented later exploit at the final step the general Whitney method which is of finite depth. However, the operators constructed in between are, in general, of infinite depth.

The last fact may be seen as a drawback of the methods involved, but the next counterexample due to Fefferman [F-2007b, Thm. 2] shows that it is, in general, an unavoidable characteristic of the multivariable extension methods.

Theorem 10.39. *There exists a countable compact set $S_0 \subset \mathbb{R}^2$ for which every operator $T \in \mathrm{Ext}\big(S_0\,; C_b^1(\mathbb{R}^2)\big)$ is of infinite depth.*

Proof. The desired set is the union of finite sets F_j, $j \in \mathbb{Z}_+$, where $F_0 := \{(0,0)\}$ and $F_j := \{x_{j,k}\}_{k\in\mathbb{N}\cup\{\infty\}}$, where $x_{j,k} := \big(2^{-j} + 10^{-j-k}, (-1)^k 10^{-k}\big)$ and $x_{j,\infty} := (2^{-j}, 0)$.

Clearly, the set

$$S_0 := \bigcup_{j\in\mathbb{Z}_+} F_j \tag{10.53}$$

is countable and compact. Its basic properties are given by the next result whose proof will be postponed to the end.

Proposition 10.40. (a) *Suppose that the subset $S \subset S_0$ is such that for every $j \geq 1$,*

$$\mathrm{card}(S \cap F_j) < \infty.$$

Then there exists a function $\psi \in C_b^1(\mathbb{R})$ such that

$$S \subset \big\{(x,y) \in \mathbb{R}^2\,; y = \psi(x)\big\}.$$

(b) *Let $\{Y^n\}_{n\in\mathbb{N}}$ be a sequence of finite subsets in S_0 of a fixed cardinality. Then there exists an integer $j_0 \geq 1$ and an increasing infinite sequence $\{n_\nu\}_{\nu\in\mathbb{N}} \subset \mathbb{N}$ such that for every $j \geq j_0$ and $\nu \geq 1$,*

$$\mathrm{card}(Y^{n_\nu} \cap F_j) < \infty.$$

Now we suppose, on the contrary, that there exists a linear operator $T \in \mathrm{Ext}\big(S_0\,; C_b^1(\mathbb{R}^2)\big)$ of depth d. Then, given $x^n \in \mathbb{R}^2$, we may find points $y_1^n, \dots, y_d^n$ in S_0 and numbers $\lambda_1^n, \dots, \lambda_d^n$ such that

$$Tf(x^n) = \sum_{i=1}^{d} \lambda_i^n f(y_i^n)$$

for all $f \in C_b^1(\mathbb{R}^2)\big|_{S_0}$.

In particular, for every $n \geq 1$, we have

$$Tf(x^n) = 0 \text{ for } x^n := \Big(0, \frac{1}{n}\Big), \tag{10.54}$$

since $f \in C_b^1(\mathbb{R}^2)\big|_{S_0}$ and $f(y_i^n) = 0$ for $1 \leq i \leq d$.

Further, let j_0 and $\{n_\nu\}$ be chosen for the sequence $Y^n := \{y_i^n\}_{1\le i\le d}$, $n \in \mathbb{N}$, as in Proposition 10.40 (b). We set $Y := \bigcup_{\nu\in\mathbb{N}} Y^{n_\nu}$ and define its subsets $\widehat{Y}$ and $\widetilde{Y}$ by

$$\widehat{Y} := Y \cap \left(\bigcup_{j<j_0} F_j\right) \quad \text{and} \quad \widetilde{Y} := Y \backslash \widehat{Y}. \tag{10.55}$$

The set $\widetilde{Y} \cap F_j$ is finite for $j \ge j_0$ by Proposition 10.40 (b) and is empty for $j < j_0$ by (10.55). Hence Proposition 10.40 (a) implies that there exists a function $\psi \in C_b^1(\mathbb{R})$ such that $y = \psi(x)$ for all $(x, y) \in \widetilde{Y}$.

Now let $\theta : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function equal to 1 in the neighborhood of origin, and equal to zero on every F_j with $j < j_0$. We define the function

$$H(x, y) := \theta(x, y)(y - \psi(x)), \quad x, y \in \mathbb{R}^2, \tag{10.56}$$

and set $h := H|_{S_0} \in C_b^1(\mathbb{R}^2)\big|_{S_0}$.

Bounded C^1 functions H and Th are both equal to f on S_0. Hence, $G := H - Th$ equals zero on S_0, and this would imply

$$\nabla(Th)(0, 0) = \nabla H(0, 0). \tag{10.57}$$

Actually, for a given $j \ge 1$,

$$G(x_{j,\infty}) = G(x_{j,k}) = 0 \text{ for all } k \ge 1.$$

Consequently, we get

$$0 = \lim_{k\to\infty} \frac{G(x_{j,2k}) - G(x_{j,\infty})}{10^{-j-k}} = \left(\frac{\partial G}{\partial x} + 10^{-j}\, \frac{\partial G}{\partial y}\right)(x_{j,\infty})$$

and, similarly,

$$0 = \lim_{k\to\infty} \frac{G(x_{j,2k+1}) - G(x_{j,\infty})}{10^{-j-k}} = \left(\frac{\partial G}{\partial x} - 10^{-j}\, \frac{\partial G}{\partial y}\right)(x_{j,\infty}),$$

see (10.53).

From these equalities we conclude that $\nabla G(x_{j,\infty}) := \left(\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}\right)(x_{j,\infty}) = 0$. Taking the limit as $j \to \infty$ we get $\nabla G(0, 0) = 0$, i.e., (10.57) has been proved.

Now we straightforwardly evaluate ∇H and $\nabla(Th)$ at $(0, 0)$ to show that they are unequal.

In fact, $H|_{\widetilde{Y}} = 0$ by (10.56) and (10.57), and $h = 0$ on $\widehat{Y}$, since $\theta = 0$ on F_j for $j < j_0$, see (10.55) and (10.56). Then $H = 0$ on Y, hence $h := H|_{S_0}$ is equal to zero on Y. By (10.54) and the definition of Y, this implies that $Th(x^{n_\nu}) = 0$ for all $\nu \ge 1$.

Recalling that $x^n := \left(0, \frac{1}{n}\right)$ we then conclude that

$$\frac{\partial(Th)}{\partial y}(0,0) = 0.$$

However, $\theta = 1$ in a neighborhood of the origin, and definition (10.56) yields

$$\frac{\partial H}{\partial y}(0,0) = 1.$$

Thus (10.57) is not true, a contradiction to the claim that the operator T is of finite depth.

Proof of Proposition 10.40. We begin with assertion (a). Since $S \subset S_0$ is such that for every $j \geq 1$ the set $S \cap F_j$ is finite, we may pick an integer M_j so that

$$M_j > \max\{k\,; x_{j,k} \in S\}. \tag{10.58}$$

Using these numbers we define a sequence of functions $\{\varphi_j\}_{j\geq 1} \subset C^1(\mathbb{R})$ whose sum yields the required function ψ satisfying

$$S \subset \{(x,y)\,; y = \psi(x)\}.$$

The definition of these φ_j is given within the proof of

Lemma 10.41. *There exists a sequence $\{\varphi_j\}_{j\geq 1} \subset C^1(\mathbb{R})$ satisfying the following conditions:*

(a) *For some constant $c > 0$ independent of j,*

$$\|\varphi\|_{C^1(\mathbb{R})} \leq c\,10^{-j}.$$

(b) *Every point $x_{j,k}$ with $1 \leq k \leq M_j$ lies in the graph of φ_j.*

(c) $\operatorname{supp}\varphi_j \cap \operatorname{supp}\varphi'_j = \emptyset$ *whenever $j \neq j'$.*

Proof. Let $\theta, \widetilde{\theta}$ be functions of $C^1(\mathbb{R})$ such that $\theta(x) := 0$ for $x \leq \frac{1}{2}$ and $\theta(x) := 1$ for $x \geq 1$ while $\widetilde{\theta}(x) := 1$ for $|x| \leq \frac{1}{2}$ and $\theta(x) := 0$ for $|x| \geq \frac{2}{3}$.

One easily checks that the function ψ_M given on $\mathbb{R}$ by

$$\psi_M(x) := \theta(10^M x)\widetilde{\theta}(x)x\cos\big(\pi\log_{10}|x|\big)$$

satisfies the conditions

$$\operatorname{supp}\psi_M \subset (0,1); \tag{10.59}$$

$$\psi_M(10^{-k}) = (-1)^k 10^{-k} \text{ for } 1 \leq k \leq M; \tag{10.60}$$

$$\|\psi_M\|_{C^1(\mathbb{R})} \le c, \tag{10.61}$$

where $c > 0$ is independent of M.

Using this we set, for $j \ge 1$,

$$\varphi_j(x) := 10^{-2j}\psi_{M_j}\big[10^j(x - 2^{-j})\big]. \tag{10.62}$$

Then (10.61) yields assertion (a) of the lemma, while a straightforward computation based on (10.58) and (10.60) gives

$$\varphi_j(2^{-j} + 10^{-j-k}) = (-1)^k 10^{-2j-k}$$

provided that $1 \le k \le M_j$.

This clearly means that the points $x_{j,k}$ with $1 \le k \le M_j$ lie in the graph $\big\{(x,y)\,;y = \varphi_j(x)\big\}$, i.e., (b) is true.

Finally, $\operatorname{supp}\varphi_j \subset \big(2^{-M_j}, 2^{-M_j} + 10^{-M_j}\big)$ by (10.59) and (10.62), and these intervals do not intersect if $j \ne j'$.

Hence, (c) and the lemma have been proved. □

Now we set $\psi := \sum_{j\in\mathbb{N}}\varphi_j$. Due to (10.61), (10.62) and Lemma 10.41 (c), the sums of φ_j's and of their derivatives are uniformly convergent; hence, $\psi \in C_b^1(\mathbb{R})$.

Further, assertions (b), (c) of the lemma yield that

$$x_{j,k} \in \big\{(x,y)\,;y = \psi(x)\big\} \text{ whenever } x_{j,k} \in S. \tag{10.63}$$

Moreover, for $(x,y) := x_{j,\infty}$ or $(0,0)$ we have $y = 0$ and, by (10.62) and (10.59), $\varphi_j(x) = 0$ for all $j \ge 1$. Hence, for these points y equals $\psi(x)$, i.e., (10.63) also holds for $x_{j,\infty}$ and $(0,0)$.

From this, (10.63) and (10.53) we conclude that $\psi(x) = y$ for all $(x,y) \in S\,(\subset S_0)$.

(b) Let $Y^n := \{y_1^n, \dots, y_d^n\} \subset \mathbb{R}^2$, $n \in \mathbb{N}$, be a sequence of d-tuples in S_0. We must find an integer $j_0 \ge 1$ and an increasing sequence $\{n_\nu\}_{\nu\ge1} \subset \mathbb{N}$ such that for every $j \ge j_0$ and $\nu \ge 1$,

$$\operatorname{card}(Y^{n_\nu} \cap F_j) < \infty.$$

Given the set $J \subset \mathbb{N}$, we will say that J is a "trap" if either $J = \emptyset$ or

$$\operatorname{card}\big\{n \in \mathbb{N}\,; Y^n \cap F_j \ne \emptyset \text{ for each } j \in J\big\} = \infty. \tag{10.64}$$

Since the sequence $\{F_j\}_{j\ge1}$ is disjoint and $\operatorname{card} Y^n = d$, no trap can have more than d elements.

Let $\widetilde{J}$ be a trap of maximal cardinality. Denoting by $\mathcal{N}$ the left-hand side of (10.64) for $J = \widetilde{J}$, we then have $\operatorname{card}\mathcal{N} = \infty$, i.e., $\mathcal{N} = \{n_1 < \cdots < n_\nu < \dots\}$, and, moreover, for $j \notin \widetilde{J}$,

$$\operatorname{card}\{n \in \mathcal{N}\,; Y^n \cap F_j\} < \infty.$$

Further, card $\widetilde{J} \le d$, and therefore there exists j_0 bigger than every $j \in \widetilde{J}$.

From these facts we conclude that, given $j \ge j_0$, there exist at most finitely many ν for which Y^{n_ν} intersects F_j. □

Theorem 10.39 has been proved. □

10.2 Trace and extension problems for Markov sets

We present complete solutions to the main problems for spaces $X := \Lambda^{k,\omega}(\mathbb{R}^n)$ and the class $\Sigma := \mathrm{Mar}(\mathbb{R}^n)$ of Markov sets, see Definition 9.28, i.e., find putatively sharp bounds (of polynomial growth in n and k) of the finiteness constants $\mathcal{F}_\Sigma(X)$, construct the corresponding linear extension operator and characterize the trace spaces $X|_F$ for subsets F of cardinality $\mathcal{F}_\Sigma(X)$. These will be attained by means of two results of Approximation Theory – the description of the trace space $X|_S$ with $S \in \Sigma$ via local polynomial approximation, see Theorem 9.8, and a Chebyshev type approximation result derived from the classical Helly theorem.

10.2.1 Uniform approximation by finite-dimensional subspaces

Let L_n be an n-dimensional subspace of the space $C(K)$ of continuous functions on a compact metric space K. We set for $f \in C(K)$ and $\widetilde{K} \subset K$,

$$e(f\,;\widetilde{K}) := \min_{\ell \in L_n} \sup_{\widetilde{K}} |f - \ell|.$$

Theorem 10.42 (Remez–Shnirelman). *For every $f \in C(K)\backslash L_n$ there exists a subset $\widetilde{K} \subset K$ of cardinality at most $n+1$ such that*

$$e(f\,;K) = e(f\,;\widetilde{K}).$$

Proof. The basic argument has been outlined in Appendix B of Volume I, see Theorem B.4 there. Because of importance of this result for the subsequent derivation we present a detailed proof.

Let us set

$$e_n := \sup\bigl\{e(f\,;\widetilde{K})\,;\widetilde{K} \subset K \quad \text{and} \quad \operatorname{card}\widetilde{K} \le n+1\bigr\}.$$

Clearly, we get

$$e_n \le e(f\,;K). \tag{10.65}$$

We show that, in fact, (10.65) is an equality. To this end we consider the family $\{V(t)\}_{t\in K}$ of subsets in L_n given, for $t \in K$, by

$$V(t) := \bigl\{\ell \in L_n\,;|f(t) - \ell(t)| \le e_n\bigr\}.$$

Since L_n is linear, every $V(t)$ is convex; it is also nonempty by the definition of e_n.

Now we establish the following two facts.

Claim A. If points t_i, $0 \le i \le n$, are mutually distinct, then

$$\bigcap_{0\le i\le n} V(t_i) \ne \emptyset.$$

Claim B. There exist points $\bar{t}_i$, $0 \le i \le n$, such that the set $\bigcap_{0\le i\le n} V(\bar{t}_i)$ is bounded.

The equality in (10.65) follows from these claims that simply assert that the assumptions of the Helly Theorem 1.22 of Volume I hold for the family of convex sets $\{V(t)\}_{t\in K}$ and therefore this family has a common point, say $\tilde{\ell}$. Then, by the definition of $V(t)$,

$$|f(t) - \tilde{\ell}(t)| \le e_n \text{ for all } t \in K,$$

hence, $e(f\,;K) := \max_K |f - \tilde{\ell}| \le e_n$. Together with (10.65) this yields

$$e(f\,;K) = \sup_{\operatorname{card}\widetilde{K}\le n+1} e(f\,;\widetilde{K}). \tag{10.66}$$

Now we prove the claims. Given mutually distinct points t_i, $0 \le i \le n$, we choose the element $\bar{\ell} \in L_n$ so that

$$\inf_{\ell\in L_n} \max_i \bigl|f(t_i) - \ell(t_i)\bigr| = \max_i \bigl|f(t_i) - \bar{\ell}(t_i)\bigr|.$$

By the definition of e_n we then have, for $0 \le i \le n$,

$$\bigl|f(t_i) - \bar{\ell}(t_i)\bigr| \le e_n,$$

that is, $\bar{\ell}$ is a common point of the sets $V(t_i)$, $0 \le i \le n$. Hence, Claim A is true.

To prove the second claim we need

Lemma 10.43. *There exist points $t_i \in K$, $1 \le i \le n$, such that the determinant*

$$\det\bigl(g_i(t_j)\bigr)_{1\le i,j\le n} \ne 0. \tag{10.67}$$

Here $\{g_i\}_{1\le i\le n}$ is a basis of the space L_n.

Proof. (by induction on n) For $n = 1$ the result is trivial. Let (10.67) be true for a linearly independent family $\{g_i\}_{1\le i\le n-1} \subset C(K)$. Then, for some points $\bar{t}_i$, $1 \le i \le n-1$,

$$\Delta := \det\bigl(g_i(\bar{t}_j)\bigr)_{1\le i,j\le n-1} \ne 0. \tag{10.68}$$

Now let $\{g_i\}_{1\le i\le n}$ be the basis of L_n. Considering the determinant in (10.67) as a function of t_n with $t_i = \bar{t}_i$ for $1 \le i \le n-1$ and assuming that it equals zero, we have, for all $t \in K$,

$$\lambda_1 g_1(t) + \cdots + \lambda_n g_n(t) = 0,$$

where λ_i is the co-factor of the term $g_i(t_n)$ in the determinant. In particular, $\lambda_n = \Delta \ne 0$, see (10.68), in contradiction to the linear independence of the family $\{g_i\}_{1\le i\le n}$.

Hence, (10.67) holds for some $t_n := \bar{t}_n$ and $t_i = \bar{t}_i$, $1 \le i \le n-1$. $\square$

Now let $\bar{t}_i$, $1 \le i \le n$, be chosen so that (10.67) holds. We show that $\bigcap_{i=1}^{n} V(\bar{t}_i)$ is bounded, i.e., Claim B is true.

Let $T : L_n \to \mathbb{R}^n$ be an affine transform given by

$$T\ell := \big(f(\bar{t}_i) - \ell(\bar{t}_i)\big)_{1 \le i \le n}.$$

Due to (10.67), T is a bijection onto $\mathbb{R}^n$. Moreover, by the definition of the sets $V(t)$, the intersection $\bigcap_{i=1}^{n} V(\bar{t}_i)$ is the preimage under T of the cube $\{x \in \mathbb{R}^n ; |x_i| \le e_n\}$ and therefore is bounded.

Finally, the function $F_\ell : (t_0, \dots, t_n) \mapsto \max_{0 \le i \le n} |g(t_i) - \ell(t_i)|$, where $\ell \in L_n$, is continuous on the compact set K^{n+1}, and therefore the function $(t_0, \dots, t_n) \mapsto e\big(f; \{t_0, \dots, t_n\}\big)$, being the infimum over L_n of the family $\{F_\ell\}_{\ell \in L_m}$ of continuous functions, is upper semicontinuous. Hence, by the Weierstrass theorem, there exists an $(n+1)$-tuple of points in K (some of which may coincide) where this function attains its maximum. In other words, there exists a subset $\overline{K} \subset K$ of cardinality at most $n+1$ such that

$$e(f\,; \overline{K}) = \max\big\{e(f;\, \widetilde{K})\,; \operatorname{card} \widetilde{K} \le n+1\big\}.$$

Together with (10.66) this proves the theorem. □

10.2.2 Finiteness constants for Markov sets

Here we deal with the space $\Lambda^{k,\omega}(\mathbb{R}^n)$ which is recalled to be defined by the norm

$$f \mapsto \sup_{\mathbb{R}^n} |f| + |f|_{\Lambda^{k,\omega}(\mathbb{R}^n)} := \sup_{\mathbb{R}^n} |f| + \sup_{t>0} \frac{\omega_k(t\,; f)}{\omega(t)},$$

where ω is a k-majorant. Let us also recall, see Theorem 2.10, that this family of spaces contains the smoothness space $C^\ell \Lambda^{k-\ell,\widetilde{\omega}}(\mathbb{R}^n)$, $1 \le \ell < k$, where $\widetilde{\omega} : t \mapsto \frac{\omega(t)}{t^\ell}$ is assumed to be a quasipower majorant. In particular, the smoothness spaces $C^{\ell,\omega}(\mathbb{R}^n)$ and $C^\ell \Lambda^{2,\omega}(\mathbb{R}^n)$ with a quasipower ω are members of the family.

Further, the basic family of subsets in $\mathbb{R}^n$ we deal with is that of Markov sets denoted for brevity by $M (:= \operatorname{Mar}(\mathbb{R}^n))$, see Definition 9.19.

Theorem 10.44. *The space $\Lambda^{k,\omega}(\mathbb{R}^n)$ has the Uniform Finiteness Property with respect to the class M and*

$$\mathcal{F}_M\big(\Lambda^{k,\omega}(\mathbb{R}^n)\big) \le 1 + \dim \mathcal{P}_{k-1,n} = \binom{n+k-1}{k-1} + 1.$$

Proof. We use the notation

$$N := \binom{n+k-1}{k-1} + 1, \quad X := \Lambda^{k,\omega}(\mathbb{R}^n)$$

and fix a set $S \in M$ satisfying the Markov inequality of Definition 9.19 with the constant γ. In accordance with Definition 10.10 we should prove that the inequality

$$\delta_N(f\,;S\,;X) < 1 \tag{10.69}$$

implies that the function $f : S \to \mathbb{R}$ belongs to $X|_S$ and its trace norm is bounded by a constant $c = c(n, k, \gamma)$.

To this end we pick a cube Q from the family $\mathcal{K}_S$ (of cubes centered at S and of radius at most $2\operatorname{diam} S$); Theorem 10.42 applied to $\mathcal{P}_{k-1,n}\big|_{Q\cap S}$ yields a subset $\Phi \subset Q \cap S$ of cardinality at most $1 + \dim\big(\mathcal{P}_{k-1,n}\big|_{Q\cap S}\big) \le N$, such that

$$E_k(Q \cap S\,; f) = E_k(\Phi\,; f).$$

By the definition of δ_N, inequality (10.69) implies existence of a function $g_\Phi \in X$ such that

$$g_\Phi|_\Phi = f|_\Phi \quad \text{and} \quad \|g_\Phi\|_X \le 1.$$

Together with the previous equality this yields

$$E_k(Q \cap S\,; f) \le E_k(Q\,; g_\Phi). \tag{10.70}$$

In turn, Theorem 2.37 of Volume I, see also (9.3), estimates the right-hand side from above by $w(k, n)\omega_k(\operatorname{diam} Q\,; g_\Phi)$ which is obviously bounded by

$$w(k,n)\omega(\operatorname{diam} Q)|g_\Phi|_{\Lambda^{k,\omega}(\mathbb{R}^n)} \le w(k,n)\omega(\operatorname{diam} Q)\|g_\Phi\|_X \le w(k,n)\omega(\operatorname{diam} Q).$$

Combining this with (10.70) we obtain

$$\sup_{Q\in\mathcal{K}_S} \frac{E_k(Q \cap S\,; f)}{\omega(\operatorname{diam} Q)} \le w(k,n). \tag{10.71}$$

Further, due to Theorem 9.30, there exists a function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ extending f and satisfying

$$|\tilde{f}|_{\Lambda^{k,\omega}(\mathbb{R}^n)} \le c(k,n,\gamma) \sup_{Q\in\mathcal{K}_S} \frac{E_k(Q \cap S\,; f)}{\omega(\operatorname{diam} Q)}\,. \tag{10.72}$$

The function $\tilde{f}$ is given by

$$\tilde{f} = \sum_{Q\in\mathcal{K}_S} p_Q(f)\varphi_Q,$$

where $p_Q(f)$ is a polynomial of degree $k-1$ satisfying

$$\max_{Q\cap S}\big|f - p_Q(f)\big| \le 2E_k(Q \cap S\,; f)$$

and $\{\varphi_Q\}$ is a partition of unity such that supports of φ_Q cover $\mathbb{R}^n \backslash S$ with order at most $c(n)$. Hence,

$$\sup_{\mathbb{R}^n} |\tilde{f}| \le c(n) \sup_{Q \in \mathcal{K}_S} \max_{\operatorname{supp} \varphi_Q} |p_Q(f)|.$$

By the Whitney construction, see Volume I, Section 2.2, $\operatorname{supp} \varphi_Q \subset \frac{9}{8}Q$. Therefore, by the Remez type inequality, see, e.g., (9.58), we have

$$\max_{\operatorname{supp} \varphi_Q} |p_Q(f)| \le c(k,n)\gamma \max_{Q \cap S} |p_Q(f)|,$$

while the latter maximum is bounded by

$$\max_{Q \cap S} |f| + 2E_k(Q \cap S\,; f) \le 3 \max_{Q \cap S} |f|.$$

These estimates then lead to the inequality

$$\sup_{\mathbb{R}^n} |\tilde{f}| \le c(k,n) \sup_S |f|$$

which together with (10.71) and (10.72) yield

$$\|\tilde{f}\|_X := \sup_{\mathbb{R}^n} |\tilde{f}| + |\tilde{f}|_{\Lambda^{k,\omega}(\mathbb{R}^n)} \le c(k,n,\gamma)\Big(\sup_S |f| + 1\Big).$$

It remains to derive from (10.69) the desired estimate for $\sup\limits_S |f|$.

Let $g_\Phi \in X$ be an extension of f such that

$$\|g_\Phi\|_X \le 2\big\|f|_\Phi\big\|_{X|_\Phi} := 2\|f\|_\Phi.$$

Then for the class of all subsets $\Phi \subset S$ of cardinality at most N,

$$\sup_S |f| = \sup_\Phi \max_\Phi |f| \le \sup_\Phi \|g_\Phi\|_X \le 2 \sup_\Phi \|f\|_\Phi := 2\delta_N(f\,; S\,; X) < 2.$$

Hence we have

$$\sup_S |f| \le \sup_{\mathbb{R}^n} |\tilde{f}| \le c(k,n,\gamma),$$

and therefore

$$\|f\|_{X|_S} \le \|\tilde{f}\|_X \le c(k,n,\gamma),$$

as required.

The theorem has been proved. □

10.2.3 Finiteness constants for weak Markov sets

In the case of spaces $C^{k,\omega}(\mathbb{R}^n)$, the class of subsets having the finiteness property with constants polynomially depending on k and n can be essentially widened. It is introduced by

Definition 10.45. A closed set $S \subset \mathbb{R}^n$ is said to be weak k-Markov if there exists a dense subset $S_0 \subset S$ such that for every $x \in S_0$,

$$\varliminf_{r\to 0}\left\{\sup_{p\in\mathcal{P}_{k,n}\setminus\{0\}}\left(\frac{\sup_{Q_r(x)}|p|}{\sup_{S\cap Q_r(x)}|p|}\right)\right\} < \infty. \tag{10.73}$$

The class just introduced is denoted by $\mathrm{Mar}_k^*(\mathbb{R}^n)$.

Remark 10.46. Let us recall, see [JW-1984], that a closed set $S \subset \mathbb{R}^n$ belongs to the class of Markov sets denoted by $\mathrm{Mar}(\mathbb{R}^n)$ if for some constant $c > 0$ and every $x \in S$, $0 < r \leq \operatorname{diam} S$ and $p \in \mathcal{P}_{1,n}$ the ratio in (10.73) is bounded by c.

It was proved in [JW-1984, Ch.2] that if the above condition holds for polynomials of degree 1, then it does for polynomials of every degree k. Hence, for all $k \geq 1$,

$$\mathrm{Mar}(\mathbb{R}^n) \subset \mathrm{Mar}_k^*(\mathbb{R}^n); \tag{10.74}$$

however, $\mathrm{Mar}(\mathbb{R}^n)$ is, as we will see, a small part of $\mathrm{Mar}_k^*(\mathbb{R}^n)$.

Now, we discuss the basic properties of weak Markov sets beginning with

Proposition 10.47. (a) *If* $\{S_i\}_{i\in I} \subset \mathrm{Mar}_k^*(\mathbb{R}^n)$, *then* $\overline{\cup_{i\in I} S_i} \in \mathrm{Mar}_k^*(\mathbb{R}^n)$.

(b) *If* $S_i \in \mathrm{Mar}_k^*(\mathbb{R}^{n_i})$, $i = 1, 2$, *then* $S_1 \times S_2 \in \mathrm{Mar}_k^*(\mathbb{R}^{n_1+n_2})$.

(c) *If* $L : \mathbb{R}^n \to \mathbb{R}^n$ *is a linear automorphism or a linear projection onto an affine subspace of* $\mathbb{R}^n$, *then* $\overline{L(S)}$ *belongs to* Mar_k^* *whenever* S *does.*

(d) *Let* $S \subset \mathrm{Mar}_k^*(\mathbb{R}^n)$ *and* $\varphi : S_\varepsilon \to \mathbb{R}^n$ *be a homeomorphism of an* ε*-neighborhood of* S. *Assume that the set of points in* S *where the derivative* $D\varphi$ *exists and is invertible is open and dense. Then* $\varphi(S) \in \mathrm{Mar}_k^*(\mathbb{R}^n)$.

Proof. Assertions (a)–(c) follow directly from the definition.

(d) Let S_0 denote the set of points in S where (10.73) holds and $S_1 \subset S$ be the set of points where $D\varphi$ exists and is invertible. Since S_0 is dense and S_1 is open and dense in S, the set $\varphi(S_0 \cap S_1)$ is dense in $\varphi(S)$. Therefore it suffices to prove

$$\text{if } \hat{x} := \varphi(x) \in \varphi(S_0 \cap S_1), \text{ then (10.73) holds for } \hat{x} \text{ and } \varphi(S). \tag{10.75}$$

To prove this we first note that, due to invariance of $\mathrm{Mar}_k^*(\mathbb{R}^n)$ with respect to linear automorphisms of $\mathbb{R}^n$, we may (and will) assume that $\hat{x} = x$ and $D\varphi$ is the identity map at $x\,(\in S_1)$. Hence, we get

$$\varphi(y) = y + R(x,y), \text{ where } \|R(x,y)\| = o(\|x-y\|) \text{ as } y \to x. \tag{10.76}$$

In particular, for some $\rho := \rho(r) \geq r$ such that $\lim_{r\to 0} \frac{\rho(r)}{r} = 1$,

$$\varphi^{-1}(Q_r(x)) \subset Q_\rho(x) \text{ and } \varphi(Q_{\rho/4}(x)) \subset Q_{r/2}(x). \tag{10.77}$$

These imply that for $p \in \mathcal{P}_{k,n}$,

$$\max_{Q_r(x)} |p| = \max_{\varphi^{-1}(Q_r(x))} |p \circ \varphi| \leq \max_{Q_\rho(x)} |p \circ \varphi|.$$

Further, for sufficiently small $r > 0$ the point $\varphi(y)$ belongs to $Q_{2\rho}(x)$ whenever $y \in Q_\rho(x)$, see (10.76). Hence, the maximum on the right-hand side does not exceed $\max_{Q_{2\rho}(x)} |p|$ which, in turn, due to the Remez type inequality [BrG-1973], see Appendix G to Chapter 2 of Volume I, is bounded by $c(k,n)\left(\frac{2\rho}{\rho/4}\right)^{nk} \max_{Q_{\rho/4}(x)} |p|$. Therefore we conclude that

$$\max_{Q_r(x)} |p| \leq c_1(n,k) \max_{Q_{\rho/4}(x)} |p|. \tag{10.78}$$

To proceed we now use condition (10.73) for the point x asserting that there exist a decreasing sequence $\{r_i\}_{i\geq 1}$ and a constant $\gamma = \gamma(x, S, k) > 0$ such that for every $p \in \mathcal{P}_{k,n}$ and $i \in \mathbb{N}$,

$$\max_{Q_{r_i}(x)} |p| \leq \gamma \max_{S \cap Q_{r_i}(x)} |p|. \tag{10.79}$$

Now let $\hat{r}_i$ satisfy $r_i = \frac{1}{4}\rho(\hat{r}_i)$. Changing variables $y \mapsto \varphi^{-1}(y)$ and applying (10.77) we then obtain

$$\max_{S \cap Q_{r_i}(x)} |p| = \max_{\varphi(S) \cap \varphi(Q_{r_i}(x))} |p \circ \varphi^{-1}| \leq \max_{\varphi(S) \cap Q_{\hat{r}_i/2}(x)} |p \circ \varphi^{-1}|.$$

For small $\hat{r}_i$ points $\varphi^{-1}(y)$ belong to $Q_{\hat{r}_i}(x)$ whenever $y \in Q_{\hat{r}_i/2}(x)$; therefore applying subsequently (10.76), the classical Markov polynomial inequality and (10.78), (10.79) we obtain that the right-hand side of the above inequality is bounded by

$$\begin{aligned}
&\max_{\varphi(S) \cap Q_{\hat{r}_i/2}(x)} |p| + \max_{\varphi(S) \cap Q_{\hat{r}_i/2}(x)} |p \circ \varphi^{-1} - p| \leq \max_{\varphi(S) \cap Q_{\hat{r}_i/2}(x)} |p| + \max_{Q_{\hat{r}_i}(x)} \|\nabla p\| \cdot o(\hat{r}_i) \\
&\leq \max_{\varphi(S) \cap Q_{\hat{r}_i/2}(x)} |p| + \frac{o(\hat{r}_i)}{\hat{r}_i} \cdot \max_{Q_{\hat{r}_i}(x)} |p| \leq \max_{\varphi(S) \cap Q_{\hat{r}_i/2}(x)} |p| + \frac{o(\hat{r}_i)}{\hat{r}_i} \cdot \max_{Q_{r_i}(x)} |p| \\
&\leq \max_{\varphi(S) \cap Q_{\hat{r}_i/2}(x)} |p| + \frac{\gamma \cdot o(\hat{r}_i)}{\hat{r}_i} \cdot \max_{S \cap Q_{r_i}(x)} |p|.
\end{aligned}$$

Since $\lim\limits_{i\to\infty} \dfrac{\gamma \cdot o(\hat{r}_i)}{\hat{r}_i} = 0$, for all sufficiently large i we obtain from here that

$$\max_{S \cap Q_{r_i}(x)} |p| \leq \max_{\varphi(S) \cap Q_{\hat{r}_i/2}(x)} |p| + \frac{1}{2} \cdot \max_{S \cap Q_{r_i}(x)} |p|.$$

Therefore for such i,

$$\max_{S\cap Q_{r_i}(x)} |p| \le 2 \max_{\varphi(S)\cap Q_{\hat{r}_i}(x)} |p|.$$

Combining this estimate with (10.78) and (10.79) we get for such i and all $p \in \mathcal{P}_{k,n}$ the inequality (recall that $\hat{x} = x$)

$$\max_{Q_{\hat{r}_i}(\hat{x})} |p| \le \gamma_1 \max_{\varphi(S)\cap Q_{\hat{r}_i}(\hat{x})} |p|,$$

where $\gamma_1 := 2c_1(n,k)\gamma$.

Thus, for every point $\hat{x}$ of the dense subset $\varphi(S_0 \cap S_1)$ of $\varphi(S)$ condition (10.73) is true, i.e., $\varphi(S) \in \mathrm{Mar}_k^*(\mathbb{R}^n)$. □

The next result requires a more involved derivation.

Theorem 10.48. *Every closed subset $S \subset \mathbb{R}^n$ can be presented as disjoint union of a weak k-Markov subset (possibly empty) and a set of Hausdorff dimension at most $n-1$.*

Proof. Due to Proposition 10.47 (a) there exists the maximal weak k-Markov subset of S. Denote it by S_{max} and show that

$$\dim_{\mathcal{H}}(S \setminus S_{max}) \le n - 1$$

proving the result.

Assume, on the contrary, that the Hausdorff dimension of this set is greater than $n-1$. Since by definition

$$\dim_{\mathcal{H}}(S \setminus S_{max}) := \sup\{d > 0\,;\, \mathcal{H}_d(S \setminus S_{max}) = \infty\},$$

where $\mathcal{H}_d$ is the Hausdorff d-measure, there exists a subset $\Omega \subset S \setminus S_{max}$ such that $\mathcal{H}_\delta(\Omega) = \infty$ for some $\delta > n-1$. In turn, by the classical Besicovich theorem [Bes-1952], there exists a compact subset $C \subset \Omega$ such that

$$0 < \mathcal{H}_\delta(C) < \infty.$$

By another Besicovich theorem, see, e.g., [Fe-1969], the upper density of every point $x \in C \setminus C_0$, for some $C_0 \subset C$ such that $\mathcal{H}_\delta(C_0) = 0$, satisfies

$$2^{-\delta} \le \varlimsup_{r\to 0} \frac{\mathcal{H}_\delta(S \cap Q_r(x))}{(2r)^\delta} \le 1.$$

Since $\mathcal{H}_\delta(C_0) = 0$, there exists a compact subset $C_1 \subset C \setminus C_0$ such that $0 < \mathcal{H}_\delta(C_1) < \infty$. Applying to C_1 the latter Besicovich theorem and using compactness of this set one obtains that there exist a subset $C_2 \subset C_1$ and a constant $a > 0$ such that

$$\text{(a)} \quad \mathcal{H}_\delta(C_2) = 0,$$

$$\text{(b)} \quad 1 \le \varlimsup_{r\to 0} \frac{\mathcal{H}_\delta(C_1 \cap Q_r(x))}{r^\delta} \quad \text{for every} \quad x \in C_1 \setminus C_2, \tag{10.80}$$

$$\text{(c)} \quad \mathcal{H}_\delta(C_1 \cap Q_r(x)) \le ar^\delta \quad \text{for every} \quad x \in C_1,\ r > 0.$$

Hence for every $x \in C_1 \setminus C_2$ there exists a decreasing to zero sequence of positive numbers $R(x)$ such that

$$\frac{1}{2} < \frac{\mathcal{H}_\delta(C_1 \cap Q_r(x))}{r^\delta} \quad \text{for all} \quad r \in R(x). \tag{10.81}$$

Now let $r \in R(x)$. By virtue of Theorem 9.25 (which uses inequalities (10.81) and (10.80) (c)) for $Q := Q_r(x)$, $\widetilde{C} := \overline{C_1 \setminus C_2}$ and every polynomial $p \in \mathcal{P}_{k,n}$,

$$\sup_Q |p| \le \gamma \sup_{Q \cap \widetilde{C}} |p|, \tag{10.82}$$

where γ depends only on k, n, a, δ.

We conclude that $\widetilde{C}$ is subject to Definition 10.45 and therefore $S_{\max} \cup \widetilde{C}$ is a weak k-Markov set containing in S, a contradiction with maximality of S_{max}. □

Example 10.49. (a) Let us recall that a closed set $S \subset \mathbb{R}^n$ is (Ahlfors) *d-regular* ($0 \le d \le n$) if for every cube $Q_r(x)$ with $x \in S$ and $0 < r < \operatorname{diam} S$,

$$c_0 r^d \le \mathcal{H}_d(S \cap Q_r(x)) \le c_1 r^d,$$

where $c_0, c_1 > 0$ are constants independent of x and r.

It was proved in [JW-1984, Ch.2] that a d-regular compact set with $d > n - 1$ is Markov. This, Proposition 10.47 (a) and (10.74) imply that the closure of the union of d_i-regular compact sets $S_i \subset \mathbb{R}^n$ with $d_i > n - 1$, $i \in I$, is weak k-Markov for all $k \ge 1$.

(b) Let a closed set $S \subset \mathbb{R}^n$ have a base (in the relative topology) consisting of sets of Hausdorff dimension greater than $n - 1$. Then S is weak k-Markov for all $k \ge 1$. For otherwise, nonempty open set $S \setminus S_{\max}$, where $S_{\max}$ is the maximal weak k-Markov subset of S, contains an element of the base with Hausdorff dimension greater than $n - 1$. This contradicts Theorem 10.48.

Using this fact we can prove, e.g., that the graph of the Weierstrass nowhere differentiable function

$$\sum_{n=0}^{\infty} a^n \cos(b^n x), \qquad \text{where} \quad 0 < a < 1,\ b > 1,\ ab > 1,$$

is weak k-Markov for all $k \ge 0$.

Indeed, it was proved by Przytycki and Urbański [PU-1989] that this graph satisfies the above base topology assumption (for $n = 2$).

(c) It might be concluded from the previous example that weak k-Markov sets with $k \ge 1$ should be sufficiently massive. However, they may have arbitrarily small Hausdorff dimension. For instance, specializing of throwing cubes we may obtain a Cantor set C_n in $\mathbb{R}^n$ which is d-regular with arbitrary small

$d > 0$ and is a direct product of $\frac{d}{n}$-regular one-dimensional Cantor sets. The latter are Markov by the result formulated in (a) (for $n = 1$). Hence, C_n is weak k-Markov by Proposition 10.47 (b).

(d) If a closed set $S \subset \mathbb{R}^n$ has an isolated point, say x_0, then it does not belong to $\mathrm{Mar}_k^*(\mathbb{R}^n)$ for any $k \ge 1$ (but, clearly, belongs to $\mathrm{Mar}_0^*(\mathbb{R}^n)$ as every closed set). In fact, for $p(x) := (x - x_0)^\alpha$, $x \in \mathbb{R}^n$, where $|\alpha| = k \ge 1$, and all sufficiently small $r > 0$ we have

$$\sup_{Q_r(x_0)} |p| > 0 \quad \text{while} \quad \sup_{S \cap Q_r(x_0)} |p| = 0.$$

(e) Clearly, $\mathrm{Mar}_0^*(\mathbb{R}^n)$ consists of all nonempty closed sets in $\mathbb{R}^n$. It is worth noting that $\mathrm{Mar}_1^*(\mathbb{R})$ consists of all closed subsets of the real line without isolated points (including those of Hausdorff dimension 0).

Remark 10.50. Unlike the case of Markov sets the classes $\mathrm{Mar}_k^*(\mathbb{R}^n)$ with different $k \ge 1$ do not coincide. We briefly describe the corresponding example for the case $k = 1$ and 2 (due to Proposition 10.47(b) it suffices to consider the case of $n = 1$).

Set $c_k := 2^{-k^2}$, $k \in \mathbb{N}$, and write for an ordered subset $K := \{k_1, \dots, k_n\} \subset \mathbb{N}$ (i.e., $k_1 < k_2 < \cdots < k_n$), where n may be $+\infty$,

$$c_K := c_{k_1} + c_{k_1} c_{k_2} + \cdots + c_{k_1} \cdots c_{k_j} + \cdots . \tag{10.83}$$

It can be checked that $S := \{c_K \, ; \, K \subset \mathbb{N}\}$ is a compact subset of $\mathbb{R}$ without isolated points and every $x \in S$ can be uniquely presented in the form (10.83). Moreover, $c_K < c_{\widetilde{K}}$ if and only if $K > \widetilde{K}$ (in the lexicographic order, i.e., there exists $\ell \ge 1$ such that $k_i = \widetilde{k}_j$ for $1 \le i \le \ell$ and $k_{\ell+1} > \widetilde{k}_{\ell+1}$).

Using these properties, presentation (10.83) and a compactness argument we can derive from inequalities

$$\sup_{Q_{r_i}(x)} |p| \le \gamma(x) \sup_{S \cap Q_{r_i}(x)} |p|, \quad x \in S,$$

that hold for some sequence $\{r_i\} \subset \mathbb{R}_+$ tending to 0 and all $p \in \mathcal{P}_{2,1}$, the following inequality,

$$\max_{[0,1]} |p| \le \gamma(x) \max\{|p(0)|, |p(1)|\}$$

which is clearly untrue.

Hence, $S \notin \mathrm{Mar}_2^*(\mathbb{R})$ but belongs to $\mathrm{Mar}_1^*(\mathbb{R})$ by Example 10.49(d).

In the following result concerning the finiteness property for weak Markov sets we set for brevity

$$M_k^* := \mathrm{Mar}_k^*(\mathbb{R}^n), \quad X := C^{k,\omega}(\mathbb{R}^n).$$

Theorem 10.51. *It is true that* $\mathcal{F}_{M_k^*}(X) \le 2\binom{n+k}{n}$.

Proof. We begin with three basic facts exploited in the derivation. The first is a simple reformulation of the Whitney–Glaeser Theorem 2.19 of Volume I.

Proposition 10.52. *Given a set $S \subset \mathbb{R}^n$ and a family of polynomials $\{p_x\}_{x\in S} \subset \mathcal{P}_{k,n}$, there exists a function $F \in X$ whose Taylor polynomials of order k satisfy*

$$T_x^k(F) = p_x \text{ for all } x \in S \tag{10.84}$$

if and only if for some constant $\lambda > 0$ and all $x, y \in S$ and $z \in \{x, y\}$

$$\max_{|\alpha|\le k} \left|D^\alpha p_x(x)\right| \le \lambda \quad \text{and} \quad \max_{|\alpha|\le k} \frac{|D^\alpha(p_x - p_y)(z)|}{\|x-y\|^{k-|\alpha|}} \le \lambda\omega\big(\|x-y\|\big). \tag{10.85}$$

Moreover, the equivalence

$$\inf\big\{\|F\|_X\,; T_x^k(F) = p_x,\ x \in S\big\} \approx \inf\lambda$$

holds with constants depending only on k, n.

Proof. It suffices to write

$$p_x(y) := \sum_{|\alpha|\le k} f_\alpha(x)\,\frac{(y-x)^\alpha}{\alpha!}$$

and note that condition (10.85) is equivalent to the assumptions of Theorem 2.19 of Volume I for the k-jet $\{f_\alpha\}_{|\alpha|\le k}$. □

Remark 10.53. Whitney's extension construction exploits only a countable subfamily of the family $\{p_x\}_{x\in S}$. In particular, it suffices to take a subfamily indexed by a dense subset of S, see, e.g., Lemma 10.59 below.

The second fact is a compactness result for the space $C^{k,\omega}(\mathbb{R}^n)$.

Proposition 10.54. *Let a sequence $\{f_j\}_{j\in\mathbb{N}} \subset C^{k,\omega}(\mathbb{R}^n)$ be bounded. Then there exists a subsequence $\{f_j\}_{j\in J}$ and a function $f \in C^{k,\omega}(\mathbb{R}^n)$ such that for every closed cube $Q \subset \mathbb{R}^n$,*

$$\lim_{j\in J} \|f - f_j\|_{C^k(Q)} = 0$$

and, moreover,

$$\max_{|\alpha|=k} \sup_{t>0} \frac{\omega_1(t\,; D^\alpha f)}{\omega(t)} \le \sup_{j\in\mathbb{N}} \|f_j\|_{C^{k,\omega}(\mathbb{R}^n)}.$$

Proof. Without loss of generality, we assume that the sequence lies in the unit ball of $C^{k,\omega}(\mathbb{R}^n)$. Then for every $j \ge 1$ and $|\alpha| = k$ we have, for a fixed closed cube $Q \subset \mathbb{R}^n$,

$$\max_Q |D^\alpha f_j| \le 1 \quad \text{and} \quad \sup_{t>0} \frac{\omega_1(t\,; D^\alpha f_j)}{\omega(t)} \le 1. \tag{10.86}$$

Disposing the set of multi-indices $\{\alpha\,;|\alpha|=k\}$ in some order $\alpha^1,\alpha^2,\dots$ we apply the Arcela–Ascoli compactness theorem to the sequence $\{D^\alpha f_j\}_{j\in\mathbb{N}}$ with $\alpha=\alpha^1$. We then find a subsequence $\{D^\alpha f_j\}_{j\in J_1}$ and a function $f_\alpha\in C(Q)$ with this α such that

$$\lim_{j\in J_1}\|f_\alpha-D^\alpha f_j\|_{C(Q)}=0. \tag{10.87}$$

Then we apply the Arcela–Ascoli criterion to the sequence $\{D^\alpha f_j\}_{j\in J_1}$ with $\alpha=\alpha^2$ to find a subsequence $\{D^\alpha f_j\}_{j\in J_2}$, where $J_2\subset J_1$, and a function $f_\alpha\in C(Q)$ so that (10.87) holds for $j\in J_2$ and $\alpha=\alpha^2$.

Proceeding this way we finally obtain a subsequence $\{f_j\}_{j\in J_d}$, where $d:=\operatorname{card}\{\alpha\,;|\alpha|=k\}$, and a family of functions $\{f_\alpha\}_{|\alpha|=k}\subset C(Q)$ such that $\lim_{j\in J_d}\|D^\alpha f_j-f_\alpha\|_{C(Q)}=0$ for all α.

We rename J_d by $I(k)$ and proceed with the derivation for $|\alpha|=k-1$. Using the norm

$$\|f\|^*:=\max_{|\alpha|\le k}\sup_{\mathbb{R}^n}|D^\alpha f|+|f|_{C^{k,\omega}(\mathbb{R}^n)}$$

equivalent to the original norm $\|f\|_{C^{k,\omega}(\mathbb{R}^n)}\Big(:=\sup_{\mathbb{R}^n}|f|+|f|_{C^{k,\omega}(\mathbb{R}^n)}\Big)$, we derive from the boundedness of $\big\{\|f_j\|^*\big\}_{j\in I(k)}$ an inequality analogous to (10.86), but now for $j\in I(k)$ and $|\alpha|=k-1$:

$$\max_Q|D^\alpha f_j|\le c\quad\text{and}\quad\sup_{t>0}\frac{\omega_1(t\,;D^\alpha f_j)}{t}\le c,$$

where c depends only on k,n.

Then we apply the already used argument to find a subsequence $I(k-1)\subset I(k)$ and a family of functions $\{f_\alpha\}_{|\alpha|=k-1}\subset C(Q)$ such that for every $|\alpha|=k-1$,

$$\lim_{j\in I(k-1)}\|D^\alpha f_j-f_\alpha\|_{C(Q)}=0.$$

Proceeding this way for $k-2,\dots,0$, we finally obtain a subsequence $\{f_j\}_{j\in I(0)}$ and a family $\{f_\alpha\}_{|\alpha|\le k}\subset C(Q)$ such that for every $|\alpha|\le k$,

$$\lim_{j\in I(0)}\|D^\alpha f_j-f_\alpha\|_{C(Q)}=0.$$

Let us show that $\{f_j\}_{j\in I(0)}$ is the desired subsequence. To this end we write the Taylor formula for f_j and estimate the remainder using the second inequality of (10.86). Then, for $x,y\in Q$, we have

$$f_j(x)=\sum_{|\alpha|\le k}\frac{D^\alpha f_j(x)}{\alpha!}(x-y)^\alpha+R_k(f_j\,;x,y)$$

where

$$\big|R_k(f_j\,;x,y)\big|\le c(k,n)\|x-y\|^k\omega\big(\|x-y\|\big).$$

Passing to the limit along the subsequence $I(0)$ we then obtain for these x, y,

$$\left| f_0(x) - \sum_{|\alpha| \le k} \frac{f_\alpha(x)}{\alpha!} (x-y)^\alpha \right| \le c(k,n)\|x-y\|^k \omega\big(\|x-y\|\big).$$

According to Theorem 2.34 of Volume I, this inequality implies that $f_0 \in C^k(Q)$ and, moreover, $D^\alpha f_0 = f_\alpha$.

Finally, passing to the limit in the second inequality (10.86) and then letting Q to $\mathbb{R}^n$, we obtain the remaining required inequality

$$\max_{|\alpha|=k} \sup_{t>0} \frac{\omega_1(t\,; D^\alpha f_0)}{\omega(t)} \le 1 = \sup_{j \in \mathbb{N}} \|f_j\|_{C^{k,\omega}(\mathbb{R}^n)}.$$

The result has been proved. □

The third fact concerns interpolating properties of subsets satisfying the Markov type inequality.

Proposition 10.55. *Let S be a subset of the closed unit cube $Q := Q_1(0)$ in $\mathbb{R}^n$ such that for some constant $\gamma > 0$ and all polynomials $p \in \mathcal{P}_{k,n}$,*

$$\max_Q |p| \le \gamma \sup_S |p|. \tag{10.88}$$

Then there exists a finite subset $F \subset S$ and a constant $c = c(k,n,\gamma) > 0$ such that

(a) $\operatorname{card} F = \dim \mathcal{P}_{k,n}$;

(b) *For every $p \in \mathcal{P}_{k,n}$,*

$$\sup_Q |p| \le c \sup_F |p|.$$

In particular, F is a minimal interpolating set for polynomials of degree k on $\mathbb{R}^n$.

Proof. We set, for brevity, $d := \dim \mathcal{P}_{n,k}$ and denote by $\{g_j\}_{1\le j\le d}$ the set of monomials $\{x^\alpha\}_{|\alpha|\le k}$ ordered lexicographically. Given a finite subset $\{x^j\}_{1\le j\le d} \subset S \subset Q$ and a polynomial $p \in \mathcal{P}_{k,n}$, we consider the linear system

$$\sum_{i=1}^d a_i g_i(x^j) = p(x^j), \quad 1 \le j \le d.$$

Let us assume for a moment that the matrix $G := \big(g_i(x^j)\big)_{1\le i,j\le d}$ is invertible. Then by the Kramer rule, for $1 \le i \le d$,

$$a_i = \frac{\sum_{j=1}^d \varphi_{ij} p(x^j)}{\det G},$$

where φ_{ij} are polynomials in variables $g_i(x^j)$ of degree $d-1$ whose coefficients equal ± 1. Since all $x^j \in Q$ and every monomial g_i is bounded in absolute value by 1 on $Q := Q_1(0)$, we then get, for some $c = c(k,n)$,

$$|a_i| \le c\,\frac{\max\limits_{1\le j\le d} |p(x^j)|}{|\det G|}\,.$$

If we show that there exists a subset $\{x^j\}_{1\le j\le d}$ in the set S such that

$$|\det G| \ge c_1 > 0,$$

where c_1 depends only on k, n and γ, then this d-tuple satisfies the assertion of the proposition. Actually, if $p \in \mathcal{P}_{k,n}$, it can be written as $p = \sum\limits_{i=1}^{d} a_i g_i$ and therefore

$$\max_Q |p| \le d \max_i |a_i| \le \frac{dc}{c_1} \max_{\{x^j\}} |p|,$$

as required.

Hence it remains to prove

Lemma 10.56. *There exists a subset $F := \{x^j\}_{1\le j\le d} \subset S$ and a constant $c = c(k,n,\gamma) > 0$ such that*

$$\big|\det\big(g_i(x^j)\big)\big| \ge c.$$

Proof. We will exploit a (Veronese) map $V : \mathbb{R}^n \to \mathbb{R}^d$ given by

$$V(x) := \big(g_i(x)\big)_{1\le i\le d}.$$

Let $\widehat{Q}$ denote the symmetric convex hull of $V(Q)$, i.e., $\widehat{Q} := \mathrm{conv}\big(V(Q) \cup (-V(Q))\big)$, and $\widehat{S}$ is the symmetric convex hull of $V(S)$. We verify that

$$\dim \widehat{Q} = d. \tag{10.89}$$

It suffices to show that $V(Q)$ is not contained in a hyperplane of $\mathbb{R}^d$, say H. But if, on the contrary, $V(Q) \subset H$, then for some nonzero vector $\ell \in \mathbb{R}^d$ and all $x \in Q$,

$$\langle \ell, V(x)\rangle := \sum_{i=1}^{d} \ell_i g_i(x) = 0.$$

This means that the d-cube Q is contained in a $(d-1)$-dimensional polynomial surface, a contradiction.

By the definition of a dual space, $(\mathbb{R}^d)^*$ can be identified with the space of polynomials $\mathcal{P}_{k,n}$. Then inequality (10.88) implies that for every $\varphi \in (\mathbb{R}^d)^*$,

$$\sup_{V(Q)} |\varphi| \le \gamma \sup_{V(S)} |\varphi|.$$

Since the supremum of a linear functional on a subset of $\mathbb{R}^d$ equals its supremum on the symmetric convex hull of this subset, we conclude that

$$\sup_{\widehat{Q}} |\varphi| \leq \gamma \sup_{\widehat{S}} |\varphi| \tag{10.90}$$

for every $\varphi \in (\mathbb{R}^d)^*$.

We derive from this that

$$\widehat{Q} \subset \gamma\widehat{S} := \{\gamma x\,; x \in \widehat{S}\}. \tag{10.91}$$

If, on the contrary, there exists a point $x \in \widehat{Q} \backslash \widehat{S}$, then the Hahn–Banach theorem implies existence of the functional $\varphi \in (\mathbb{R}^d)^*$ separating x and $\gamma\widehat{S}$, i.e., such that

$$\sup_{\gamma\widehat{S}} |\varphi| < |\varphi(x)| \big(\leq \sup_{\widehat{Q}} |\varphi|\big).$$

But the left-hand side equals $\gamma \sup_{\widehat{S}} |\varphi|$ and the inequality contradicts (10.90).

Further, $\det\big(g_i(x^j)\big)$ is, up to the factor $\pm\frac{1}{d!}$, equal to the d-volume of the simplex $\operatorname{conv}\big(V(x^1), \dots, V(x^d)\big)$. Hence, it suffices to find a d-tuple $\{x^1, \dots, x^d\} \subset S$ such that

$$\big|\operatorname{conv}\{V(x^1), \dots, V(x^d)\}\big| \geq c(k, n, \gamma) > 0, \tag{10.92}$$

where $|\cdot|$ stands hereafter for the Lebesgue d-measure.

To do this, we use the John theorem [Jo-1948] to find an ellipsoid $\mathcal{E} \subset \mathbb{R}^d$ centered at 0 such that

$$\mathcal{E} \subset \widehat{S} \subset \sqrt{d}\mathcal{E}. \tag{10.93}$$

Due to (10.91) and (10.89), the set $\widehat{S}$ is of dimension d, and therefore $\mathcal{E}$ also is.

Let $\Delta := \operatorname{conv}\{s^j\}_{1\leq j\leq d}$ be the simplex of maximal volume inscribed into $\mathcal{E}$. To evaluate its volume, we use the map $(s_1, \dots, s_d) \mapsto (a_1 s_1, \dots, a_d s_d)$ where the numbers a_i are the halflengths of the principal axes of $\mathcal{E}$. Then the maximal simplex Δ is the image of the regular d-simplex inscribed in the unit Euclidean $(d-1)$-sphere. Denoting the volume of the latter by σ_d we then have

$$|\Delta| = \sigma_d \prod_{i=1}^{d} a_i = \frac{\sigma_d}{\beta_d}\, |\mathcal{E}|,$$

where β_d is the volume of the unit Euclidean d-ball. This, (10.93) and (10.90) yield

$$|\Delta| \geq d^{-\frac{d}{2}} \frac{\sigma_d}{\beta_d}\, |\widehat{S}| \geq c(k, n)\gamma^{-d} |\widehat{Q}|.$$

On the other hand, $\dim \widehat{Q} = d$, and therefore its d-volume is not zero and obviously depends only on $d = d(k,n)$ and n. Hence, we finally have

$$\big|\det(s^1, \dots, s^d)\big| \geq \tilde{c}(k,n)\gamma^{-d} > 0. \tag{10.94}$$

We use this to find the desired d-tuple satisfying (10.92) and in this way complete the proof. By the Carathéodory theorem, see, e.g., [DGK-1963], every point s^j of the convex hull $\widehat{S} := \operatorname{conv}\big(V(S) \cup (-V(S))\big)$ is a convex combination of $d+1$ points of the initial set. Hence, there exist a $(d+1)$-tuple of points $\{x^{ji}\}_{1\leq i\leq d+1} \subset S$ and a sequence $\{\lambda_{ji}\}_{1\leq i\leq d} \subset \mathbb{R}$ such that

$$s^j = \sum_{i=1}^{d+1} \lambda_{ji} V(x^{ji}) \quad \text{and} \quad \sum_{i=1}^{d+1} |\lambda_{ji}| = 1,\ 1 \leq j \leq d+1.$$

Since the determinant is a multilinear function of its entries, we then have

$$\big|\det(s^1, \dots, s^d)\big| \leq \max\big|\det(y^1, \dots, y^d)\big|,$$

where the maximum is taken over all d-tuples $\{y^j\}_{1\leq j\leq d}$ such that

$$y^j \in \{V(x^{j1}), \dots, V(x^{j\,d+1})\}.$$

If the maximum is attained at $\{\bar{y}^1, \dots, \bar{y}^d\}$, then the above inequality and (10.94) give

$$\big|\det(\bar{y}^1, \dots, \bar{y}^d)\big| \geq \tilde{c}(k,n)\gamma^{-d} > 0.$$

The left-hand side divided by $d!$ equals $|\operatorname{conv}(\bar{y}^1, \dots, \bar{y}^d)|$. Hence, the desired inequality (10.92) holds for $V(x^i) := \bar{y}^i$ and the constant $\frac{\tilde{c}(n,k)\gamma^{-d}}{d!}$. □

The proposition has been proved. □

Now we are in a position to prove the required result.

Let a set S belong to $M_k^* := \operatorname{Mar}_k^*(\mathbb{R}^n)$ and let the trace of a function $f : S \to \mathbb{R}$ to every subset of S of cardinality at most $2\binom{n+k}{n}$ admit an extension to a function from the unit ball of the space X. We must prove that the function f belongs to the trace space $X|_S$ and its trace norm is bounded by a constant $c = c(n,k)$.

To this end we construct a family of polynomials $\{T_x^k\}_{x\in S}$ of degree k satisfying the assumptions of the Whitney-Glaeser Proposition 10.52 with $\lambda = c(k,n) > 0$ such that $T_x^k(x) = f(x)$ for all $x \in S$. As was noted in Remark 10.53 we may work with a dense subset S_0 of S. In particular, S_0 may be taken as the subset of $S\,(\in M_k^*)$ subject to Definition 10.45. Since it suffices to obtain the required extension for $f|_{S_0}$, we may and will assume below that $S_0 = S$, i.e., every point of S satisfies condition (10.73) of this definition. If the family $\{T_x^k\,;\, x \in S\}$ would be constructed the cited proposition were implied that there exists a function $F \in X$

of the norm bounded by a constant depending only on k, n such that $F(x) = T_x^k(x)$ for all $x \in S$. This clearly means that F is an extension of f and

$$\|f\|_{X|_S} \le \|F\|_X \le c(k,n),$$

as required.

We define the desired polynomial T_x^k for $x \in S$ using a compactness argument. Due to Definition 10.45, there exist a sequence of positive numbers $\{r_j\}$ tending to 0 and a constant $\gamma(x) > 0$ such that for every polynomial $p \in \mathcal{P}_{k,n}$ and $r = r_j$,

$$\max_{Q_r(x)} |p| \le \gamma(x) \sup_{Q_r(x)\cap S} |p|.$$

Applying Proposition 10.55 to the polynomial $x \mapsto p(rx)$, where $p \in \mathcal{P}_{k,n}$, we derive from the previous inequality existence of a subset $S_r(x) \subset Q_r(x) \cap S$ such that

(a) card $S_r(x) = \binom{n+k}{n}$;

(b) for every $p \in \mathcal{P}_{k,n}$ and $r = r_j$, $j \in \mathbb{N}$,

$$\max_{Q_r(x)} |p| \le c\big(k, n, \gamma(x)\big) \sup_{S_r(x)} |p|. \tag{10.95}$$

By the assumption of the theorem there exists a function $F_j \in X$, $j \in \mathbb{N}$, such that

$$F_j = f \text{ on } S_{r_j}(x) \tag{10.96}$$

and, moreover,

$$\|F_j\|_X \le 1. \tag{10.97}$$

In view of the last inequality, the family $\{F_j\}_{j\in\mathbb{N}}$ satisfies the compactness criterion of Proposition 10.54. Hence, a subsequence of $\{F_j\}_{j\in\mathbb{N}}$ is C^k convergent on every closed cube of $\mathbb{R}^n$ to a function $F \in C^k(\mathbb{R}^n)$ such that $\|F\|_X \le 1$.

Now the desired polynomial T_x^k is determined as the Taylor polynomial of F at x of degree k. To prove that T_x^k is well-defined, we should show that it is independent of the choice of a sequence satisfying (10.96) and (10.97).

Let $\{F_j'\}_{j\in\mathbb{N}}$ be such a sequence distinct from $\{F_j\}_{j\in\mathbb{N}}$. Then

$$\|F_j - F_j'\|_X \le 2 \tag{10.98}$$

and, moreover,

$$F_j - F_j' = 0 \text{ on } S_{r_j}(x). \tag{10.99}$$

Let T_j and T'_j be the Taylor polynomials at x of degree k for F_j and F'_j, respectively. According to the Taylor formula (the necessary condition of Proposition 10.52) and (10.98), for every $z \in S_{r_j}(x) \subset Q_{r_j}(x)$,

$$\big|(T_j - T'_j)(z)\big| \le c(k,n)\|x-z\|^k\,\omega\big(\|x-z\|\big) \le c(k,n)r_j^k\,\omega(r_j). \tag{10.100}$$

This and (10.95) then imply

$$\max_{Q_{r_j}(x)} |T_j - T'_j| \le c\big(k,n,\gamma(x)\big)r_j^k\,\omega(r_j).$$

Now we apply the polynomial inequality of Lemma 9.5 with $n = 1$ for the restriction of $T_j - T'_j$ to a straight line passing through x. Along with the previous inequality this yields

$$\begin{aligned} &\max_{Q_1(x)} |T_j - T'_j| \le c(n,k)\Big(\frac{1}{r_j}\Big)^k \max_{Q_{r_j}(x)} |T_j - T'_j| \\ &\le c(n,k)c\big(n,k,\gamma(x)\big)\omega(r_j). \end{aligned} \tag{10.101}$$

Finally, let $J \subset \mathbb{N}$ be a subsequence such that for some functions $F, F' \in C^k(\mathbb{R}^n)$,

$$\lim_J F_j = F, \quad \lim_J F'_j = F' \quad \text{(convergence in } C^k(Q_1(x))\text{)}.$$

Then T_j and T'_j tend to the Taylor polynomials at x of order k for F and F', respectively. But due to (10.101) these polynomials coincide.

Hence, the polynomial T_x^k, $x \in S$, does not depend on the choice of $\{F_j\}$, as required.

Now we show that the family $\{T_x^k\}_{x\in S}$ satisfies the assumptions of Proposition 10.52. To this end given two fixed points $x \ne y$ of S and $r_j, \tilde{r}_j > 0$ we find the corresponding finite subsets $S_{r_j}(x)$ and $S_{\tilde{r}_j}(y)$, $j \in \mathbb{N}$. Since

$$\operatorname{card}\big(S_{r_j}(x) \cup S_{\tilde{r}_j}(y)\big) \le 2\binom{n+k}{n},$$

the assumption of the theorem implies existence of a function F_j such that

$$F_j = f \text{ on } S_{r_j}(x) \cup S_{\tilde{r}_j}(y)$$

and, moreover,

$$\|F_j\|_X \le 1 \text{ for all } j \in \mathbb{N}.$$

Employing again the just used compactness argument we find a subsequence $\{F_j\}_{j\in J}$ and a function $F \in C^k(\mathbb{R}^n)$ such that the limit of $\{F_j\}_{j\in J}$ in $C^k(Q)$ equals $F|_Q$ for any cube Q (in particular, for a cube containing x and y).

Passing then to the limit as $r_j, \tilde{r}_j \to 0$ in the previous two relations we get

$$F(x) = f(x), \;\; F(y) = f(y) \text{ and } \|F\|_X \le 1. \tag{10.102}$$

Now let $T_x^k(F)$ and $T_y^k(F)$ be the Taylor polynomials of degree k at the points x and y of S, respectively. Both of them are defined by the sequence $\{F_j\}_{j\in\mathbb{N}}$ and therefore belong to the above introduced family $\{T_z^k\}_{z\in S}$, i.e., $T_z^k(F) = T_z^k$ for $z \in \{x, y\}$.

Since $\|F\|_X \le 1$, the Taylor formula for F yields for $z \in \{x, y\}$

$$\big|(T_x^k - T_y^k)(z)\big| \le c(k,n)\|x-y\|^k\,\omega\big(\|x-y\|\big).$$

This and the first equality in (10.102) show that the sufficiency condition of Proposition 10.52 holds. Hence, there exists the required function $g \in X := C^{k,\omega}(\mathbb{R}^n)$ which extends f and is such that $\|g\|_X \le c(n,k)$.

The theorem has been proved. □

10.2.4 Linearity problem for Markov sets

In the formulation and the proof of the next result we use the notation $X := \Lambda^{k,\omega}(\mathbb{R}^n)$ and $M := \mathrm{Mar}(\mathbb{R}^n)$.

Theorem 10.57. *Let $S \in M$. Then there exists a linear extension operator from $X|_S$ into X whose norm and depth are bounded by a constant depending only on S, k, n* [1].

Proof. Given a cube $Q \in \mathcal{K}_S$, we apply Proposition 10.55 and Theorem 9.21 to find a subset denoted by F_Q satisfying the conditions:

(a) $F_Q \subset Q \cap S$ and $\operatorname{card} F_Q = \dim \mathcal{P}_{k-1,n}$;

(b) for every polynomial $p \in \mathcal{P}_{k-1,n}$,

$$\max_Q |p| \le c \max_{F_Q} |p| \tag{10.103}$$

where $c = c(k,n,S)$.

The latter condition implies that if $p|_{F_Q} = 0$ for a polynomial $p \in \mathcal{P}_{k-1,n}$, then $p = 0$. Hence, for every function $f \in \ell_\infty(S)$ there exists a (unique) polynomial denoted by $J_Q(f)$ that interpolates f on F_Q.

Now we introduce the desired extension operator by setting, for $f : S \to \mathbb{R}$,

$$Tf := \begin{cases} f & \text{on } S, \\ \sum\limits_{Q\in\mathcal{W}_S} J_{\widehat{Q}}(f)\varphi_Q & \text{on } S^c. \end{cases}$$

Here $\widehat{Q} \in \mathcal{K}_S$ is obtained by shifting the Whitney cube Q to match its center c_Q with the closest to Q point of S.

[1] More precisely, the norm is bounded by the constant $c(k,n,\gamma)$, where $\gamma = \gamma(k,n,S)$ is the constant in the Markov type inequality of Theorem 9.21, while the depth is bounded by $c(k,n)$.

Let us compare this with the extension operator of Theorem 9.3. As a matter of fact, the only point required in the proof of Theorem 9.3 is a family of polynomials $\{p_Q\}_{Q\in\mathcal{W}_S} \subset \mathcal{P}_{k-1,n}$ whose members satisfy the inequality

$$\sup_{\widehat{Q}\cap S} |f - p_Q| \le cE_k(\widehat{Q}\cap S\,; f)$$

where $c = c(k, n, S)$.

Hence, if the family $\{J_{\widehat{Q}} f\}_{Q\in\mathcal{W}_S}$ meets this property, then, by the argument of Theorem 9.3, T will be a (clearly linear) extension operator whose norm is bounded by a constant $c = c(k, n, \gamma)$.

To prove the required inequality, we use the interpolation property of the operator $J_{\widehat{Q}}$ and write

$$\begin{aligned}\sup_{\widehat{Q}\cap S} |f - J_{\widehat{Q}}(f)| &= \sup_{\widehat{Q}\cap S}\big|\big(f - p_{\widehat{Q}}(f)\big) + J_{\widehat{Q}}\big(p_{\widehat{Q}}(f) - f\big)\big| \\ &\le \big(1 + \|J_{\widehat{q}}\|\big)\big\|f - p_{\widehat{Q}}(f)\big\|_{\ell_\infty(\widehat{Q}\cap S)}\end{aligned}$$

where $\|J_{\widehat{Q}}\|$ is the norm of $J_{\widehat{Q}}$ in $\ell_\infty(\widehat{Q}\cap S)$.

By (10.103), $\|J_{\widehat{Q}}\| \le \gamma$ and therefore

$$\sup_{\widehat{Q}\cap S}\big|f - J_{\widehat{Q}}(f)\big| \le (1+\gamma)E_k(\widehat{Q}\cap S\,; f),$$

as required.

It remains to estimate the depth of T, i.e., the number of terms in the equality

$$Tf(x) = \sum \lambda_i(x) f(x^i) \tag{10.104}$$

where $x \in S^c$ and x^i are points of S. To this end we, given $x \in S^c$, denote by $\mathcal{W}_S(x)$ the set $\big\{\operatorname{supp}\varphi_Q\,; Q \in \mathcal{W}_S \text{ and } \operatorname{supp}\varphi_Q \ni x\big\}$. According to the Whitney construction, see Volume I, Section 2.2, $\operatorname{card}\mathcal{W}_S(x) \le c(n)$.

Hence, the decomposition

$$Tf(x) = \sum_{Q\in\mathcal{W}_S(x)} J_{\widehat{Q}}(f)(x)\varphi_Q(x)$$

consists of at most $c(n)$ terms.

Further, by the Lagrange interpolation formula,

$$(J_{\widehat{Q}} f)(x) = \sum_{y\in F_{\widehat{Q}}} \ell_y(x) f(y)$$

where ℓ_y is a polynomial of degree $k-1$ which equals 1 at y and 0 on $F_{\widehat{Q}}\backslash\{y\}$.

Inserting this into the sum for $Tf(x)$ we bound the number of terms in (10.104) by $(\operatorname{card} F_{\widehat{Q}})(\operatorname{card}\mathcal{W}_S(x)) \le \binom{n+k-1}{n} c(n)$.

This completes the proof. $\square$

10.2.5 Linearity problem for weak Markov sets

Now we present a solution of the linearity problem for traces of spaces $C^{k,\omega}(\mathbb{R}^n)$ to weak Markov sets.

Theorem 10.58. *If $S \in M_k^*$ $(:= \mathrm{Mar}_k^*(\mathbb{R}^n))$, then there exists a linear bounded extension operator from $X|_S := C^{k,\omega}(\mathbb{R}^n)|_S$ to $X := C^{k,\omega}(\mathbb{R}^n)$ whose norm and depth are bounded by constants depending only on n and k.*

Proof. The desired linear extension operator $E : X|_S \to X$ where $S \in M_k^*$ is a modification of the classical Whitney extension construction. The latter is used, e.g., in the proof of Proposition 10.52 to recover the required function $F \in X$ there by the formula

$$F(x) := \begin{cases} p_x(x) & \text{for } x \in S, \\ \displaystyle\sum_{Q \in \mathcal{W}_S} p_{x(Q)}(x)\varphi_Q(x) & \text{for } x \in S^c := \mathbb{R}^n \setminus S. \end{cases} \tag{10.105}$$

We recall some properties of the involved ingredients in a form used in the forthcoming derivation. In their formulations λQ, $\lambda > 0$, stands for $Q_{\lambda r}(x)$ provided $Q := Q_r(x)$, and all distances are measured in the ℓ_∞ norm of $\mathbb{R}^n$, i.e., $\|x\| := \max_{1 \le i \le n} |x_i|$.

Lemma 10.59. (a) *$\mathcal{W}_S$ is a family of closed dyadic cubes in S^c with pairwise disjoint interiors covering S^c.*

(b) *$\mathcal{W}_S^* := \{Q^* := \frac{9}{8}Q\,;\, Q \in \mathcal{W}_S\}$ is a family of cubes in S^c covering this set with multiplicity*[2]

$$\mathrm{mult}(\mathcal{W}_S^*) \le c(n). \tag{10.106}$$

(c) *For every $Q := Q_r(x) \in \mathcal{W}_S$ and $y \in Q^*$ the point $x(Q)$ belongs to S and*

$$\|x(Q) - y\| \approx d(Q, S) \approx r \tag{10.107}$$

with constants of equivalence depending only on n.

(d) *$\{\varphi_Q\}_{Q \in \mathcal{W}_S}$ is a C^∞ partition of unity subordinate to $\mathcal{W}_S^*$ such that for every $\alpha \in \mathbb{Z}_+^n$ and $Q := Q_r(x) \in \mathcal{W}_S$,*

$$\max_{\mathbb{R}^n} |D^\alpha \varphi_Q| \le c(\alpha, n) r^{-|\alpha|}. \tag{10.108}$$

Remark 10.60. $x(Q)$ is an almost closest to Q point of S, e.g., we may take any $x(Q) \in S$ satisfying

$$d(x(Q), Q) \le 2d(S, Q). \tag{10.109}$$

[2] i.e., every point of $x \in S^c$ is covered by at most $c(n)$ cubes from $\mathcal{W}_S^*$.

Now let $f \in X|_S$ where

$$\|f\|_{X|_S} \le 1 \tag{10.110}$$

and S_0 be a dense subset of $S \in M_k^*$ subject to Definition 10.45. If we select points $x(Q)$ from S_0, then for every $Q \in \mathcal{W}_S$ there exist a decreasing to zero sequence $\mathcal{R}_Q \subset \mathbb{R}_+$ and a family of finite subsets $\{S_r(x(Q))\}_{r\in\mathcal{R}_Q}$ of S such that

$$S_r(x(Q)) \subset S \cap Q_r(x(Q)) \quad \text{and} \quad \operatorname{card} S_r(x(Q)) = \binom{n+k}{k}, \tag{10.111}$$

and for every $p \in \mathcal{P}_{k,n}$,

$$\max_{Q_r(x(Q))} |p| \le c(k,n,x(Q)) \sup_{S_r(x(Q))} |p|, \tag{10.112}$$

see Proposition 10.55.

In the subsequent derivation $c(\cdot)$ denotes a positive constant depending only on parameters in the brackets. It may change from line to line or in the same line.

Now for $r \in \mathcal{R}_Q$ by $p_{r,Q}$ we denote a (unique) polynomial interpolating f at points of $S_r(x(Q))$ and compare this with the Taylor polynomial $T^k_{x(Q)}$ constructed for f in the proof of Theorem 10.51 (as $f \in X|_S$ it clearly satisfies the finiteness condition of this theorem). We have for $z \in S_r(x(Q))$,

$$\begin{aligned} |\big(T^k_{x(Q)} - p_{r,Q}\big)(z)| &= |\big(T^k_{x(Q)} - f\big)(z)| = |\big(T^k_{x(Q)} - T^k_z\big)(z)| \\ &\le c(k,n)\|x(Q) - z\|^k \omega(\|x(Q) - z\|), \end{aligned}$$

where as in (10.100) the inequality follows from the Taylor formula and (10.110).

Estimating the right-hand side by (10.107) and (10.112) we then have

$$\max_{Q_r(x(Q))} |T^k_{x(Q)} - p_{r,Q}| \le c(k,n,x(Q)) r^k \omega(r). \tag{10.113}$$

Now let r_Q stand for the ℓ_∞ radius (length of halfedge) of Q and $\rho(Q)$ be the smallest of numbers ρ such that $Q \subset Q_\rho(x(Q))$. Due to (10.107)

$$\rho(Q) \approx r_Q \approx d(Q,S) \tag{10.114}$$

with constants of equivalence depending only on n. Using then the univariate Remez inequality, cf. (10.101), and (10.113) and (10.114) we then obtain

$$\begin{aligned} &\max_Q |T^k_{x(Q)} - p_{r,Q}| \le c(k,n) \left(\frac{\rho(Q)}{r}\right)^k \cdot \max_{Q_r(x(Q))} |T^k_{x(Q)} - p_{r,Q}| \\ &\le c(k,n,x(Q)) \left(\frac{r_Q}{r}\right)^k \cdot r^k \omega(r) = c(k,n,x(Q)) r_Q^k \omega(r). \end{aligned}$$

Since $\omega(r) \to 0$ as $r \to 0$ we may choose $r \in \mathcal{R}_Q$ to be a number so that the right-hand side is bounded by $\widehat{r}^{k+1} \min\{1, \omega(\widehat{r}_Q)\}$, where $\widehat{r}_Q := \min\{1, r_Q\}$. Denoting $p_{r,Q}$ with this r by p_Q we then prove

Lemma 10.61. *For every $Q \in \mathcal{W}_S$ there exists a polynomial p_Q interpolating f at some points of S and satisfying*

$$\max_Q |T^k_{x(Q)} - p_Q| \le \widehat{r}^{k+1} \min\{1, \omega(\widehat{r}_Q)\}, \tag{10.115}$$

where $r_Q := \min\{1, r_Q\}$.

Now we define the desired extension of the function f by setting

$$Ef := \begin{cases} f & \text{on } S, \\ \sum\limits_{Q \in \mathcal{W}_S} p_Q \varphi_Q & \text{on } S^c. \end{cases} \tag{10.116}$$

This clearly defines a linear extension operator acting from $X|_S$ and we first show that it acts into X. For this aim we choose an extension of f, say F, from X such that

$$\|F\|_X \le 2\|f\|_{X|_S} \ (\le 2) \tag{10.117}$$

and then use the Taylor polynomials of F to write

$$f_1 := \begin{cases} f & \text{on } S, \\ \sum\limits_{Q \in \mathcal{W}_S} \big(T^k_{x(Q)} F\big) \varphi_Q & \text{on } S^c. \end{cases}$$

Due to the proof of Theorem 10.51, Taylor polynomials at points of S of any X extension of f are uniquely defined by f. Hence, we have the equality

$$T^k_{x(Q)} = T^k_{x(Q)} F$$

which, in turn, implies

$$Ef = f_1 + f_2,$$

where

$$f_2 := \begin{cases} 0 & \text{on } S, \\ \sum\limits_{Q \in \mathcal{W}_S} \big(p_Q - T^k_{x(Q)}\big) \varphi_Q & \text{on } S^c. \end{cases} \tag{10.118}$$

Since the family $\{T^k_x F\}_{x \in S}$ clearly satisfies the assumptions of Proposition 10.52 with $\lambda := \|F\|_X$, this proposition and (10.117) imply that $f_1 \in X$ and

$$\|f_1\|_X \le c(k, n)\|F\|_X \le c(k, n). \tag{10.119}$$

It remains to prove a similar inequality for f_2. Differentiating f_2 at a point $x \in S^c$ and applying (10.115), the Markov polynomial inequality and (10.108) we get for $|\alpha| \le k+1$,

$$|D^\alpha f_2|(x) \le \sum_{\mathcal{W}_S^* \ni Q \ni x} \left(\sum_{\gamma \le \alpha} |D^\gamma \big(p_Q - T^k_{x(Q)}\big)|(x) \cdot |D^{\alpha-\gamma}\varphi_Q|(x) \right)$$

$$\le c(k,n) \sum_{\mathcal{W}_S^* \ni Q \ni x} \frac{\widehat{r}_Q^{\,k+1-|\gamma|} \min\{1, \omega(\widehat{r}_Q)\}}{r_Q^{|\alpha|-|\gamma|}} \le c(k,n) \sum_{\mathcal{W}_S^* \ni Q \ni x} \widehat{r}_Q^{\,k+1-|\alpha|} \min\{1, \omega(\widehat{r}_Q)\}.$$

Setting for brevity $d(x) := \min\{d(x,S), 1\}$ and applying (10.106) and (10.107) we then conclude that

$$|D^\alpha f_2|(x) \le c(k,n) d(x)^{k+1-|\alpha|} \min\{1, \omega(d(x))\} \tag{10.120}$$

provided $x \in S^c$ and $|\alpha| \le k+1$.

We derive from here that f_2 is $(k+1)$-times differentiable and all its derivatives equal zero on S. Indeed, it follows from (10.120) with $|\alpha| = 1$ and equality $f_2|_S = 0$ that every derivative of f_2 of the first order exists and equals zero at all points of S. Using this and (10.120) with $|\alpha| = 2$ we prove the same for derivatives of the second order and so forth.

Now we estimate the X norm of f_2. From (10.120) immediately follows that

$$\max_{|\alpha| \le k} \sup_{\mathbb{R}^n} |D^\alpha f_2| \le c(k,n) \min\{1, \omega(1)\} \le c(k,n)$$

and it remains to estimate modulii of continuity of the k-th derivatives. For $|\alpha| = k$ we have from (10.120) with $|\alpha| = k+1$,

$$|D^\alpha f_2(x) - D^\alpha f_2(y)| \le n \max_{|\beta| = k+1} |D^\beta f_2| \cdot \|x - y\| \le c(k,n)\omega(1)\|x - y\|.$$

Hence, from here and decrease of $t \mapsto \frac{\omega(t)}{t}$ as t increases we get for $t \le 1$,

$$\frac{\omega(t; D^\alpha f_2)}{\omega(t)} \le c(k,n)\omega(1)\frac{t}{\omega(t)} \le c(k,n).$$

Also, if $\|x - y\| \ge 1$ and $|\alpha| = k+1$ we have by (10.120),

$$|D^\alpha f_2(x) - D^\alpha f_2(y)| \le c(k,n)\omega(1).$$

Since $t \mapsto \omega(t)$ is a nondecreasing function, the previous two inequalities imply

$$\sup_{0<t<\infty} \frac{\omega(t; D^\alpha f_2)}{\omega(t)} \le c(k,n).$$

Combining the inequalities so obtained we then get

$$\max_{|\alpha| \le k} \left(\sup_{\mathbb{R}^n} |D^\alpha f_2| + \sup_{0<t\le\infty} \frac{\omega(t; D^\alpha f_2)}{\omega(t)} \right) \le c(k,n).$$

But the left-hand side is the X norm of f_2. Therefore, the last inequality, (10.119) and (10.110) imply the required inequality

$$\|Ef\|_X := \|f_1 + f_2\|_X \le c(k,n)\|f\|_{X|_S}.$$

It remains to show that E is of finite depth. By definition, see (10.116), for $x \in S^c$,

$$(Ef)(x) = \sum_{\mathcal{W}_S^* \ni Q \ni x} p_Q(x)\varphi_Q(x).$$

Moreover, $p_Q = f(x_Q^i)\ell_i^Q$, where $\{x_Q^i\}$ is a subset of S with $\binom{n+k}{k}$ points and ℓ_i^Q are the fundamental polynomials of Lagrange interpolation. Hence, the previous equality can be rewritten as

$$(Ef)(x) = \sum_{\mathcal{W}_S^* \ni Q \ni x} \sum_i f(x_Q^i)\lambda_i^Q(x),$$

where all λ_i^Q are C_0^∞ functions and the number of points in the family $\{x_Q^i\} \subset S$ is at most $\binom{n+k}{k}$ mult $\mathcal{W}_S^* \le c(n)\binom{n+k}{k}$. This means that E is of depth bounded by some constant $c(k,n)$.

The theorem has been proved. □

10.2.6 Divided difference characteristic for trace spaces of $\Lambda^{k,\omega}(\mathbb{R}^n)$ on Markov sets

The final step in the study of the trace problem is, in this case, characterization of functions $f \in \Lambda^{k,\omega}(\mathbb{R}^n)\big|_S$, where $S \in \mathrm{Mar}(\mathbb{R}^n)$, via their behavior on finite subsets of cardinality at most $1 + \dim \mathcal{P}_{k-1,n}$. To accomplish this, we introduce a certain multivariate analog of divided difference, see Volume I, Section 2.3 for its definition. This notion, called *reduced divided difference* for reasons explained later, is defined over finite subsets of a special structure which is described by

Definition 10.62. A finite set $F \subset \mathbb{R}^n$ is said to be k-minimal if

(a) $\dim\big(\mathcal{P}_{k,n}\big|_F\big) = -1 + \mathrm{card}\, F$;

(b) for every proper subset $F' \subset F$,

$$\dim\big(\mathcal{P}_{k,n}\big|_{F'}\big) = \mathrm{card}\, F'.$$

To illustrate this definition we present

Example 10.63. (a) Every set $F \subset \mathbb{R}$ of cardinality $k+2$ is k-minimal. Actually, a polynomial $p \in \mathcal{P}_{k,1}$ is uniquely defined by its $k+1$ values; therefore $\dim\big(\mathcal{P}_{k,1}\big|_F\big) = k+1 = -1 + \mathrm{card}\, F$. On the other hand, for every subset $F' \subset F$ of $\mathrm{card}\, F' := k'+1$ where $k' \le k$ and every degree x^s where $s > k'$ there exists a polynomial $p_s \in \mathcal{P}_{k',1}$ such that $p_s(x) = x^s$ on F' (by Lagrange interpolation). Hence, $\dim\big(\mathcal{P}_{k,1}\big|_{F'}\big) = \mathrm{card}\, F'$, as required.

(b) Every subset F of an affine d-plane in $\mathbb{R}^n$, say A_d, containing $d+2$ points is 1-minimal if all its proper subsets are generic. Specifically, $\dim\big(\mathcal{P}_{1,n}\big|_{A_d}\big)$ clearly is $d+1$; moreover, if $F' \subsetneqq F$ is generic, then the affine hull of F' is of dimension $-1 + \operatorname{card} F'$ and therefore $\dim\big(\mathcal{P}_{1,n}\big|_{F'}\big) = \operatorname{card} F'$.

(c) By definition, the cardinality of a $(k-1)$-minimal set in $\mathbb{R}^n$ is bounded by $1 + \dim \mathcal{P}_{k-1,n}$ (the upper bound of the finiteness constant of $\Lambda^{k,\omega}(\mathbb{R}^n)$).

Now let $F := \{x^0, \dots, x^d\} \subset \mathbb{R}^n$ be k-minimal. Then there exists a basis $\{g_1, \dots, g_d\} \subset \big\{x^\alpha|_F\big\}_{|\alpha|\le k}$ of the trace space $\mathcal{P}_{k,n}|_F$. We introduce the $d \times (d+1)$ matrix $\big(g_i(x^j)\big)$ and denote by $\Delta_j(F)$ its minor obtained by removing the j-th column. Under this notation the desired concept is introduced by

Definition 10.64. The reduced divided difference of a function $g : \mathbb{R}^n \to \mathbb{R}$ over the k-minimal set F denoted by $D_F(g)$ is given by

$$D_F(g) := \sum_{j=0}^{d} (-1)^j \Delta_j(F) g(x^j). \tag{10.121}$$

We further write

$$\|D_F\| := \sum_{j=0}^{d} \big|\Delta_j(f)\big|$$

and define the *normed* divided difference by setting

$$\widehat{D}_F(g) := \frac{D_F(g)}{\|D_F\|}\,. \tag{10.122}$$

The next simple result motivates this definition.

Proposition 10.65. (a) *If $F \subset \mathbb{R}^n$ is k-minimal, then D_F annihilates polynomials of degree k.*

(b) *If $F \subset \mathbb{R}$ is k-minimal, then for $g : \mathbb{R} \to \mathbb{R}$,*

$$D_F(g) = (-1)^{k+1} \Delta(F) g[F],$$

where $\Delta(F) := \prod_{x,y\in F,\, x\ne y} (x-y)$ and $g[F]$ is the k-th divided difference over F (see Section 2.3 of Volume I for its definition).

Proof. (a) Due to (10.121),

$$D_F(g) = -\det\big(g_i(x^j)\big)_{0\le i,\, j\le d}$$

where $d := -1 + \operatorname{card} F$ and we set $g_0 := g|_F$.

If now $g(x) := x^\alpha$ for $|\alpha| \le k$, then $g_0 = \sum_{i=1}^{d} \lambda_i g_i$ and therefore the determinant is zero.

(b) Let $F := \{t_0, \dots, t_{k+1}\} \subset \mathbb{R}$ (i.e., F is k-minimal). Then

$$g[F] = \sum_{i=0}^{k+1} \frac{1}{w'(t_i)}\, g(t_i) \tag{10.123}$$

where $w(t) := \prod_{i=0}^{k+1} (t - t_i)$, see, e.g., Volume I, Proposition 2.25.

On the other hand, $\Delta_j(F)$ is the Vandermond determinant for the family of polynomials $\{t^i\}_{0\le i\le k}$ and the point set $F\backslash\{t_j\}$. Hence,

$$\Delta_j(F) = \prod (t_i - t_{i'})$$

where the product is taken over the set of all pairs (i, i') satisfying $0 \le i < i' \le k{+}1$ and $i, i' \ne j$.

This, in turn, may be written as

$$\Delta_j(F) = \frac{(-1)^{k-j}}{w'(t_j)} \prod_{0\le i<i'\le k+1} (t_i - t_{i'}).$$

Inserting this in (10.121) and comparing the result with (10.122) we obtain the required formula. □

Now we present the main result of this part.

Theorem 10.66. *A function g defined on a Markov set $S \subset \mathbb{R}^n$ belongs to the trace space $\Lambda^{k,\omega}(\mathbb{R}^n)\big|_S$ if and only if for some constant $c > 0$ and every $(k-1)$-minimal subset $F \subset S$,*

$$\big|\widehat{D}_F(g)\big| \le c\,\omega(\operatorname{diam} F) \quad \text{and} \quad \max_F |g| \le c.$$

Moreover, the equivalence

$$\|f\|_{\Lambda^{k,\omega}(\mathbb{R}^n)|_S} \approx \inf c$$

holds with constants independent of f.

Proof. We obtain the theorem by combining Theorem 9.30 characterizing the trace space in question with a quantitative version of the Remez–Shnirelman Theorem 10.42 which is of interest by itself.

First, as in the latter theorem, let K be a compact metric space and L be a finite-dimensional subspace of the space $C(K)$. Repeating Definition 10.62 for L in place of $\mathcal{P}_{k,n}$, we introduce the notion of an L-minimal subset of K. Finally, we recall that for $\widetilde{K} \subset K$,

$$e(\widetilde{K}\,; f) := \inf_{\ell\in L} \max_{\widetilde{K}} |f - \ell|.$$

Proposition 10.67. *Given $f \in C(K)\backslash L$, there exist an L-minimal subset $\widehat{K} \subset K$ and a unique, up to a constant factor, linear functional $H \in C(\widehat{K})^*$ such that*

$$e(f\,; K) = e(f\,; \widehat{K}) = \frac{\big|H(f|_{\widehat{K}})\big|}{\|H\|}\,. \tag{10.124}$$

Proof. By virtue of Theorem 10.42, there exists a subset $K_1 \subset K$ of cardinality at most $1 + \dim L$ such that

$$e(K_1\,;f) = e(K\,;f).$$

If $\dim\bigl(L|_{K_1}\bigr) < \dim L$, we apply Theorem 10.42 to the subspace $L|_{K_1}$ of the space $C(K_1)$ to find a subset $K_2 \subset K_1$ of cardinality at most $1 + \dim\bigl(L|_{K_1}\bigr)$ such that

$$e(K_2\,;f) = e(K_1\,;f).$$

Proceeding this way we finally find a subset K_p $(\subset K_{p-1} \subset \cdots \subset K_1)$ of cardinality, say, $s+1$ such that

$$e(K_p\,;f) = e(K_{p-1}\,;f) = \cdots = e(K\,;f) \tag{10.125}$$

and, moreover,

$$\dim\bigl(L|_{K_p}\bigr) = s.$$

Then $L|_{K_p}$ is a proper linear subspace of codimension 1 in the space $C(K_p)$ $\bigl(= \ell_\infty(K_p)\bigr)$. Hence, there exists a unique, up to a constant factor, linear functional $H \in C(K_p)^* = \ell_1(K_p)$ such that

$$L|_{K_p} = \bigl\{g \in C(K_p)\,; H(g) = 0\bigr\}.$$

Further, let $K_p := \{t_1, \dots, t_{s+1}\}$ and $\{\delta_j\}_{1\le j\le s+1}$ be the canonical basis of the space $\ell_1(K_p)$, i.e., $\delta_j(t_i) = 0$ if $i \ne j$ and 1 if $i = j$. Then for some nontrivial sequence $\{a_j\}_{1\le j\le s+1}$,

$$H = \sum_{j=1}^{s+1} a_j \delta_j.$$

Without loss of generality, we assume that $a_j \ne 0$ for $1 \le j \le r+1$ and $a_j = 0$ for $r+1 < j \le s+1$ (if such j exists). We then set $\widehat{K} := \{t_1, \dots, t_{r+1}\}$ and show that (10.124) holds for this H and $\widehat{K}$.

By the definition of $\widehat{K}$, for $g \in C(K_p)$,

$$H(g) = H\bigl(g|_{\widehat{K}}\bigr).$$

In particular, $\|H\| = \|H\|_{C(\widehat{K})^*}$ and

$$L|_{\widehat{K}} = \bigl\{g \in C(\widehat{K})\,; H(g) = 0\bigr\}.$$

This implies that for $\ell \in L|_{\widehat{K}}$,

$$\bigl|H(f|_{\widehat{K}})\bigr| = \bigl|H\bigl(f|_{\widehat{K}} - \ell\bigr)\bigr| \le \|H\|\,\|f - \ell\|_{C(\widehat{K})}.$$

Taking the infimum over ℓ we get

$$\frac{\bigl|H(f|_{\widehat{K}})\bigr|}{\|H\|} \le e(\widehat{K}\,;f).$$

On the other hand, let $m \in C(K_p)$ be such that

$$\|m\|_{C(K_p)} = 1 \quad \text{and} \quad H(m) = \|H\|.$$

Then we set

$$\ell^* := f|_{K_p} - \frac{H(f|_{K_p})}{H(m)}\, m.$$

Clearly, $H(\ell^*) = 0$, i.e., ℓ^* belongs to $L|_{K_p}$. Therefore

$$e(K_p\,; f) \le \|f - \ell^*\|_{C(K_p)} = \frac{\big|H(f|_{K_p})\big|}{\|H\|} = \frac{\big|H(f|_{\widehat{K}})\big|}{\|H\|}\,.$$

Together with the previous inequality and the embedding $\widehat{K} \subset K_p$ this yields the required result

$$e(\widehat{K}\,; f) = \frac{\big|H(f|_{\widehat{K}})\big|}{\|H\|}\,.$$

It remains to show that $\widehat{K}$ is L-minimal, i.e., that

$$\dim(L|_{\widehat{K}}) = -1 + \operatorname{card} \widehat{K}$$

and that for every proper subset $K' \subset \widehat{K}$,

$$\dim(L|_{K'}) = \operatorname{card} K'.$$

The first assertion follows from the definition of $\widehat{K}$. To prove the second one, we use the map $g \mapsto \big(g(t_i)\big)_{1 \le i \le r+1}$ to identify the space $C(\widehat{K})$ with the space ℓ_∞^{r+1}. Then $L|_{\widehat{K}}$ is identified with the subspace

$$\Big\{x \in \ell_\infty^{r+1}\,; \sum_{i=1}^{r+1} a_i x_i = 0\Big\}, \tag{10.126}$$

where all a_i (the coordinates of H) are nonzero.

In turn, $L|_{K'}$ under this identification coincides with the orthogonal projection of (10.126) onto a subspace spanned by the $\operatorname{card} K'$ coordinate axes of $\mathbb{R}^{r+1}$. Since all $a_i \ne 0$, none of the coordinate axes lies in (10.126). Hence, the dimension of the projection equals $\min\{\operatorname{card} K', r\} = \operatorname{card} K'$, as required.

The result has been proved. □

Now we apply the proposition to the case of $L := \mathcal{P}_{k,n}$; an L-minimal set is now subject to Definition 10.62, i.e., is k-minimal.

Corollary 10.68. *Let $F \subset \mathbb{R}^n$ be k-minimal and $g : F \to \mathbb{R}$. Then*

$$E_{k+1}(F\,; g) = \big|\widehat{D}_F(g)\big|.$$

Proof. If $F := \{x^1, \dots, x^{s+1}\}$, then there exists a basis, say $\{g_j\}_{1\le j\le s}$, of the trace space $\mathcal{P}_{k,n}\big|_F$ such that every $g_j \in \{x^\alpha\big|_F\}_{|\alpha|\le k}$.

We set $g_{s+1} := g$ and introduce a linear functional $g \mapsto D(g)$ on $C(F)$ by

$$D(g) := \det\bigl(g_j(x^i)\bigr)_{1\le i,\, j\le s+1}.$$

Then $\mathcal{P}_{k,n}\big|_F$ is the null-space of the functional D. Actually, if $g \in \mathcal{P}_{k,n}\big|_F$, then g is a linear combination of the basis functions g_j, and therefore $D(g) = 0$.

Conversely, if $D(g) = 0$, then we get for some nontrivial linear combination $\mu g(t) + \sum\limits_{j=1}^{s+1} \lambda_j g_j(t) = 0$ for all $t \in F$. Since $\{g_j\}$ is a basis, $\mu \ne 0$ and, therefore, g is a linear combination of g_j, i.e., $g \in \mathcal{P}_{k,n}\big|_F$.

Because of the uniqueness of such a functional up to a constant factor, Proposition 10.67 with $H := D$ yields

$$E_{k+1}(F\,; g) = \frac{|D(g)|}{\|D\|} = \frac{\Bigl|\sum\limits_{j=1}^{s+1} (-1)^j \Delta_j(F) g(x^j)\Bigr|}{\sum\limits_{j=1}^{s+1} |\Delta_j(F)|}.$$

Here Δ_j is the $s \times s$ minor of the matrix $\bigl(g_j(x^i)\bigr)_{\substack{1\le i\le s+1\\ 1\le j\le s}}$ obtained by removing the jth column.

Comparing the right-hand side with the definition of $\hat{\delta}_F(g)$, see (10.122), we obtain the result. □

Now we are ready to prove Theorem 10.66.

(Necessity) Let $g \in \Lambda^{k,\omega}(\mathbb{R}^n)\big|_S$ and let its trace norm be less than 1. We should show that for every $(k-1)$-minimal set $F \subset S$,

$$\bigl|\widehat{D}_F(g)\bigr| \le c\omega(\operatorname{diam} F), \tag{10.127}$$

where $c = c(k,n,S)$. To this end we choose a cube $Q \in \mathcal{K}_S$ containing F so that its diameter equals $\operatorname{diam} F$. If $\tilde{g} \in \Lambda^{k,\omega}(\mathbb{R}^n)$ is an almost optimal extension of g, then

$$E_k(F\,; g) \le E_k(Q\,; \tilde{g}) \le c(k,n,S)\,\omega(\operatorname{diam} F)$$

by Theorem 9.30. Combining this with Corollary 10.68 we obtain (10.127). Since, moreover, $\sup\limits_F |g|$ is trivially bounded by 1, the result is positive.

(Sufficiency) Let $g : S \to \mathbb{R}^n$, where $S \in \operatorname{Mar}(\mathbb{R}^n)$, be such that for every $(k-1)$-minimal set $F \subset S$,

$$\bigl|\widehat{D}_F(g)\bigr| \le \omega(\operatorname{diam} F) \quad \text{and} \quad \sup_F |g| \le 1. \tag{10.128}$$

We should prove that $g \in \Lambda^{k,\omega}(\mathbb{R}^n)\big|_S$ and that its trace norm is bounded by $c = c(k, n, S)$.

Let $Q \in \mathcal{K}_S$ and $F := F_Q$ be a $(k-1)$-minimal subset of the compact set $Q \cap S$ such that

$$E_k(F\,; g) = E_k(Q \cap S\,; g),$$

see Proposition 10.67.

Since by this proposition the left-hand side is $\big|\widehat{D}_F(g)\big|$, we conclude from here and (10.128) that

$$E_k(Q \cap S\,; g) \le \omega(\operatorname{diam} F) \le \omega(\operatorname{diam} Q).$$

Due to this and the second inequality of (10.128),

$$\sup_S |g| + \sup_{Q\in\mathcal{K}_S} \frac{E_k(Q \cap S\,; g)}{\omega(\operatorname{diam} Q)} \le 2.$$

By Theorem 9.30, the left-hand side is equivalent to the trace norm of g in $\Lambda^{k,\omega}(\mathbb{R}^n)\big|_S$ with constants of equivalence depending only on k, n, S.

Theorem 10.66 has been proved. □

10.2.7 Concluding remarks

(a) All basic results for Markov sets remain true for locally Markov ones. Recall that this means that the Markov inequality

$$\|\nabla p\| \le \frac{c}{r} \sup_{Q\cap S} |p|$$

for $p \in \mathcal{P}_{1,n}$ and $Q \in \mathcal{K}_S$ is true only for sufficiently small r.

(b) The following conjecture seems to be true:

$$\mathcal{F}_{\mathrm{Mar}(\mathbb{R}^n)}(\Lambda^{k,\omega}) = 1 + \dim \mathcal{P}_{k-1,n}$$

and the same is valid for the space $C^{k-1,\omega}(\mathbb{R}^n)$ and weak $(k-1)$-Markov sets.

(c) More generally, we conjecture that the finiteness and linearity properties and the divided difference characteristic hold for the trace space $C^{\ell}\Lambda^{k,\omega}(\mathbb{R}^n)\big|_S$, where S is locally (maybe even weak ℓ-) Markov.

10.3 $C^{k,\omega}(\mathbb{R}^n)$ spaces: finiteness and linearity

We begin with the sharp finiteness result for $k = 1$ due to Yu. Brudnyi and Shvartsman [BSh-2001b]. In the next subsection we discuss the general Fefferman finiteness theorem [F-2005a], [F-2003] formulated (equivalently) as a selection result for a set-valued map from a subset of $\mathbb{R}^n$ into the set of so-called (ω, k)-convex subsets of $\mathcal{P}_{k,n}$. Its proof yields an effective but enormously large upper bound $N^* = N^*(k, n)$ for the corresponding finiteness constant. Fefferman's theorem immediately implies the Uniform Finiteness Property for $C^{k,\omega}(\mathbb{R}^n)$ with $\mathcal{F}(C^{k,\omega}(\mathbb{R}^n))$ bounded by N^* and the second finiteness constant $\gamma(C^{k,\omega}(\mathbb{R}^n))$ bounded by some $c = c(k, n)$ (exponentially depending on N^*). The first estimate can be essentially improved using the results due independently and in different ways by Shvartsman [Shv-2008] and Bierstone and Milman [BM-2007]. They in particular estimate the first finiteness constant $\mathcal{F}_S(C^{k,\omega}(\mathbb{R}^n))$ for finite subsets S by $2^{\dim \mathcal{P}_{k,n}}$. The norms of the extension operators in their proofs tend to infinity along with card S; therefore the Fefferman theorem does not follow from their results. Shvartsman's proof will be outlined in the second part of subsection 10.3.2.

The final subsection is devoted to the linear extension problem for $C^{k,\omega}(\mathbb{R}^n)$. For $k = 1$ its solution was due to Yu. Brudnyi and Shvartsman [BSh-1999]; their methods can be applied to more general situations, see subsection 10.5.2 below. A new approach leading to the solution to the linear problem for $C^{k,\omega}(\mathbb{R}^n)$ with any k was discovered by Fefferman [F-2009b] whose main results along with the related concepts will be formulated and discussed in this subsection.

Unfortunately, we cannot describe (even very sketchily) the basic ideas of Fefferman's impressive proofs hidden very deeply under the surface of multileveled, voluminous and highly technical constructions.

10.3.1 Sharp finiteness constant for $C^{1,\omega}(\mathbb{R}^n)$

Our main result is

Theorem 10.69. *The space $C^{1,\omega}(\mathbb{R}^n)$ has the Uniform Finiteness Property and, moreover,*

$$\mathcal{F}\big(C^{1,\omega}(\mathbb{R}^n)\big) = 3 \cdot 2^{n-1}.$$

Proof. The inequality

$$\mathcal{F}\big(C^{1,\omega}(\mathbb{R}^n)\big) \geq 3 \cdot 2^{n-1}.$$

is a direct consequence of Theorem 10.16.

The lengthy proof of the converse inequality will be firstly done for the somewhat less technical case of the homogeneous space $\dot{C}^{1,\omega}(\mathbb{R}^n)$. Then we shall indicate small changes in the proof leading to the result for $C^{1,\omega}(\mathbb{R}^n)$.

The main tools for the proof of the inequality

$$\mathcal{F}\big(\dot{C}^{1,\omega}(\mathbb{R}^n)\big) \leq 3 \cdot 2^{n-1} \tag{10.129}$$

are the Whitney–Glaeser Theorem 2.19 for $k = 1$ reformulated as a Lipschitz selection result, and Shvartsman's Lipschitz selection Theorem 5.56 of Volume I.

We obtain the aforementioned reformulation in several steps, the first of which is a straightforward consequence of Theorem 2.19 of Volume I.

Lemma 10.70. *A function $f : S \to \mathbb{R}$ belongs to the trace space $\dot{C}^{1,\omega}(\mathbb{R}^n)\big|_S$ if and only if there exists a 1-jet $\vec{g} = (g_1, \dots, g_n) : S \to \mathbb{R}^n$ such that for some constant $\lambda > 0$ and all points $x, y \in S$ the following holds:*

(a) $\big\|\vec{g}(x) - \vec{g}(y)\big\| \le \lambda\omega\big(\|x - y\|\big)$;

(b) $\big|f(x) - f(y) - \langle \vec{g}(x), x - y\rangle\big| \le \lambda\|x - y\|\omega\big(\|x - y\|\big)$.

Moreover, for some constants of equivalence depending only on n,

$$\inf \lambda \approx |f|_{C^{1,\omega}(\mathbb{R}^n)|_S}. \tag{10.130}$$

Hereafter S is a closed subset of $\mathbb{R}^n$ and

$$\|x\| := \max_{1\le i\le n} |x_i|, \ \langle x, y\rangle := \sum_{i=1}^{n} x_i y_i.$$

The next step involves a metric space $(\mathcal{M}, d_\omega)$ whose underlying set is given by

$$\mathcal{M} := \big\{(x, y) \in S \times S\,; x \ne y\big\} \tag{10.131}$$

and the metric d_ω given at $\big(m := (x, y),\ m' := (x', y')\big)$, where $m \ne m'$, by

$$d_\omega(m, m') := \omega\big(\|x - y\|\big) + \omega\big(\|x' - y'\|\big) + \omega\big(\|x - x'\|\big).$$

Since ω is a 1-majorant, it is subadditive and the triangle inequality holds for d_ω.

Lemma 10.71. *A function $f : S \to \mathbb{R}$ belongs to the trace space $\dot{C}^{1,\omega}(\mathbb{R}^n)\big|_S$ if and only if there exists a map $\vec{G}_f : \mathcal{M} \to \mathbb{R}^n$ such that for some constant $\lambda > 0$ and every $m := (x, y) \in \mathcal{M}$ the following is true:*

(i) $L(\vec{G}_f) := \sup\limits_{m\ne m'} \dfrac{\|\vec{G}_f(m) - \vec{G}_f(m')\|}{d_\omega(m, m')} \le \lambda$;

(ii) $\langle \vec{G}_f(m), x - y\rangle = f(x) - f(y)$.

Moreover, (10.130) *holds with these λ.*

Proof. (Necessity) Let $f \in \dot{C}^{1,\omega}(\mathbb{R}^n)\big|_S$ and $\lambda > 0$, $\vec{g} : S \to \mathbb{R}^n$ and f satisfy the assumptions of Lemma 10.70. Given $m := (x, y) \in \mathcal{M}$ we define an affine hyperplane $H(f\,; m)$ by

$$H(f\,; m) = \big\{z \in \mathbb{R}^n\,; \langle z, x - y\rangle = f(x) - f(y)\big\} \tag{10.132}$$

and denote by $\vec{G}_f(m)$ the orthogonal projection of the point $\vec{g}(x)$ on this hyperplane.

Let us show that the so-defined map $\vec{G}_f$ is as required. By the well-known formula measuring the Euclidean distance from a point to a hyperplane we then have

$$\big\|\vec{g}(x) - \vec{G}_f(m)\big\|_2 \cdot \|x-y\|_2 = \big|f(x) - f(y) - \langle \vec{g}(x), x-y\rangle\big|,$$

where $\|\cdot\|_2$ stands for the Euclidean norm. This and Lemma 10.70 imply

$$\begin{aligned}\big\|\vec{g}(x) - \vec{G}_f(m)\big\| &\le \big\|\vec{g}(x) - \vec{G}_f(m)\big\|_2 \\ &\le \lambda \frac{\|x-y\|}{\|x-y\|_2}\,\omega\big(\|x-y\|\big) \le \lambda\omega\big(\|x-y\|\big).\end{aligned} \tag{10.133}$$

Now let $m := (x,y)$, $m' := (x',y')$ be points of $\mathcal{M}$. Then (10.133) and Lemma 10.70 yield

$$\begin{aligned}\big\|\vec{G}_f(m) - \vec{G}_f(m')\big\| &\le \big\|\vec{G}_f(m) - \vec{g}(m)\big\| + \big\|\vec{g}(m) - \vec{g}(m')\big\| + \big\|\vec{g}(m') - \vec{G}_f(m')\big\| \\ &\le \lambda\Big(\omega\big(\|x-y\|\big) + \omega\big(\|x-x'\|\big) + \omega\big(\|x'-y'\|\big)\Big) \\ &=: \lambda d_\omega(m,m').\end{aligned}$$

This proves (i) whereas (ii) follows directly from the definition of $\vec{G}_f(m)$.

(Sufficiency) Assume that $f : S \to \mathbb{R}$ satisfies the sufficiency conditions of Lemma 10.71 with the given $\vec{G}_f : \mathcal{M} \to \mathbb{R}^n$ and $\lambda > 0$. We should find a map $\vec{g} : S \to \mathbb{R}^n$ satisfying, together with f, the conditions of Lemma 10.70 with constant $c(n)\lambda$.

To define this $\vec{g}$ we use the metric projection $Pr_S : \ell_\infty^n \to S \subset \ell_\infty^n$ defined by

$$\big\|x - Pr_S(x)\big\| = \inf_{y \in S\setminus\{x\}} \|x-y\|.$$

More precisely, for each x this condition determines a nonempty subset of S (since S is closed) and we fix a point of this subset (denoted by $Pr_S(x)$). In particular, $Pr_S(x) = x$ if x is a nonisolated point of S, and $Pr_S(x) \in S\setminus\{x\}$ otherwise.

Given a nonisolated point $x \in S$, we fix a sequence $\{x^i\}_{i\in\mathbb{N}} \subset S\setminus\{x\}$ such that $\lim_{i\to\infty} x^i = Pr_S(x)$; for an isolated point $x \in S$ we set $x^i := Pr_S(x)$ for all $i \in \mathbb{N}$.

By $\{m_i\}_{i\in\mathbb{N}}$, where $m_i := (x, x^i)$, we then denote the sequence in $\mathcal{M}$ associated to x and $\{x^i\}_{i\in\mathbb{N}}$.

Now we define the desired map $\vec{g}$ at $x \in S$ by

$$\vec{g}(x) := \lim_{i\to\infty} \vec{G}_f(m_i).$$

Let us show that $\vec{g}(x)$ is independent of the choice of $\{x^i\}_{i\in\mathbb{N}}$. Actually, due to condition (i) we get for $m' := (x,y')$, $m'' := (x,y'')$,

$$\|\vec{G}_f(m') - \vec{G}_f(m'')\| \le \lambda d_\omega(m',m'') =: \lambda\big(\omega(\|x-y'\|) + \omega(\|x-y''\|)\big)$$

and the right-hand side tends to zero as $y', y'' \to x$.

Now we check that Lemma 10.70 holds for this $\vec{g}$. To this end we fix points $x, \hat{x} \in S$, $x \neq \hat{x}$, and let $\{m_i\}_{i\in\mathbb{N}}$, $\{\hat{m}_i\}_{i\in\mathbb{N}}$ be the sequences in $\mathcal{M}$ associated to them. Then condition (i) implies

$$\begin{aligned}
&\|\vec{g}(x) - \vec{g}(\hat{x})\| := \lim_{i\to\infty} \|\vec{G}_f(m_i) - \vec{G}_f(\hat{m}_i)\| \\
&\le \lambda \overline{\lim_{i\to\infty}}\, d_\omega(m_i, \hat{m}_i) := \lambda \overline{\lim_{i\to\infty}} \big(\omega(\|x - x^i\|) + \omega(\|\hat{x} - \hat{x}^i\|) + \omega(\|x - \hat{x}\|)\big) \\
&= \lambda\big(\omega(\|x - Pr_S(x)\|) + \omega(\|\hat{x} - Pr_S(\hat{x})\|) + \omega(\|x - \hat{x}\|)\big).
\end{aligned}$$

Since by the definition of Pr_S

$$\max\{\|x - Pr_S(x)\|, \|\hat{x} - Pr_S(\hat{x})\|\} \le \|x - \hat{x}\|,$$

the previous inequality gives

$$\|\vec{g}(x) - \vec{g}(\hat{x})\| \le 3\lambda\omega(\|x - \hat{x}\|).$$

This proves assertion (a) of Lemma 10.70.

To prove (b) we use (ii) to write, for $m := (x, y) \in \mathcal{M}$ and the associated sequence to x, $\{m_i\}_{i\in\mathbb{N}} \subset \mathcal{M}$, the identity

$$\begin{aligned}
&|f(x) - f(y) - \langle \vec{g}(x), x - y\rangle| = |\langle \vec{G}_f(m) - \vec{g}(x), x - y\rangle| \\
&= \lim_{i\to\infty} |\langle \vec{G}_f(m) - \vec{G}_f(m_i), x - y\rangle|.
\end{aligned}$$

Estimating the right-hand side by the Cauchy inequality and then applying (i) we bound it by

$$\begin{aligned}
&\lambda\|x - y\|_2 \cdot \lim_{i\to\infty} d_\omega(m, m_i) := \lambda\|x - y\|_2 \cdot \lim_{i\to\infty} \big(\omega(\|x - y\|) + \omega(\|x - x^i\|)\big) \\
&= \lambda\|x - y\|_2 \cdot \big(\omega(\|x - y\|) + \omega(\|x - Pr_S(x)\|)\big) \le 2\lambda\|x - y\|_2\, \omega(\|x - y\|) \\
&\le 2\sqrt{n}\lambda\|x - y\|\, \omega(\|x - y\|).
\end{aligned}$$

This proves assertion (b) and, hence, Lemma 10.71. □

Now we reformulate the last result in the desired form. First, given $f \in \ell_\infty(S)$ we introduce a set-valued function L_f sending points of the metric space $(\mathcal{M}, d_\omega)$ into the set of affine hyperplanes $\mathrm{Aff}_{n-1}(\mathbb{R}^n)$; specifically, we set

$$L_f(m) := H(f; m)\,(:= \{z \in \mathbb{R}^n\,;\, \langle z, x - y\rangle = f(x) - f(y)\}). \tag{10.134}$$

By Lemma 10.71 $\vec{G}_f(m) \in L_f(m)$ for all $m \in \mathcal{M}$ and, moreover, $\vec{G}_f$ is a Lipschitz map. Hence, Lemma 10.71 is equivalent to

Lemma 10.72. *A function $f : S \to \mathbb{R}^n$ belongs to the trace space $\dot{C}^{1,\omega}(\mathbb{R}^n)|_S$ if and only if the map L_f admits a Lipschitz selection $\vec{G}_f : \mathcal{M} \to \mathbb{R}$.*

Moreover, the following holds with the constants of equivalence depending only on n:

$$\inf |\vec{G}_f|_{\mathrm{Lip}} \approx |f|_{C^{1,\omega}(\mathbb{R}^n)},$$

where the infimum is taken over all such selections G_f.

Now we pass to the second step of the proof based on Theorem 5.56 of Volume I. Since this theorem concerns set-valued functions on metric graphs, we turn the space $(\mathcal{M}, d_\omega)$ into a metric graph whose (geodesic) metric is equivalent to d_ω. To this end we equip $\mathcal{M}$ with a graph structure regarding its points as vertices and joining two distinct points $m := (x, y)$, $m' := (x', y')$ by an edge if the set $\{x, y\} \cap \{x', y'\}$ is nonempty. Then we define a "premetric" $\psi_\omega : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ given on the vertex set by

$$\psi_\omega(m, m') := \omega(\|x - y\|) + \omega(\|x' - y'\|)$$

and then introduce the required geodesic metric on $\mathcal{M}$ by setting

$$\hat{d}_\omega(m, m') := \inf \sum_{i=0}^{k-1} \psi_\omega(m_i, m_{i+1}),$$

where the infimum is taken over all paths $\{m_i\}_{0\le i\le k}$ in the graph joining m and m' and all k (recall that $\{m_i\}_{0\le i\le k}$ is a *path joining* m, m' if $m = m_0$, $m' = m_k$ and every pair m_i, m_{i+1} is joined by an edge).

Lemma 10.73. $\frac{1}{2}\hat{d}_\omega \le d_\omega \le 2\hat{d}_\omega$.

Proof. Connecting $m := (x, y)$, $m' := (x', y')$ by the path $\{m, m'', m'\}$ where $m'' := (x, x')$ we get

$$\begin{aligned}\hat{d}_\omega(m, m') &\le \psi_\omega(m, m'') + \psi_\omega(m'', m') \\ &:= \omega(\|x - y\|) + 2\omega(\|x - x'\|) + \omega(\|x' - y'\|) \le 2d_\omega(m, m').\end{aligned}$$

To prove the converse we fix a path $\{m_i := (x^i, y^i)\}_{0\le i\le k}$ joining $m\,(:= m_0)$ and $m'\,(:= m_k)$ and use subadditivity of the 1-majorant ω to write

$$\begin{aligned}&\sum_{i=0}^{k-1} \psi_\omega(m_i, m_{i+1}) := \omega(\|x - y\|) + 2\sum_{i=1}^{k-1} \omega(\|x^i - y^i\|) + \omega(\|x' - y'\|) \\ &\ge \frac{1}{2}\left\{\omega(\|x - y\|) + \omega(\|x' - y'\|) + \sum_{i=0}^{m} \omega(\|x^i - y^i\|)\right\} \\ &\ge \frac{1}{2}\left\{\omega(\|x - y\|) + \omega(\|x' - y'\|) + \omega\left(\sum_{i=0}^{k} \|x^i - y^i\|\right)\right\}.\end{aligned}$$

Further, since m_i, m_{i+1} are joined by an edge, i.e., $\{x^i, y^i\} \cap \{x^{i+1}, y^{i+1}\} \neq \emptyset$, we get

$$\sum_{i=0}^{k} \|x^i - y^i\| \geq \|x^0 - x^k\| =: \|x - x'\|.$$

This, finally, yields

$$\begin{aligned}\hat{d}_\omega(m, m') &:= \inf_{\{m_i\}} \sum_{i=0}^{k-1} \psi_\omega(m_i, m_{i+1}) \\ &\geq \frac{1}{2}\{\omega(\|x - y\|) + \omega(\|x' - y'\|) + \omega(\|x - x'\|)\} := \frac{1}{2} d_\omega(m, m').\end{aligned}$$

The result has been proved. □

At the final step of the proof we apply the Lipschitz selection Theorem 5.56 of Volume I to the set-valued map $L_f : (\mathcal{M}, \hat{d}_\omega) \to \text{Aff}_{n-1}(\mathbb{R}^n)$, see (10.134), and formulate the result so obtained as

Proposition 10.74. *If the trace of the L_f to every subset of cardinality at most 2^n with no isolated points admits a 1-Lipschitz selection, then L_f itself has a Lipschitz selection $\vec{G}_f : (\mathcal{M}, \hat{d}_\omega) \to \mathbb{R}^n$ with Lipschitz constant bounded by $c(n)$.*

Let us recall that $\mathcal{M}' \subset \mathcal{M}$ has no isolated points if, being regarded as a subgraph of the graph $\mathcal{M}$, it is connected.

The straightforward application of the proposition gives the constant 2^{n+1} as an upper bound for $\mathcal{F}(\dot{C}^{1,\omega}(\mathbb{R}^n))$. To improve this estimate we exploit the next simple fact.

Lemma 10.75. *Let Γ be a (simple) graph with V vertices and E edges. If Γ has no isolated edges, then*

$$2V \leq 3E. \tag{10.135}$$

Proof. The minimal graph with no isolated edges contains two edges and three vertices, i.e., (10.135) is equality in this case. Let us show by induction on E that for *connected* Γ and $E \geq 3$, (10.135) is a strict inequality. Since $E \geq 3$, we may remove from Γ a suitable edge to obtain a connected subgraph with $E - 1$ edges and V or $V - 1$ vertices. In the worst case, $2(V - 1) \leq 3(E - 1)$ by the induction hypothesis with equality for $E = 3$. This yields

$$2V \leq 3E - 1 < 3E,$$

as required.

If, however, Γ is a disjoint union of connected subgraphs, then none of them contains isolated edges and we obtain the result by summing the inequalities for the subgraphs. □

Now we are ready to prove the basic result, inequality (10.129). In other words, we establish the following

Claim. Let $f : S \to \mathbb{R}$ be such that the trace of f to a subset $S' \subset S$ of cardinality at most $3 \cdot 2^{n-1}$ admits a $\dot{C}^{1,\omega}$ extension $f|_{S'} : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$|f|_{S'}|_{C^{1,\omega}(\mathbb{R}^n)} \leq 1. \tag{10.136}$$

Then f belongs to $\dot{C}^{1,\omega}(\mathbb{R}^n)|_S$ and its trace norm is bounded by a constant $c = c(n)$.

To prove this result we combine Lemma 10.70 with the criterion of Lemma 10.72. So, let $\mathcal{M}' \subset \mathcal{M}$ be a connected metric subgraph of the metric graph $(\mathcal{M}, d_\omega)$. Assuming that card $\mathcal{M}' \leq 2^n$ we check that the trace $L_f|_{\mathcal{M}'}$ admits a $c(n)$-Lipschitz selection.

To do this we associate to $\mathcal{M}'$ the subset

$$S_{\mathcal{M}'} := \{z \in S\ ;\ z \in \{x, y\} \quad \text{for some} \quad (x, y) \in \mathcal{M}'\}$$

and equip $S_{\mathcal{M}'}$ with a graph structure by joining vertices $x, y \in S_{\mathcal{M}'}$ with an edge if $(x, y) \in \mathcal{M}'$. Since $\mathcal{M}'$ is connected, this graph has no isolated edges. Then by Lemma 10.75 the numbers of its vertices card $S_{\mathcal{M}'}$ and edges, say E, are related by the inequality

$$\operatorname{card} S_{\mathcal{M}'} \leq \frac{3}{2} E \leq \frac{3}{2} \operatorname{card} \mathcal{M}' \leq 3 \cdot 2^{n-1}.$$

This and (10.136) imply that the trace $g := f|_{S_{\mathcal{M}'}}$ admits a $\dot{C}^{1,\omega}$-extension to $\mathbb{R}^n$ of norm bounded by $c(n)$.

Further, due to the assertion of Lemma 10.72 applied to the function g, the associated-to-g set-valued function $L_g : \widetilde{S}_{\mathcal{M}'} \to \mathrm{Aff}_{n-1}(\mathbb{R}^n)$ where $\widetilde{S}_{\mathcal{M}'} := \{(x, y) \in S_{\mathcal{M}'} \times S_{\mathcal{M}'}\ ;\ x \neq y\}$, see (10.134), admits a $c(n)$-Lipschitz selection.

On the other hand, $\mathcal{M}' \subset \widetilde{S}_{\mathcal{M}'}$; moreover, by the definitions of the latter set and L_f we have

$$L_g = L_f \quad \text{on} \quad \mathcal{M}'.$$

Hence, $L_f|_{\mathcal{M}'}$ also has a $c(n)$-Lipschitz selection for every connected subgraph $\mathcal{M}'$ of cardinality $\leq 2^n$. Due to Lemma 10.70 this, in turn, implies existence of such a selection for L_f itself. Finally, by the sufficiency part of Lemma 10.72 the last fact implies that $f \in \dot{C}^{1,\omega}(\mathbb{R}^n)|_S$ and its trace norm is bounded by $c(n)$.

Thus the result has been proved for the space $\dot{C}^{1,\omega}(\mathbb{R}^n)$.

Now we explain how to adapt this proof to the space $C^{1,\omega}(\mathbb{R}^n)$ which we prefer to equip with the equivalent norm

$$\|f\|_{C^{1,\omega}(\mathbb{R}^n)} := \sup_{\mathbb{R}^n} |f| + \max_{|\alpha|=1} \sup_{\mathbb{R}^n} |D^\alpha f| + |f|_{C^{1,\omega}(\mathbb{R}^n)}.$$

Here we assume without loss of generality that $\omega \le 1$ replacing otherwise ω by $\widetilde{\omega}(t) := \min(1, t)$, $t > 0$. Because of the inequality

$$|f|_{C^{1,\omega}(\mathbb{R}^n)} := \max_{|\alpha|=1} \sup_{t>0} \frac{\omega(t; D^\alpha f)}{\omega(t)} \le |f|_{C^{1,\widetilde{\omega}}(\mathbb{R}^n)} + 2 \max_{|\alpha|=1} \sup_{\mathbb{R}^n} |D^\alpha f|$$

this replacement gives an equivalent norm.

The Whitney-Glaeser Theorem 2.19 of Volume I gives an analog of Lemma 10.70 with additional inequalities

$$\sup_S |f| \le \lambda \quad \text{and} \quad \max_{1\le i\le n} \sup_S |g_i| \le \lambda.$$

After a small change in the proof this leads to an analog of Proposition 10.74, where the selection $\vec{G}_f : (\mathcal{M}, d_\omega) \to \mathrm{Aff}_{n-1}(\mathbb{R}^n)$ of the set-valued function L_f is now a *bounded* Lipschitz function and, moreover,

$$\|f\|_{C^{1,\omega}(\mathbb{R}^n)|_S} \approx \sup_S |f| + \inf_{\vec{G}_f} \left\{ \sup_{m\in\mathcal{M}} \|\vec{G}_f(m)\| + \|\vec{G}_f\|_{\mathrm{Lip}} \right\};$$

here the infimum is taken over all such selections G_f.

This result is then reformulated using a metric graph $(\mathcal{M}^*, d^*_\omega)$ with the underlying set $\mathcal{M}^* := \mathcal{M} \cup \{\infty\}$ and the metric given by

$$d^*_\omega(m, m') := \begin{cases} d_\omega(m, m') & \text{if } m, m' \in \mathcal{M}, \\ 0 & \text{otherwise.} \end{cases}$$

Further, we extend the map L_f to $\mathcal{M}^*$ by setting

$$L^*_f(m) := \begin{cases} L_f(m) & \text{if } m \in M, \\ \{0\} & \text{if } m = \infty. \end{cases}$$

Hence, L^*_f maps $\mathcal{M}^*$ into the set $\mathrm{Aff}_{n-1}(\mathbb{R}^n) \cup \{0\} \subset \mathrm{Aff}(\mathbb{R}^n)$.

It may be straightforwardly checked that the proof of Lemma 10.72 after small changes gives the following

Lemma 10.76. *A function $f : S \to \mathbb{R}^n$ admits a $C^{1,\omega}$-extension to $\mathbb{R}^n$ if and only if the map L^*_f has a Lipschitz selection $\vec{G}^*_f : (\mathcal{M}^*, d^*_\omega) \to \mathbb{R}^n$.*

Moreover, up to the constants of equivalence depending only on n,

$$\|f\|_{C^{1,\omega}(\mathbb{R}^n)|_S} \approx \sup_S |f| + \inf_{\vec{G}^*_f} \|\vec{G}^*_f\|_{\mathrm{Lip}}$$

where the infimum is taken over all such selections $\vec{G}^*_f$.

At the final step we turn $(\mathcal{M}^*, d^*_\omega)$ into a metric graph $(\mathcal{M}^*, \widehat{d^*_\omega})$ in a way analogous to that for $(\mathcal{M}, \widehat{d}_\omega)$. Specifically, we regard the latter as a subgraph of the former and then join the vertex $\infty \in \mathcal{M}^*$ by an edge with each vertex of $\mathcal{M}$. Moreover, the geodesic metric $\widehat{d^*_\omega}$ is introduced similarly to $\widehat{d}_\omega$ but with the premetric ψ_ω replaced by

$$\psi^*_\omega := \begin{cases} \omega(\|x-y\|) + \omega(\|x'-y'\|) & \text{if } m, m' \in \mathcal{M}, \\ 2 & \text{otherwise.} \end{cases}$$

The inequality

$$\frac{1}{2}\widehat{d^*_\omega} \le d^*_\omega \le 2\widehat{d^*_\omega}$$

is proved similarly to that of Lemma 10.73 (using, in addition, the inequality $\omega \le 1$ to estimate $d^*(m, m')$ for m or $m' = \infty$).

Starting from this point the proof repeats line by line that for $\dot{C}^{1,\omega}(\mathbb{R}^n)$ with L^*_f and $(\mathcal{M}^*, \widehat{d^*_\omega})$ substituted for L_f and $(\mathcal{M}, \widehat{d}_\omega)$ leading to the required result for $C^{1,\omega}(\mathbb{R}^n)$.

Theorem 10.69 has been proved. □

10.3.2 Fefferman's finiteness theorem

We devote this part to a special case of the Fefferman general result whose form is most relevant to the main direction of the present book. The general result will be discussed in the next subsection.

Theorem 10.77. *The space $C^{k,\omega}(\mathbb{R}^n)$ has the Uniform Finiteness Property.*

That is to say, given $S \subset \mathbb{R}^n$ and $g : S \to \mathbb{R}$ there exist constants $N^* \in \mathbb{N}$ and $\gamma^* > 0$ depending only on k and n such that, if every trace $g|_F$ to a subset of S with at most N^* points admits an extension $g_F \in C^{k,\omega}(\mathbb{R}^n)$ of norm at most 1, then g extends to a function of $C^{k,\omega}(\mathbb{R}^n)$ of norm at most γ^*.

Clearly, N^* and γ^* are upper bounds for the finiteness constants $\mathcal{F}(C^{k,\omega}(\mathbb{R}^n))$ and $\gamma(C^{k,\omega}(\mathbb{R}^n))$, see Definition 10.10. Fefferman's proof estimates the first constant by

$$N^* \le \left(\dim \mathcal{P}_{k,n} + 1\right)^{3\cdot 2^{\dim \mathcal{P}_{k,n}}} \tag{10.137}$$

while the second constant γ^* exponentially depends on N^*.

Using (10.137) one can essentially reduce the upper bound for the finiteness constant of $C^{k,\omega}(\mathbb{R}^n)$. This result was independently due to Bierstone and Milman [BM-2007] and Shvartsman [Shv-2008] who proved the following.

Theorem 10.78. *Let a function g be defined on a finite set $S \subset \mathbb{R}^n$. Assume that every trace $g|_F$ to a subset of cardinality at most $2^{\dim \mathcal{P}_{k,n}}$ admits an extension $g_F \in C^{k,\omega}(\mathbb{R}^n)$ whose norm is at most 1. Then g extends to a function of $C^{k,\omega}(\mathbb{R}^n)$ whose norm is bounded by a constant $\gamma = \gamma(k, n, \operatorname{card} S)$.*

As a consequence of this and the previous theorems we immediately obtain

Corollary 10.79.

$$\mathcal{F}(C^{k,\omega}(\mathbb{R}^n)) \le 2^{\dim \mathcal{P}_{k,n}}.$$

Shvartsman's Proof of Theorem 10.78 *(outlined).* As in Theorem 10.69 two basic ingredients of the proof are a suitably reformulated Whitney-Glaeser Theorem 2.19 of Volume I, cf. Lemma 10.72, and a Lipschitz selection result for *finite* metric spaces given by Theorem 5.52 of Volume I.

So, let $S \subset \mathbb{R}^n$ be a finite set and $f : S \to \mathbb{R}^n$ be given. We reformulate Theorem 2.19 of Volume I as a Lipschitz selection result for maps between metric spaces. The first of them has S as an underlying set and its metric δ_ω is given for $x, y \in S$ by

$$\delta_\omega(x, y) := \omega(\|x - y\|). \tag{10.138}$$

The second space has as the underlying set the disjoint union

$$\mathcal{M}_f := \bigsqcup_{x \in S} \mathcal{H}_f(x) := \bigsqcup_{x \in S} \{p \in \mathcal{P}_{k,n}\, ;\, p(x) = f(x)\}. \tag{10.139}$$

To motivate the definition of a metric on $\mathcal{M}_f$ we present the criterion of Theorem 2.19 of Volume I in a special form.

First, due to Remark 9.15 we can equivalently formulate this theorem as follows.

A function $f : S \to \mathbb{R}$ extends to a function $\tilde{f} \in C^{k,\omega}(\mathbb{R}^n)$ of norm bounded by a certain constant $c = c(n, k)$ if and only if there exists a family of polynomials $\{p_x\}_{x \in S} \subset \mathcal{P}_{k,n}$ such that

(a) *for every $x \in S$ and $|\alpha| \le k$,*

$$f(x) = p_x(x) \quad \textit{and} \quad |D^\alpha p_x(x)| \le 1; \tag{10.140}$$

(b) *for every $x, y \in S$, $z \in \{x, y\}$ and $|\alpha| \le k$,*

$$|D^\alpha\big(p_x - p_y\big)|(z) \le \|x - y\|^{k - |\alpha|}\omega(\|x - y\|). \tag{10.141}$$

Further, we rewrite the above inequalities using a special function $\varphi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$, $|\alpha| \le k$, where $\varphi_\alpha(t) := t$ for $|\alpha| = k$ and φ_α is the inverse to the function $t \mapsto t^{k-|\alpha|}\omega(t)$, $t > 0$, for $|\alpha| < k$. Since φ_α are increasing, the inequality (10.141) may be equivalently written as

$$\max_{|\alpha| \le k} \max_{z \in \{x, y\}} \psi_\alpha\big(D^\alpha(p_x - p_y)(z)\big) \le \omega(\|x - y\|), \tag{10.142}$$

where ψ_α is given by

$$\psi_\alpha(t) := \omega\big(\varphi_\alpha(|t|)\big), \quad t \in \mathbb{R}. \tag{10.143}$$

Hence, the above result is equivalent to that with (10.142) in place of (10.141).

This motivates the definition of the required metric on the set $\mathcal{M}_f$ given for $p \in \mathcal{H}_f(x)$ and $p' \in \mathcal{H}_f(x')$ by

$$\widehat{d}_\omega(p, p') := \max_{|\alpha|\le k} \max_{z\in\{x,y\}} \{\omega(\|x - x'\|), \psi_\alpha\big(D^\alpha(p - p')(z)\big)\}. \tag{10.144}$$

In fact, $\widehat{d}_\omega$ is a pseudometric (with factor 2 in the triangle inequality) and it should be replaced by an equivalent geodesic metric defined in the usual way:

$$d_\omega(p, p') := \inf \sum_{i=1}^{k} \widehat{d}_\omega(p_i, p_{i+1}),$$

where the infimum is taken over all chains $\{p_i \in \mathcal{H}_f(x_i)\}_{1\le i\le k}$ joining p and p' and all k.

The basic fact about d_ω is the inequality

$$d_\omega \le \widehat{d}_\omega \le 3^n d_\omega \tag{10.145}$$

whose proof exploits concavity of the function ψ_α, see [Shv-2008, Thm. 2.1].

Now we present the required formulation of the Whitney-Glaeser criterion where we assume without loss of generality that the function $f : S \to \mathbb{R}$ in question satisfies $\sup_S |f| \le 1$ and set

$$\mathcal{M}_f^* := \bigsqcup_{x\in S} \{p \in \mathcal{H}_f(x)\,;\, \max_{|\alpha|\le k} |D^\alpha p(x)| \le 1\}.$$

Because of the condition on f the set $\mathcal{M}_f^*$ is nonempty.

Proposition 10.80. *The function f extends to a function $\tilde{f} \in C^{k,\omega}(\mathbb{R}^n)$ of norm bounded by a certain constant $c(k,n)$ if and only if there exists a Lipschitz selection map $F : (S, \delta_\omega) \to (\mathcal{M}_f^*, d_\omega)$ with Lipschitz constant at most* 1.

Proof. (Necessity) If $\tilde{f} \in C^{k,\omega}(\mathbb{R}^n)$ is the assumed extension of f, then we define the required map F by setting for $x \in S$,

$$F(x) := T_x^k,$$

where the right-hand side is the Taylor polynomial of $\tilde{f}$ at x of degree k. By definition,

$$F(x)(x) = \tilde{f}(x) = f(x) \quad \text{for all} \quad x \in S,$$

i.e., $F(x) \in \mathcal{H}_f(x)$.

Further, choosing $c(k, n) := 1$ we also get for all $|\alpha| \le k$ and $x \in S$,

$$\sup_S |D^\alpha F(x)| := \sup_S |D^\alpha \tilde{f}| \le c(k, n) = 1,$$

i.e., F maps S into $\mathcal{M}_f^*$.

Finally, due to necessity of condition (10.142) and relations (10.143)-(10.145) we obtain for $x, x' \in S$,

$$d_\omega(F(x), F(x')) \le \widehat{d}_\omega(F(x), F(x'))$$
$$:= \max_{|\alpha|\le k} \max_{z\in\{x,x'\}} \{\omega(\|x-x'\|), \psi_\alpha(D^\alpha(T_x^k - T_{x'}^k)(z))\} \le \omega(\|x-x'\|) =: \delta_\omega(x,x').$$

Hence, F is a 1-Lipschitz selection map from (S, δ_ω) into $\mathcal{M}_f, d_\omega)$, as required.

(Sufficiency) Let $F \in \operatorname{Lip}(S, \mathcal{M}_f)$ be a 1-Lipschitz selection map. By virtue of (10.145) and (10.144), inequality (10.142) or, equivalently, condition (10.141), is then true. Further, $F(x) \in \mathcal{M}_f^* \cap \mathcal{H}_f(x)$, i.e., this polynomial equals $f(x)$ at x and, moreover, $\max_{|\alpha|\le k} |D^\alpha F(x)(x)| \le 1$. Hence, the family $\{F(x)\}_{x\in S}$ satisfies the Whitney-Glaeser conditions (10.140) and (10.141). Theorem 2.19 of Volume I then implies existence of an extension $\tilde{f} \in C^{k,\omega}(\mathbb{R}^n)$ for f with norm bounded by $c(k,n)$. □

At the next step we use a version of the Lipschitz selection Theorem 5.52 of Volume I adjusted to our settings. For its formulation we equip the set $\mathcal{P}_{k,n}\times S$ with a metric $\widehat{\rho}_\omega$ similar to that on $\mathcal{M}_f$. Specifically, we set for (p,x), $(p',x') \in \mathcal{P}_{k,n}\times S$,

$$\widehat{\rho}_\omega\big((p,x),(p',x')\big) := \max_{|\alpha|\le k} \max_{z\in\{x,x'\}} \{\omega(\|x-x'\|), |D^\alpha(p-p')(z)|\}$$

and by ρ_ω we denote the geodesic metric generated by $\widehat{\rho}_\omega$, cf. (10.144).

Now we apply Theorem 5.52 of Volume I giving a Lipschitz selection of a set-valued function from a finite metric space into the set of convex subsets of an n-dimensional linear space. In our case the finite metric space is (S, δ_ω) and the linear space is $\mathcal{P}_{k,n}$ and this theorem immediately implies

Proposition 10.81. *Let J be a set-valued map on the metric space (S, δ_ω) given for $x \in S$ by*

$$J(x) := (G(x), x),$$

where $G(x)$ is a nonempty convex subset of $\mathcal{P}_{k,n}$.

Assume that the restriction $J|_{S'}$ to every subset $S' \subset S$ of cardinality at most $2^{\dim \mathcal{P}_{k,n}}$ extends to a 1-Lipschitz map from (S, δ_ω) into $(\mathcal{P}_{k,n}\times S, \rho_\omega)$.

Then there exists a selection of J with Lipschitz constant depending only on k, n and $\operatorname{card} S$.

Now we are in a position to complete our discussion. Using the identification

$$\mathcal{H}_f(x) \leftrightarrow \{(p,x) \in \mathcal{P}_{k,n}\times S\,;\, p(x) = f(x)\}$$

we embed $\mathcal{M}_f$ into $\mathcal{P}_{k,n}\times S$. Then the restriction of the metric ρ_ω to the image of $\mathcal{M}_f$ coincides with d_ω, i.e., $(\mathcal{M}_f, d_\omega)$ may be identified with a metric subspace of $(\mathcal{P}_{k,n}\times S, \rho_\omega)$. Hence, a set-valued map $F : (S, \delta_\omega) \to \mathcal{M}_f^*$ defined by

$$F(x) := \{p \in \mathcal{H}_f(x)\,;\, \max_{|\alpha|\le k} |D^\alpha p(x)| \le 1\}$$

gives rise to the associated map $\mathcal{F}(x) := (F(x), x)$, $x \in S$.

Now let us assume that the function $f : S \to \mathbb{R}$ in question is such that for every $S' \subset S$ with at most $2^{\dim \mathcal{P}_{k,n}}$ points the trace $f|_{S'}$ extends to a function $f_{S'} \in C^{k,\omega}(\mathbb{R}^n)$ of norm bounded by the constant $c(k,n)$ from Proposition 10.80. Due to the necessity part of this proposition the set-valued map associated to $f_{S'}$, say $F_{S'}$, acting from (S', δ_ω) into $(\mathcal{M}_f^*, d_\omega)$, has a 1-Lipschitz selection. By virtue of the definition of F this means that this map satisfies assumptions of Proposition 10.81. In accordance with this result there exists a c-Lipschitz selection of $\mathcal{F}$, where c depends only on k and n. Then the sufficiency part of Proposition 10.80 implies that f extends to a function from $C^{k,\omega}(\mathbb{R}^n)$ of norm bounded by a constant $c = c(k, n, \operatorname{card} S)$. □

10.3.3 Fefferman's general finiteness theorem

The first generalization deals with the extension of a function defined with only limited accuracy. If the measure of precision is a function $\varepsilon : S \to \mathbb{R}$, then given a function $f : S \to \mathbb{R}$ we are seeking an "ε-approximative" extension $\tilde{f} \in C^{k,\omega}(\mathbb{R}^n)$ such that

$$|f(x) - \tilde{f}(x)| \le \varepsilon(x) \quad \text{on} \quad S.$$

We derive the corresponding finiteness result for this case from the second generalization which requires some new concepts.

The first one, called below a $C^{k,\omega}$ *selection*, deals with a family of nonempty convex subsets $\{\sigma_x\}_{x\in S}$ and every σ_x is a subset in the space of polynomials $\mathcal{P}_{k,n}$. The named object is a function $f \in C^{k,\omega}(\mathbb{R}^n)$ whose Taylor polynomial $T_x^k f$ belongs to σ_x for every $x \in S$. If, e.g., every set σ_x consists of a single polynomial, say p_x, $x \in S$, we turn out to have the conditions of the Whitney-Glaeser Theorem 2.19 of Volume I. Existence of a $C^{k,\omega}$ selection for this family is, in essence, the main assertion of this theorem. Another illustration may be derived from the proof of Theorem 10.69. In this case, given a function $f : S \to \mathbb{R}$ with $\sup_S |f| \le 1$ the associated family consists of the sets

$$\sigma_x := \{p \in \mathcal{P}_{1,n}\, ;\, p(x) = f(x) \quad \text{and} \quad \max_{|\alpha|\le 1} |D^\alpha p(x)| \le 1\}$$

and the main result of the theorem states existence of a $C^{1,\omega}$ selection of this family.

In a quantitative version of the notion presented below, each σ_x is convex and *centrally symmetric* and $\gamma\sigma_x$ for $\gamma > 0$ denotes the image of σ_x under γ-homothety with respect to the center of symmetry for σ_x. Denoting the center by p_x we hence have

$$\gamma\sigma_x := \{\gamma(p - p_x) + p_x\, ;\, p \in \sigma_x\}.$$

Definition 10.82. The family $\{\sigma_x\}_{x\in S}$ has a $C^{k,\omega}$ selection of size $\gamma > 0$ if there exists a function $f \in C^{k,\omega}(\mathbb{R}^n)$ such that

$$\|f\|_{C^{k,\omega}(\mathbb{R}^n)} \le \gamma \quad \text{and} \quad T_x^k f \in \sigma_x \quad \text{for all} \quad x \in S.$$

The aim of the subsequent study is to find finiteness type sufficient conditions on a family $\{\sigma_x\}_{x\in S}$ providing existence of a $C^{k,\omega}$ selection. Two previous examples seemingly confirm accessibility of this goal. Actually, Theorem 2.19 of Volume I reformulated in an equivalent form asserts that the family $\{p_x\}_{x\in S}$ has a $C^{k,\omega}$ selection of size $c = c(k,n)$ if each of its two-point subfamilies has a $C^{k,\omega}$ selection of size 1. In turn, Theorem 10.69 asserts that the corresponding family of subsets in $\mathcal{P}_{1,n}$ has a $C^{1,\omega}$ selection of size $c(k,n)$ if each of its subfamilies of cardinality at most $3\cdot 2^{n-1}$ has a $C^{1,\omega}$ selection of size 1.

However, the next example taken from Fefferman's survey [F-2009a] demonstrates that, in fact, the problem is hopelessly difficult.

Example 10.83. Any partial differential equation with variable coefficients and with boundary conditions on a bounded domain may be written as

$$Lf = g \quad \text{on} \quad \Omega, \quad \widehat{L}f = h \quad \text{on} \quad \partial\Omega. \tag{10.146}$$

Here f is the unknown function, g and h are given, and $L, \widehat{L}$ are variable coefficient linear partial differential operators.

Taking $S := \bar{\Omega}$ we note that the condition $Lf(x) = g(x)$ for a point $x \in \Omega$ or $\widehat{L}f(x) = h(x)$ for a point x in $\partial\Omega$ simply asserts that the Taylor polynomial of f at x belongs to a particular affine subspace σ_x of the corresponding space of polynomials. Hence, the boundary value problem (10.146) has a $C^{k,\omega}$ solution if and only if the family $\{\sigma_x\}_{x\in S}$ admits a $C^{k,\omega}$ selection.

Surprisingly, Fefferman does find rather wide finiteness conditions on $\{\sigma_x\}_{x\in S}$ that provide existence of a $C^{k,\omega}$ selection. His general result implies as consequences Theorem 2.19 of Volume I and Theorem 10.69 (the latter without sharp constants) and its "approximate" version.

To formulate the theorem we need

Definition 10.84. A convex set $\sigma \subset \mathcal{P}_{k,n}$ is said to be (k,ω)-convex at a point $x \in \mathbb{R}^n$ with constant $A > 0$ if:

(a) σ is centrally symmetric with center at 0;

(b) For every $p \in \sigma$ and $q \in \mathcal{P}_{k,n}$ satisfying for some $\delta \in (0,1]$ the inequalities

$$|D^\alpha p(x)| \le \delta^{n-|\alpha|}, \quad |D^\alpha q(x)| \le \delta^{-|\alpha|}, \tag{10.147}$$

the Taylor polynomial of pq at x of degree k belongs to $A\sigma$.

To illustrate this notion we consider several examples; an algebraic structure introduced in the first example will play an essential role in the subsequent exposition of Fefferman's results.

Example 10.85. (a) Let us define a multiplication on the space $\mathcal{P}_{k,n}$ by setting for polynomials p, q of degree k and a point $x \in \mathbb{R}^n$,

$$p \odot_x q := T_x^k(pq).$$

In other words, if $p(y) = \sum_{|\alpha|\le k} p_\alpha (y-x)^\alpha$ and $q(y) = \sum_{|\alpha|\le k} q_\alpha (y-x)^\alpha$, then

$$(p \odot_x q)(y) := \sum_{|\alpha+\beta|\le k} p_\alpha q_\beta (y-x)^{\alpha+\beta},$$

i.e., the polynomials are simply multiplied in the usual way and then powers $(y-x)^\alpha$ with $|\alpha| > k$ are discarded.

Hence, $\odot_x$ is associative, commutative and distributive with respect to summation, i.e., $(\mathcal{P}_{k,n}, +, \odot_x)$ is a commutative $\mathbb{R}$-algebra. In the sequel, it will be denoted by $\mathcal{A}_x^k$ (and also by $\mathcal{A}_x$ if k can be restored).

Now let $\mathcal{L}$ be an *ideal* of $\mathcal{A}_x^k$, i.e.,

(i) $0 \in \mathcal{L}$;

(ii) for every $\lambda, \mu \in \mathbb{R}$ and $p, q \in \mathcal{L}$,

$$\lambda p + \mu q \in \mathcal{L};$$

(iii) for every $p \in \mathcal{A}_x^k$,

$$p \odot_x \mathcal{L} \subset \mathcal{L}.$$

Clearly, $\mathcal{L}$ is (k,ω)-convex at x with constant 1.

(b) Let $\gamma > 0$ be given. Then the set $\Sigma := \{p \in \mathcal{P}_{k,n}\,;\, |p(x)| \le \gamma\}$ is (k,ω)-convex at x with constant 1.

Actually, if $p \in \Sigma$, $q \in \mathcal{P}_{k,n}$ and (10.147) holds, then

$$|T_x^k(pq)(x)| = |p(x)q(x)| \le \gamma \cdot \delta^k \le \gamma.$$

(c) The set $\Sigma_k := \{T_x^k f\,;\, \|f\|_{C^{k,\omega}(\mathbb{R}^n)} \le 1 \text{ and } f|_S = 0\}$ is (k,ω)-convex at x with constant $c = c(k,n)$.

Actually, let $p = T_x^k f \in \Sigma_k$ and $q \in \mathcal{P}_{k,n}$. Multiplying f by a C^∞ test function equals 1 on the ball $B_1(x)$ and 0 outside the ball $B_2(x)$, we have for the function $\tilde{f}$ so defined

$$T_x^k(pq) = T_x^k(q\tilde{f}) \quad \text{and} \quad \|q\tilde{f}\|_{C^{k,\omega}(\mathbb{R}^n)} \le c(k,n).$$

Hence, $p \odot_x q \le \frac{1}{c(k,n)} \Sigma_k$ as required.

Now we present the generalized finiteness theorem by Fefferman [F-2005b] and some of its consequences. For its formulation we extend Definition 10.84 to general centrally symmetric sets as follows:

A set $\sigma \in \mathcal{P}_{k,n}$ is said to be (k,ω)-*convex at x with constant A* if the shifted set $\sigma - p$ for some $p \in \sigma$ is subject to Definition 10.84.

Clearly, the set σ so defined is centrally symmetric with center at p, so that, as above, $A\sigma$ is the image of σ under A-homothety with respect to p.

Theorem 10.86. *Let $S \subset \mathbb{R}^n$ be a subset. There exists an integer $N^* > 1$ such that the following is true.*

Let a set σ_x be (k,ω)-convex at x with constant A for every $x \in S$. Assume that for some constant $M > 0$ every subfamily of the family $\{\sigma_x\}_{x\in S}$ consisting of at most N^* elements has a $C^{k,\omega}$ selection of size M.

Then there exists a constant $A' = A'(k,n,A)$ such that the family $\{A'M\sigma_x\}_{x\in S}$ has a $C^{k,\omega}$ selection of size $A'M$.

Shvartsman's theorem [Shv-2008, Thm.1.6] a special case of which has been presented as Theorem 10.78 is, in fact, devoted to the settings of Theorem 10.86. It gives the following estimate for the Fefferman constant N^*:

$$N^* \le 2^{\min\{\ell+1, \dim \mathcal{P}_{k,n}\}}, \tag{10.148}$$

where $\ell := \max_{x\in S} \dim \sigma_x$.

Now we present several consequences of Fefferman's result demonstrating its power and profundity.

Corollary 10.87 (Whitney-Glaeser). $\mathcal{F}(J^{\ell,\omega}(\mathbb{R}^n)) = 2$.

Proof. Given $S \subset \mathbb{R}^n$ and a family $\{p_x\}_{x\in S} \subset \mathcal{P}_{k,n}$ such that for every pair of points $x \ne y$ from S there exists a function $f_{x,y} : \mathbb{R}^n \to \mathbb{R}$ satisfying for $z \in \{x, y\}$,

$$\|f_{x,y}\|_{C^{k,\omega}(\mathbb{R}^n)} \le 1, \quad T_z^k f_{x,y} = p_z, \tag{10.149}$$

we should find a function $f : \mathbb{R}^n \to \mathbb{R}$ which for some constant $c = c(k,n)$ and all $x \in S$ complies with the conditions

$$\|f\|_{C^{k,\omega}(\mathbb{R}^n)} \le 1 \quad \text{and} \quad T_x^k f = p_x.$$

To this end we consider the family $\{\sigma_x\}_{x\in S}$, where $\sigma_x := \{p_x\}$. Clearly, every σ_x is (k,ω)-convex with constant 1 and the number ℓ in (10.148) is 2, i.e., N^* here equals 2.

Due to (10.149) and (10.148) the assumption of Theorem 10.86 holds for this case and therefore the required f exists. □

As the next consequence we consider a generalization of Theorem 10.77 (on the Uniform Finiteness Property for $C^{k,\omega}(\mathbb{R}^n)$). In its formulation, $\varepsilon : S \to \mathbb{R}_+$ is a round-off function measuring accuracy of extension from the set $S \subset \mathbb{R}$. A function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ is said to be *an ε-approximate extension of $f : S \to \mathbb{R}$* if for all $x \in S$,

$$|f(x) - \tilde{f}(x)| \le \varepsilon(x).$$

If $\varepsilon = 0$, then $f = \tilde{f}$ on S, i.e., $\tilde{f}$ extends f to $\mathbb{R}^n$.

Corollary 10.88. *A function $f : S \to \mathbb{R}$ admits a $c\varepsilon$-approximate extension $\tilde{f} \in C^{k,\omega}(\mathbb{R}^n)$ of norm at most $c = c(k,n)$ if every restriction $f|_{S'}$ to a subset $S' \subset S$ with at most $2^{\dim \mathcal{P}_{k,n}}$ points admits an ε-approximate extension $f_{S'} \in C^{k,\omega}(\mathbb{R}^n)$ of norm at most* 1.

Proof. We set for $x \in S$,

$$\sigma_x := \{p \in \mathcal{P}_{k,n}\,;\, |p(x) - f(x)| \le \varepsilon(x)\}.$$

Then σ_x is (k,ω)-convex at x with constant 1, see Example 10.85(b). Moreover, the constant ℓ in (10.148) equals now $1 + \dim \mathcal{P}_{k,n}$, i.e., $N^* \le 2^{\dim \mathcal{P}_{k,n}}$. The application of Theorem 10.86 to this case immediately implies the result. □

If, in particular, $\varepsilon = 0$, then we conclude from here that

$$\mathcal{F}(C^{k,\omega}(\mathbb{R}^n)) \le 2^{\dim \mathcal{P}_{k,n}}.$$

This gives the improved version of Theorem 10.77.

Finally, we consider another aspect of the Finiteness Property related to Computer Science.

Problem 10.89. *Given a real function f on a finite subset $S \subset \mathbb{R}^n$ find an algorithm computing a function $\tilde{f} \in C^{k,\omega}(\mathbb{R}^n)$ such that*

$$\|\tilde{f}\|_{C^{k,\omega}(\mathbb{R}^n)} \le M \text{ and } |f(x) - \tilde{f}(x)| \le M\varepsilon(x) \text{ for all } x \in S, \tag{10.150}$$

where constant M is almost optimal.

Here almost optimality of M means that it is equivalent to the optimal constant in (10.150) with the constants of equivalence depending only on k and n.

Of course, the meaning of the Computer Science concept used in the formulation should be explained, i.e., that of *computation of a function*. A detailed explanation is presented in the paper [FK-2009] to which the reader is referred. Here we consider only one question concerning the computer realization of the Finiteness Property.

According to Corollary 10.88 the solution to Problem 10.89 may be obtained by testing the family of trace functions $f|_{S'}$, where S' runs over all subsets of S with at most $\mathcal{F}(C^{k,\omega}(\mathbb{R}^n))$ points. Even for $k = 0$, where the finiteness constant is 2, the number of the required tests is of order $O(N^2)$ where $N := \operatorname{card} S$ whereas for $k = 1$ this number becomes enormously large (equals $O(N^{3\cdot 2^{n-1}})$). However, the theorem of Callahan and Kasaraiu [CK-1995] implies the possibility to use only $O(N)$ tests for $k = 0$. The far-reaching generalization of this fact was due to Fefferman [F-2009] who proved that $O(N)$ tests is sufficient for every k. This result was then extended even to the general case of Theorem 10.86 by Fefferman and Klartag [FK-2007]. In both cases ω is a linear function but apparently the case of a general ω requires minor changes in their proof.

Restricting ourselves to the settings of Problem 10.89, we present now the latter result. In its formulation, $\|f\|_{S,\varepsilon}$ stands for the optimal M in inequality (10.150) and the constants of equivalence depend only on k and n.

Theorem 10.90 ([F-2009]). *Let S be a subset of $\mathbb{R}^n$ with N points and let $\varepsilon : S \to \mathbb{R}_+$. Then there exist a constant $C = C(k,n)$ and a family of subsets $\{S_j\}_{1\le j\le L}$ of S such that:*

(a) *Every* S_j *contains at most* C *points.*

(b) *The number of subsets* S_j *is at most* CN.

(c) *For each* $f : S \to \mathbb{R}$ *we have*

$$\|f\|_{S,\varepsilon} \approx \max_{1 \le j \le L} \|f\|_{S_j,\varepsilon}.$$

Remark 10.91. This theorem gives, in addition, some information on computational aspects of the problem.[3] Hence, to verify solvability of Problem 10.89 it suffices to use at most $O(\operatorname{card} S)$ tests. Moreover, the results of these tests may be used to compute the almost optimal constant M in (10.150).

10.3.4 Fefferman's linearity theorem

Unlike the previous subsection we begin the discussion of the general result at once and only then derive from it some special cases. The result now presented is a linearized version of Theorem 10.86. However, the settings of the latter theorem and the corresponding extension operator are highly nonlinear; the required linearization procedure deals with a linear variable voluntarily inserted in initial data. This variable is a point of some seminormed linear space $(\Xi, |\cdot|)$ (not necessarily complete) denoted below by ξ. Hence, the family $\{\sigma(x)\}_{x\in S}$, $\sigma(x) := \sigma_x$, of centrally symmetric convex sets of Theorem 10.86 is now transformed into a family $\{\sigma_\xi(x)\}_{x\in S}$, where $\sigma_\xi(x) := \sigma(x) + p_\xi(x)$ and the second term is an element of $\mathcal{P}_{k,n}$ *linearly depending* on $\xi \in \Xi$.

The problem now is to find finiteness type conditions providing existence of a $C^{k,\omega}$ selection for the family $\{\sigma_\xi(x)\}_{x\in S}$ which linearly depends on ξ. The answer given by Fefferman's theorem [F-2009b] is as follows.

Theorem 10.92. *Let every set* $\sigma(x), x \in S$, *be* (k,ω)*-convex at* x *with constant* A. *Then there exists an integer* N^* *depending only on* k *and* n *for which the following holds.*

Assume that, given $\xi \in \Xi$ *with* $|\xi| \le 1$, *for every subfamily of the family* $\{\sigma_\xi(x)\}_{x\in S}$ *with cardinality at most* N^* *there exists a* $C^{k,\omega}$ *selection of size* 1. *Then there exists a linear map* $\xi \mapsto f_\xi$ *from* Ξ *into* $C^{k,\omega}(\mathbb{R}^n)$ *such that, for* $|\xi| \le 1$, *the function* f_ξ *is a* $C^{k,\omega}$ *selection of the family* $\{\sigma_\xi(x)\}_{x\in S}$ *of size* A' *depending only on* k, n *and* A.

Now we present consequences of this powerful result placing them into descending generality order.

To formulate the first result we introduce a seminormed space $C^{k,\omega}(\sigma)$, where $\sigma : x \mapsto \sigma(x)$, $x \in S$, and the sets $\sigma(x)$ are chosen as in Theorem 10.92 *but with* 0

[3] For the versed reader: the family $\{S_j\}$ may be computed from S, ε, k, n using at most $CN \log N$ operations, and using storage at most CN.

as the common center of symmetry. Specifically, a function $f : S \to \mathbb{R}$ belongs to $C^{k,\omega}(\sigma)$ if the family $\{\sigma(x) + f(x)\}_{x\in S}$ has a $C^{k,\omega}$ selection of size M. We also set

$$|f|_{C^{k,\omega}(\sigma)} := \inf M.$$

It is easily seen, that the functional $f \mapsto |f|_{C^{k,\omega}(\sigma)}$ satisfies the triangle inequality and it is zero if $f = 0$. Hence, $C^{k,\omega}(\sigma)$ equipped with this functional is a seminormed space.

In this setting, we have

Corollary 10.93. *There exists a linear operator $T : C^{k,\omega}(\sigma) \to C^{k,\omega}(\mathbb{R}^n)$ of norm bounded by $c = c(k,n)$ such that if $|f|_{C^{k,\omega}(\sigma)} \le \frac{1}{2}$, then Tf is a $C^{k,\omega}$ selection of the family $\{\sigma(x) + f(x)\}_{x\in S}$ of size $c = c(k,n,A)$.*

Proof. One applies Theorem 10.92 with $\Xi := C^{k,\omega}(\sigma)$ equipped with the seminorm $|\xi| := (1+\varepsilon)|f|_{C^{k,\omega}(\sigma)}$ where ξ is identified with $f \in C^{k,\omega}(\sigma)$ and $\varepsilon > 0$ is arbitrary. Further, the linear map $\xi \mapsto f_\xi(x)$ is identity, i.e., for $\xi = (f(x))_{x\in S}$ one just sets $f_\xi(x) := f(x)$.

Let us check that the assumption of Theorem 10.92 holds. Actually, suppose that $\xi := f = (f(x))_{x\in S} \in \Xi$ and $|\xi| \le 1$. Then $|f|_{C^{k,\omega}(\sigma)} \le \frac{1}{1+\varepsilon} < 1$ and by definition this implies existence of a $C^{k,\omega}$ selection for the family $\{\sigma(x)+f(x)\}_{x\in S}$ of size 1, hence, also for *every* member of its subfamily. Thus, the assumption of Theorem 10.92 holds in this case. Applying this theorem we obtain a linear map $T : C^{k,\omega}(\sigma) \to C^{k,\omega}(\mathbb{R}^n)$ such that for $f = (f(x))_{x\in S} \subset C^{k,\omega}(\sigma)$ with $|f|_{C^{k,\omega}(\sigma)} \le \frac{1}{1+\varepsilon}$ the function Tf is a $C^{k,\omega}$ selection for the family $\{\sigma(x)+f(x)\}_{x\in S}$ of size $c = c(k,n,A)$. □

To derive a linearized version of the ε-approximate extension result, see Corollary 10.88, we set for $x \in S \subset \mathbb{R}^n$ and $\varepsilon : S \to \mathbb{R}$,

$$\widehat{\sigma}(x) := \{p \in \mathcal{P}_{k,n} \,;\, |p(x)| \le \varepsilon(x)\}.$$

Then $\widehat{\sigma}$ is clearly convex centrally symmetric with respect to 0, and is (k,ω)-convex at x with constant 1, see Example 10.85(b).

Applying to the $\widehat{\sigma}$ the previous result we get

Corollary 10.94. *There exists a linear operator $T : C^{k,\omega}(\widehat{\sigma}) \to C^{k,\omega}(\mathbb{R}^n)$ of norm bounded by $c(k,n)$ such that for $|f|_{C^{k,\omega}(\sigma)} \le 1$,*

$$|Tf(x) - f(x)| \le c\varepsilon(x) \quad \textit{for all} \quad x \in S$$

with c depending only on k and n.

Finally, choosing $\varepsilon = 0$ we obtain the required linear extension result for the space $C^{k,\omega}(\mathbb{R}^n)$.

Theorem 10.95. *There exists a linear operator $T : C^{k,\omega}(\mathbb{R}^n)|_S \to C^{k,\omega}(\mathbb{R}^n)$ of norm at most $c(k,n)$ such that*

$$Tf(x) = f(x) \quad \textit{for all} \quad x \in S.$$

In fact, now all sets $\widehat{\sigma}(x) = \{0\}$ and therefore $|f|_{C^{k,\omega}(\widehat{\sigma})} = 0$ for every f.

Remark 10.96. Recently G.K. Luli [Lul-2010] proved that a modified Fefferman's linear extension operator $C^{k,\omega}(\mathbb{R}^n)|_S \to C^{k,\omega}(\mathbb{R}^n)$ has finite depth (see Example 10.38 (b)) bounded by some constant $c(k, n)$.

10.4 Fefferman's solution to the classical Whitney problems

We survey Fefferman's results devoted to the trace and extension problems for the space $C_b^k(\mathbb{R}^n)$ going back to the classical Whitney papers [Wh-1934a] and [Wh-1934b]. Since the term "Whitney problems" has been used in the present book in an essentially more general context, we distinguish those for $C_b^k(\mathbb{R}^n)$ by adding the adjective "classical".

As before our main theme is the trace and extension problems, however, for the essentially more complicated case of the space $C_b^k(\mathbb{R}^n)$. One indication of this complexity is nonexistence for this case of the Uniform Finiteness Property. In the first subsection, we present another type of finiteness property following Fefferman's paper [F-2006a] and also another version of this concept, the Strong Finiteness Property, discussed earlier in subsection 2.4.3 of Volume I.

The second subsection surveys the Fefferman paper on the Linear Extension Property [F-2007a] containing the most profound result of the field.

Finally, we briefly discuss the basic features of the Fefferman approach following his paper [F-2006a].

10.4.1 Fefferman's finiteness theorem for $C_b^k(\mathbb{R}^n)$

The main object of this theorem is a family $H := \{H(x)\}_{x \in S}$, where every $H(x)$ is either the empty set or a coset of an ideal of the algebra $\mathcal{A}_x := \{\mathcal{P}_{k,n}, \odot_x\}$, see Example 10.85(b) (the empty set values of H are allowed for a technical reason). This will be called, for brevity, a *k-bundle over a set $S \subset \mathbb{R}^n$* with "fibres" $H(x)$.[4] The choice of the object reflects the fact that every space $C^{k,\omega}(\mathbb{R}^n)$ embeds continuously into $C_b^k(\mathbb{R}^n)$, since a k-bundle is (k, ω)-convex for every ω.

A k-bundle may be naturally regarded as a set-valued map from a subset of $\mathbb{R}^n$ into the set of affine subspaces of $\mathcal{P}_{k,n}$. Therefore, it is natural to ask for existence of C_b^k selections of a k-bundle. For the precise formulation of the problem we use

Definition 10.97. A function $f \in C_b^k(\mathbb{R}^n)$ is said to be a C_b^k selection of size $A > 0$ of a k-bundle $\{H(x)\}_{x \in S}$ if the Taylor polynomial $T_x^k f$ belongs to $H(x)$ for every nonempty fibre $H(x)$, $x \in S$, and, moreover,

$$\|f\|_{C_b^k(\mathbb{R}^n)} := \max_{|\alpha| \le k} \sup_{\mathbb{R}^n} |D^\alpha f(x)| \le A.$$

[4] $H(x)$ need not vary continuously in x; actually, they may vary wildly.

As in the case of $C^{k,\omega}$, the basic question concerns existence of so defined C_b^k selections. Comparing the Whitney extension Theorems 2.47 and 2.52 of Volume I for $C_b^k(\mathbb{R})$ and $C^{k,\omega}(\mathbb{R})$, respectively, we remark a new component appearing in the former case, namely, a certain limit procedure. The multivariate case clearly requires a more involved limit construction; it firstly appeared in the Glaeser geometric result [Gl-1958] devoted to a solution to the $C_b^1(\mathbb{R}^n)$ trace problem. The Glaeser construction was generalized by Bierstone, Milman and Pawlucki [BMP-2003] to characterize the traces of C_b^k functions to subanalytic subsets. Another version of their construction due to Fefferman [F-2006a] plays a considerable role in the proof of his finiteness theorem.

The mentioned construction called the *Glaeser refinement* consists of the family of operations $\{GR_N\}_{N\in\mathbb{N}}$ acting on the set of k-bundles; every GR_N turns a bundle $H := \{H(x)\}_{x\in S}$ into its subbundle $\widetilde{H} := GR_N(H)$ defined by the next condition.

For every $x^0 \in S$ a polynomial $p_0 \in \mathcal{P}_{k,n}$ belongs to a fibre $\widetilde{H}(x^0)$ if for any set $\{x^j\}_{1\le j\le N} \subset S$ there exist polynomials $p_j \in H(x^j)$, $1 \le j \le N$, such that

(*)
$$\max_{|\alpha|\le k}\max_{1\le j\le N}\frac{|D^\alpha(p_i-p_j)(x^j)|}{\|x^i-x^j\|^{k-|\alpha|}} \to 0$$

as all $x^1,\dots,x^N$ tend to x^0.

Clearly, $\widetilde{H}(x^0)$ may be empty even if $H(x^0)\neq\emptyset$. Moreover, $\widetilde{H}(x^0)=H(x^0)$ if x^0 is an isolated point of S. Hence, $GR_N(H)=H$ if H is a k-bundle over a *finite* set.

In the formulations of the subsequent results we assume that $S\subset\mathbb{R}^n$ is a *compact set*; finiteness results for arbitrary closed sets may be easily obtained from the compact set case using an appropriate partition of unity. However, this assumption is introduced for a more essential reason: all results based on Glaeser refinement cannot be achieved for noncompact sets.

Proposition 10.98. *Let H be a k-bundle over $S\subset\mathbb{R}^n$. Then the following holds:*

(a) *$GR_N(H)$ is a subbundle of H.*

(b) *If f is a C_b^k selection of H of size A, then f is that for $GR_N(H)$.*

(c) *There exists an integer $N^* = N^*(k,n)$ such that a sequence $\{H_j\}_{j\in\mathbb{Z}_+}$, where $H_0 := H$ and $H_{j+1} := GR_{N^*}(H_j)$ for $j\ge 0$ stabilizes after the $j^* := 2\dim\mathcal{P}_{k,n}+1$ initial steps, i.e.,*

$$H_{j^*} = H_{j^*+1} = \cdots .$$

Only to illustrate the concept introduced we prove assertions (a) and (b). The third statement is going back to the ingenious simple Glaeser lemma [Gl-1958] adapted to the general case in [BMP-2003]; its simple proof may be found in [F-2006a].

Proof. Let $\widetilde{H}(x) := GR_N(H)(x^0)$, $x^0 \in S$, contain more than one element (otherwise all is trivial). In case (a) we should show that $\widetilde{H}(x^0)$ is a linear space and the shifted set $-p_0 + \widetilde{H}(x^0)$ is an ideal of the $\mathbb{R}$-algebra $\mathcal{A}^k_{x^0}$ for any $p_0 \in H(x)$. The former is evident, since condition (*) holds for a linear combination if it is valid for each of its terms.

Now let $\tilde{p}_0 \in \widetilde{H}(x^0)$ and $q \in \mathcal{P}_{k,n}$ be arbitrary. We show that $\hat{p}_0 := p_0 + q(\tilde{p}_0 - p_0)$ belongs to $\widetilde{H}(x^0)$, i.e., $-p_0 + \widetilde{H}(x^0)$ is an ideal in the $\mathcal{A}^k_x$. Actually, by Definition 10.97 there exist sequences $\{p_j\}_{1\le j\le N}$ and $\{\tilde{p}_j\}_{1\le j\le N}$ related to p_0 and $\tilde{p}_0$, respectively, for which (*) holds. Then the sequence $\hat{p}_j := p_0 + q(\tilde{p}_j - p_j)$, $1 \le j \le N$, satisfies

$$\hat{p}_i - \hat{p}_j = q(\tilde{p}_i - \tilde{p}_j) - q(p_i - p_j)$$

and therefore conditions (*) for p_0 and $\tilde{p}_0$ imply condition (*) for $\hat{p}_0$.

Now let $f \in C^k_b(\mathbb{R}^n)$ be a C^k_b-selection of H of size A. To show that f is such also for $GR_N(H) =: \widetilde{H}$ we should check that for every nonisolated point $x^0 \in S$ the Taylor polynomial $T^k_{x^0} f$ belongs to $\widetilde{H}(x^0)$. Let $\{x^j\}_{1\le j\le N} \subset S$ and $p_j := T^k_{x^j} f$, $1 \le j \le N$. Then by the Taylor formula the term

$$\max_{|\alpha|\le k}\max_{1\le j\le N} \frac{|(D^\alpha T^k_{x^0} f - p_j)(x^j)|}{\|x^i - x^j\|^{k-|\alpha|}}$$

tends to zero as all x^j tend to x^0. Hence, (*) holds for $T^k_{x^0} f$ and therefore f is a C^k_b selection of size A for $\widetilde{H}$. □

Proposition 10.98 reduces the question on C^k_b selections for a k-bundle H to that for a somewhat simpler *Glaeser stable* subbundle (i.e., a k-bundle coinciding with its Glaeser refinement). This leads to the following strikingly simple result, the first part of Fefferman's theorem under consideration.

Theorem 10.99. *Let H be a k-bundle over a compact set $S \subset \mathbb{R}^n$ and $\widetilde{H}$ be a Glaeser stable subbundle of H. Then the following holds.*

(a) *H has a C^k_b selection if and only if $\widetilde{H}(x) \ne \emptyset$ for every $x \in S$.*

(b) *For every $x \in S$ the linear space $\widetilde{H}(x)$ is composed by the Taylor polynomials $T^k_x f$, where f runs over all C^k_b selections of H (or, which is the same, of $\widetilde{H}$).*

A quantitative version of this result is given by the second part of Fefferman's theorem.

Theorem 10.100. *Under the assumptions of the previous theorem there exists an integer N^* depending only on k and n such that:*

$\widetilde{H}$ has a C^k_b selection of size $A = A(k,n)$ if and only if for every subset $\{x^j\}_{1\le j\le N^*} \subset S$ there exist polynomials $p_j \in \widetilde{H}(x^j)$, $1 \le j \le N^*$, satisfying for all $|\alpha| \le k$ and $1 \le i, j \le N$ the condition

$$|D^\alpha p_j(x^j)| \le 1 \quad \textit{and} \quad |D^\alpha (p_i - p_j)(x^j)| \le \|x^i - x^j\|^{k-|\alpha|}.$$

Since a k-bundle over a finite set is Glaeser stable, the conclusions of Theorem 10.99 are evident while Theorem 10.100 has a highly non-trivial content even in this case. However, this special case can be proved essentially simpler, see Bierstone and Milman [BM-2007]. The result of these authors combined with Theorem 10.100 gives the next improvement of the upper bound for the Fefferman constant N^*:

$$N^* \leq 2^{\dim \mathcal{P}_{k,n}}.$$

Now we return to the initial question on characterization of the traces of $C_b^k(\mathbb{R}^n)$ functions to S. To formulate the result we define a k-bundle H_f over S given at $x \in S$ by

$$H_f(x) := \{p \in \mathcal{P}_{k,n}\,;\, p(x) = f(x)\}.$$

Then a straightforward consequence of the above presented results is as follows.

Theorem 10.101. *There exists an integer* $N^* \in (2\dim \mathcal{P}_{k,n}, 2^{\dim \mathcal{P}_{k,n}}]$ *such that* $f : S \to \mathbb{R}$ *is the trace of a* $C_b^k(\mathbb{R}^n)$ *function of norm at most* $c(k,n)$ *if and only if every trace* $f|_{S'}$ *to a subset* $S' \subset S$ *with at most* N^* *points extends to a function* $f_{S'} \in C_b^k(\mathbb{R}^n)$ *of norm at most* 1 *such that*

$$T_x^k f_{S'}(x) \in H^*(x) \quad \textit{for all} \quad x \in S';$$

here H^* *is the Glaeser stable subbundle of* H *(existing as* $N^* > 2\dim \mathcal{P}_{k,n}$*).*

Let us discuss shortly to what extent the results presented are constructive.

From one point, the Fefferman Theorems 10.100-10.101 are constructive, since all the involved operations may be fulfilled by doing elementary linear algebra and taking limits. On the other hand, the number of these operations is very large even for relatively small n and k (say, for n, k about 10).

From another point of view a result is recognized to be constructive if its already known special cases can be derived from it as easy consequences. However, this is not the case neither for the Whitney divided difference criterion for $C_b^k(\mathbb{R})$, see Volume I, Theorem 2.47, nor for Theorem 10.66 characterizing traces to Markov sets. Therefore, search for solutions to the finiteness problem corresponding to the latter constructivity requirement is of considerable interest.

One of them presented below is related to the Strong Finiteness Property, see Volume I, Definition 2.59. This property was used in subsection 2.4.3 of Volume I to reformulate Whitney's Theorem 2.47 there as a finiteness type result. The limit procedure presented in its formulation, in fact, reflects a compactness property of the family of extensions used there. This point in a more precise form is presented in the next multivariate result.

Theorem 10.102. *There exists a constant* $N^* \leq 2^{\dim \mathcal{P}_{k,n}}$ *such that a function* $f : S \to \mathbb{R}$ *belongs to the trace space* $C_b^k(\mathbb{R}^n)|_S$ *with norm bounded by* $c(k,n)$ *if and only if the trace of* f *to every subset of* S *with at most* N^* *points extends to a function contained in a fixed compact subset of the unit ball of* $C_b^k(\mathbb{R}^n)$*.*

Proof. The assertion clearly holds if f belongs to $C_b^k(\mathbb{R}^n)|_S$ with norm at most 1.

To prove the converse statement we choose the constant N^* by setting

$$N^* := \max_{\omega} \mathcal{F}(C^{k,\omega}(\mathbb{R}^n)); \tag{10.151}$$

according to Theorem 10.78 this constant is at most $2^{\dim \mathcal{P}_{k,n}}$.

Now let $S' \subset S$ be a subset of cardinality at most N^*. By the assumptions of the theorem, $f|_{S'}$ extends to a function $f_{S'} \in C_b^k(\mathbb{R}^n)$ such that

$$\|f_{S'}\|_{C_b^k(\mathbb{R}^n)} \le 1 \tag{10.152}$$

and, moreover, the set

$$\mathcal{K}_f := \{f_{S'}\,;\, S' \subset S \text{ and } \operatorname{card} S' \le N^*\}$$

is precompact.

We should show that $f \in C_b^k(\mathbb{R}^n)|_S$ and its trace norm is bounded by $c = c(k,n)$. To this end we fix a closed cube $\bar{Q} \subset \mathbb{R}^n$ and $|\alpha| = k$ and consider a map L_α from $C_b^k(\mathbb{R}^n)$ into $C(\bar{Q})$ given by $L_\alpha(f) := D^\alpha f|_{\bar{Q}}$. Since L_α is continuous and the continuous image of a compact set is compact, the set $\{D^\alpha g|_{\bar{Q}}\,;\, g \in \mathcal{K}_f\}$ is precompact in $C(\bar{Q})$. Then by the Arcela-Ascoli criterion there exist a 1-majorant ω_α and a constant $\bar{c}_\alpha > 0$ such that

$$\sup_{t>0} \frac{\omega(t; D^\alpha g|_{\bar{Q}})}{\omega_\alpha(t)} \le \bar{c}_\alpha$$

for all $g \in \mathcal{K}_f$. This and (10.152) imply that the family of extensions $\mathcal{K}_f$ satisfies the condition

$$\sup\left\{\|f_{S'}\|_{C^{k,\omega}(\bar{Q})}\,;\, S' \subset S \text{ and } \operatorname{card} S' \le N^*\right\} \le c, \tag{10.153}$$

where $\omega := \max_{|\alpha|=k} \omega_\alpha$ and $c := \max_{|\alpha|=k} \bar{c}_\alpha$.

The constant c here depends not only on k, n but also on the radius of $\bar{Q}$. To avoid the last dependence we assume for a moment that S is a closed subset of the cube $\bar{Q}$ of radius 1. Then we apply Fefferman's extension Theorem 10.77 for the space $C^{k,\omega}(\mathbb{R}^n)$ in these settings. Due to the choice of N^* the assumptions of this theorem hold. Hence, there exists an extension $f_{\bar{Q}}$ of the function $f : S \to \mathbb{R}$ belonging to the space $C^{k,\omega}(\mathbb{R}^n)$ with norm at most $c(k,n) > 0$.

Now let $S \subset \mathbb{R}^n$ be an arbitrary closed set. Let $\mathcal{F} = \{\bar{Q}\}$ be a cover of $\mathbb{R}^n$ by unit closed cubes such that the intersection of two cubes of this cover is either its common facet or the empty set. Applying the result proved above to the function $f|_{S\cap\bar{Q}}$, where $\bar{Q} \in \mathcal{F}$, we find its extension $f_{\bar{Q}} \in C^{k,\omega}(\mathbb{R}^n)$ such that $\|f_{\bar{Q}}\|_{C^{k,\omega}(\mathbb{R}^n)} \le c(k,n)$ for some $\omega = \omega(\bar{Q})$. Therefore,

$$\max_{|\alpha|\le k} \sup_{2\bar{Q}} |D^\alpha f_{\bar{Q}}| \le \|f_{\bar{Q}}\|_{C^{k,\omega}(\mathbb{R}^n)} \le c(k,n).$$

Here we use the norm equivalent to the initial norm of the space $C^{k,\omega}(\mathbb{R}^n)$ which includes all derivatives up to order k.

Finally, let $\{\varphi_{\bar{Q}}\}_{\bar{Q}\in\mathcal{F}}$ be a C^∞ partition of unity subordinate to the cover $\{2\bar{Q}\}_{\bar{Q}\in\mathcal{F}}$ such that $\phi_{\bar{Q}} = 1$ on $\bar{Q}$. Then $\tilde{f} = \sum_{\bar{Q}\in\mathcal{F}} \varphi_{\bar{Q}} f_{\bar{Q}}$ is a C^k function coinciding with f on S whose C_b^k-norm is bounded by a constant depending only on k and n.

The theorem has been proved. □

10.4.2 Fefferman's linearity theorems

We first present a result formulated in the context of the bundles.[5] So, let as before $I(x)$ be an ideal of the $\mathbb{R}$-algebra $\mathcal{A}_x^k := (\mathcal{P}_{k,n}, \odot_x)$. Given a family of ideals $\{I(x)\}_{x\in S}$, where S is an arbitrary subset of $\mathbb{R}^n$, we define a subset $\mathcal{J}(S)$ of $C_b^k(\mathbb{R}^n)$ by

$$\mathcal{J}(S) := \{f \in C_b^k(\mathbb{R}^n)\,;\, T_x^k f \in I(x) \quad \text{for all} \quad x \in S\}.$$

By virtue of the definition of multiplication in $\mathcal{A}_x^k$, this set is an ideal in the $\mathbb{R}$-algebra $(C_b^k(\mathbb{R}^n), \odot_x)$. In fact, if $f_1, f_2 \in \mathcal{J}(S)$ and $g_1, g_2 \in C_b^k(\mathbb{R}^n)$, then for any $x \in S$,

$$T_x^k(f_1 g_1 + f_2 g_2) = \sum_{i=1,2} T_x^k f_i \odot_x T_x^k g_i$$

and each summand (and the sum) belongs to the ideal $I(x)$, since every $T_x^k f_i$ does.

Further, $\mathcal{J}(S)$ is a closed subspace of $C_b^k(\mathbb{R}^n)$. Actually, if a sequence $\{f_i\}$ from $\mathcal{J}(S)$ converges in this space to a function f, then $T_x^k f_j \to T_x^k f$ for every $x \in S$. Since $\{T_x^k f_j\}$ is contained in the finite-dimensional (hence, closed) subspace $I(x)$, the polynomial $T_x^k f$ belongs to $I(x)$, as required.

Hence the factor-space $C_b^k(\mathbb{R}^n)/\mathcal{J}(S)$ is Banach. Let π denote the natural projection of $C_b^k(\mathbb{R}^n)$ onto $C_b^k(\mathbb{R}^n)/\mathcal{J}(S)$. The Bartle–Graves Theorem 1.6 of Volume I implies that there exists a *nonlinear* continuous map inverse to π, $\tau : C_b^k(\mathbb{R}^n)/\mathcal{J}(S) \to C_b^k(\mathbb{R}^n)$, i.e., such that $\pi\tau$ is the identity on $C_b^k(\mathbb{R}^n)/\mathcal{J}(S)$.

The next Fefferman result, see [F-2007a], gives an essentially more profound and difficult linear version of this statement.

Theorem 10.103. *Given S and $\{I(x)\}_{x\in S}$ as above there exists a linear operator from $C_b^k(\mathbb{R}^n)/\mathcal{J}(S)$ into $C_b^k(\mathbb{R}^n)$ of norm bounded by a constant $c(k,n)$ that is inverse to the natural projection π.*

We recover from here the final most spectacular Fefferman theorem by taking for $x \in S$,

$$I(x) := \{p \in \mathcal{P}_{k,n}\,;\, p(x) = 0\}.$$

[5] see the next subsection for a general result of this kind.

The corresponding space $\mathcal{J}(S)$ consists of functions $f \in C_b^k(\mathbb{R}^n)$ such that $(T_x^k f)(x) = 0$ (equivalently, $f(x) = 0$) for every $x \in S$. Hence,

$$\mathcal{J}(S) = \{f \in C_b^k(\mathbb{R}^n)\,;\, f|_S = 0\}$$

and therefore the factor-space $C_b^k(\mathbb{R}^n)/\mathcal{J}(S)$ is, as a linear space, naturally identified with the trace space $C_b^k(\mathbb{R}^n)|_S := \{f|_S\,;\, f \in C_b^k(\mathbb{R}^n)\}$. Moreover, the norm of the factor-space is, by definition, given for $f \in C_b^k(\mathbb{R}^n)$ by

$$\inf\{\|f+g\|_{C_b^k(\mathbb{R}^n)}\,;\, g|_S = 0\} = \inf\{\|h\|_{C_b^k(\mathbb{R}^n)}\,;\, h|_S = f\} =: \|f\|_{C_b^k(\mathbb{R}^n)|_S}.$$

Hence, $C_b^k(\mathbb{R}^n)/\mathcal{J}(S) = C_b^k(\mathbb{R}^n)|_S$ isometrically and Theorem 10.103 immediately implies

Theorem 10.104. *For every subset $S \subset \mathbb{R}^n$ there exists a linear extension operator $T : C_b^k(\mathbb{R}^n)|_S \to C_b^k(\mathbb{R}^n)$ of norm bounded by a constant $c = c(k, n)$.*

Though Fefferman's proofs of all these results are constructive and can be turned into algorithms for finite subsets, see [F-2009a], the constant $c(k, n)$ and the likes in the previous theorems are unspecified. This leads to a question on the (rough) asymptotics of these constants.

In particular, for the linear extension constant defined by

$$\lambda(C_b^k(\mathbb{R}^n)) := \inf\{\|T\|\,;\, T \in \mathrm{Ext}(C_b^k(\mathbb{R}^n); S)\}$$

the following conjecture seems to be plausible.

Problem 10.105. *Prove that*

$$\lambda(C_b^k(\mathbb{R}^n)) \le c^{\dim \mathcal{P}_{k,n}},$$

where c is a numerical constant.

A more far-reaching conjecture, justified only by the case of $k = 1$, states that $\lambda(C_b^k(\mathbb{R}^n))$ is bounded by a constant of polynomial growth in k and n.

10.4.3 Remarks on the proof of Theorem 10.104

Our exposition presents only the basic blocks of the proof for Theorem 10.104. The main and most difficult of them, Assertion I below, is merely formulated and referred to the corresponding pages of the paper [F-2007a]; the remaining parts are explained in a little more detail. To measure the weights of the different parts, one considers the basic, in a sense, case of a finite set S. Now Assertion I is the *only* fact that needs to be proved to establish the corresponding result.

We begin with a reformulation of Theorem 10.104 in the context of bundles. To this end one defines the ideal $I_0(x) := \{p \in \mathcal{P}_{k,n}\,;\, p(x) = 0\}$ of the $\mathbb{R}$-algebra $\mathcal{A}_x^k := (\mathcal{P}_{k,n}, \odot_x)$. Then Theorem 10.104 is equivalent to

Theorem 10.106. *For every $f \in C_b^k(\mathbb{R}^n)|_S$ there exists a C_b^k selection of the k-bundle $\{f(x) + I_0(x)\}_{x \in S}$ of size $c(k,n)\|f\|_{C_b^k(\mathbb{R}^n)|_S}$ that linearly depends on f.*

One first reduces the proof to the case of *Glaeser stable* k-bundles over *compact subsets*. Actually, if Theorem 10.106 were true for compact subsets it would be proved for arbitrary closed $S \subset \mathbb{R}^n$ by using a suitable partition of unity, cf. the proof of Theorem 10.102.

Now let S be compact. Then we apply the Glaeser refinement to the k-bundle $\{I_0(x)\}_{x \in S}$ to obtain a Glaeser stable bundle $\{I(x)\}_{x \in S}$, see assertion (a) of Proposition 10.98. By this assertion and Theorem 10.99 every $I(x)$ is nonempty and is an ideal of $\mathcal{A}_x^k$. Moreover, assertion (c) of the cited proposition implies that Theorem 10.106 holds, if it does for the k-bundle $\{f(x) + I(x)\}_{x \in S}$ in place of $\{f(x) + I_0(x)\}_{x \in S}$. Hence, the following needs to be proved.

Theorem 10.107. *Let $\{I(x)\}_{x \in S}$ be a Glaeser stable k-bundle of ideals in $\mathcal{A}_x^k$ over a compact set $S \subset \mathbb{R}^n$. Then the k-bundle $\{f(x)+I(x)\}_{x \in S}$, where $f \in C_b^k(\mathbb{R}^n)|_S$, has a C_b^k selection of size $c(k,n)\|f\|_{C_b^k(\mathbb{R}^n)|_S}$ linearly depending on f.*

At the next stage one decomposes the set S into subsets called *slices*. To this end one exploits the natural projections π_x^ℓ of $\mathcal{A}_x^k$ onto its subalgebras $\mathcal{A}_x^\ell$, $0 \le \ell \le k$, to associate to every point $x \in S$ a $(k+1)$-tuple of integers, its *type*, by setting

$$\mathit{type}(x) := \left(\dim[\pi_x^0 I(x)], \dots, \dim[\pi_x^k I(x)]\right).$$

Then for every $(k+1)$-tuple $(d_0, \dots, d_k)$ of nonnegative integers we set

$$S(d_0, \dots, d_k) := \{x \in S\,;\, \mathit{type}(x) = (d_0, \dots, d_k)\}.$$

By the definition of this set which is called a *slice*, it is nonempty if $0 \le d_0 \le d_1 \le \dots \le d_k \le \dim \mathcal{P}_{k,n}$. Hence, the *number* of nonempty slices

$$\sigma(S) := \operatorname{card}\{(d_0, \dots, d_k)\,;\, S(d_0, \dots, d_k) \ne \emptyset\}$$

is bounded by a constant depending only on k and n.

The subsequent proof is based on induction on the number of slices. The most important role in this derivation is played by *the first slice*, introduced as follows.

One defines an order on the set of $(k+1)$-tuples by assuming that

$$(d_0, \dots, d_k) \prec (d_0', \dots, d_k') \quad \text{if} \quad d_j < d_j' \text{ for the } \mathit{largest}\ j \text{ for which } d_j \ne d_j'.$$

Since $S \ne \emptyset$, the set of $(n+1)$-tuples $\{\mathit{type}(x)\,;\, x \in S\}$ has a *minimal* element with respect to the so-introduced order. Choosing one of them, denoted by $(d_0^*, \dots, d_k^*)$, one sets

$$S_0 := S(d_0^*, \dots, d_k^*)$$

and calls S_0 the *first slice*.

According to the definition of order on the set of $(k+1)$-tuples, S_0 may be equivalently characterized as follows:

A point $x_0 \in S$ belongs to the first slice S_0 if and only if:

(a_k) $I(x_0)$ is of the smallest dimension among all linear subspaces $I(x)$, $x \in S$; this number is denoted by $d_k(x_0)$.

(a_{k-1}) $\pi_{x_0}^{k-1} I(x_0)$ is of the smallest dimension among all linear subspaces $\pi_x^{k-1} I(x)$ with $x \in S$ satisfying $\dim I(x) = d_k(x_0)$; this number is denoted by $d_{k-1}(x_0)$.

(a_{k-2}) $\pi_{x_0}^{k-2} I(x_0)$ is of the smallest dimension among all linear subspaces $\pi_x^{k-2} I(x)$ with $x \in S$ satisfying $\dim I(x) = d_k(x_0)$, $\dim \pi_x^{k-1} I(x) = d_{k-1}(x_0)$ and so forth.

Further, the first slice S_0 is compact. To prove this one writes $\mathit{type}(x) := (d_0(x), \dots, d_k(x))$, where $d_j(x) := \dim[\pi_x^j I(x)]$. Let $x_0 \in S$ and $\ell \in \{0, 1, \dots, k\}$ be fixed. Since $d_\ell(x_0)$ is the dimension of $\pi_{x_0}^\ell I(x_0)$, one can choose polynomials $p_j \in \mathcal{A}_{x_0}^k$, $1 \le j \le d_\ell(x_0)$, such that the images $\pi_{x_0}^\ell p_j$ are linearly independent. Next, since $\{I(x)\}_{x \in S}$ is Glaeser stable, assertion (b) of Theorem 10.99 implies that there exist functions $g_j \in C_b^k(\mathbb{R}^n)$ such that

$$T_x^k g_j \in I(x) \quad \text{for all} \quad x \in S \quad \text{and} \quad T_{x_0}^k g_j = p_j \quad \text{for} \quad 1 \le j \le d_\ell(x_0).$$

Then the family of polynomials $T_x^\ell(T_{g_j}^k)$, $1 \le j \le d_\ell(x_0)$, is linearly independent at $x = x_0$, hence, it also is at all $x \in S$ close enough to x_0. This clearly implies that the dimension of $\pi_x^\ell I(x)$, i.e., $d_\ell(x)$, for these points x that are close to x_0 is at least $d_\ell(x_0)$, $0 \le \ell \le k$. By the definition of type one concludes from here that there exists a neighborhood of x_0, say U, such that

$$\mathit{type}(x) \succeq \mathit{type}(x_0) \quad \text{for all} \quad x \in S \cap U,$$

where the inequality sign refers to the above introduced order on the set of $(k+1)$-tuples.

Now, let $\{x^i\}_{i \in \mathbb{N}}$ be a sequence of points from the first slice $S_0 := S(d_0^*, \dots, d_k^*)$ converging to a point x. Show that $x \in S_0$, i.e., S_0 is a closed subset of the compact set S, hence, is compact. Actually, $d_\ell(x^i) = d_\ell^*$ for all ℓ and all i; moreover, as already proved, $d_\ell(x^i) \ge d_\ell(x)$ for ℓ and i big enough. Hence, $d_\ell^* \ge d_\ell(x)$ for $0 \le \ell \le k$. But the $(k+1)$-tuple $(d_0^*, \dots, d_k^*)$ is minimal, hence, $d_\ell(x)$ should be equal to d_ℓ^*, $0 \le \ell \le k$, and therefore $x \in S(d_0^*, \dots, d_k^*) =: S_0$, as required.

The subsequent proof is based on induction on the number of slices. The base of induction is the next fact.

Assertion I. Let $S \subset \mathbb{R}^n$ be compact consisting of a unique slice. Then there exists a linear operator $L : C_b^k(\mathbb{R}^n)|_S \to C_b^k(\mathbb{R}^n)$ such that:

(A) $T_x^k(Lg) \in g(x) + I(x)$ provided $x \in S$ and $g \in C_b^k(\mathbb{R}^n)|_S$.

(B) The norm of L is bounded by a constant depending only on k and n.

Here $\{I(x)\}_{x \in S}$ is the Glaeser stable k-bundle of Theorem 10.107.

Let us note that if S is finite, then $\{I(x)\}_{x\in S}$ is Glaeser stable and S consists of a unique slice. Hence, in this case, the above preparatory work is not required and Assertion I coincides with that of Theorem 10.104.

To proceed by induction one should replace condition (A) by a more complicated condition (A′) presented below. There we will briefly explain the reason of such replacement while now the induction assertion is formulated and its proof is briefly discussed.

Assertion II. Assume that Theorem 10.107 holds for compact sets with the number of slices at most $N-1$, where $N \geq 2$. Then the theorem is true for compact sets with the number of slices N.

Let S_0 be the first slice of a compact set S. Applying Assertion I to this one-slice compact set we obtain a linear operator denoted by L_0 with the following properties

(A_0) $T_x^k(L_0 g) \in g(x) + I(x)$ for all $x \in S_0$ and $g \in C_b^k(\mathbb{R}^n)|_{S_0}$.

(B_0) There exists a constant $c = c(k, n)$ so that

$$\|L_0\| \leq c(k, n).$$

At the next stage L_0 is corrected away from S_0. To do this one covers $\mathbb{R}^n \setminus S_0$ by "Whitney cubes" $\{Q_\nu\}$ with the following geometric properties:

(a) $\operatorname{diam} Q_\nu \leq 1$;

(b) $Q_\nu^* := 3Q_\nu \subset \mathbb{R}^n \setminus S_0$;

(c) $\operatorname{dist}(Q_\nu^*, S) \leq c(n) \operatorname{diam} Q_\nu$ for all Q_ν with $\operatorname{diam} Q_\nu < 1$.

According to the properties of the Whitney cover $\{Q_\nu\}$, there is a partition of unity $\{\theta_\nu\}$ satisfying:

(i) $\sum_\nu \theta_\nu = 1$ on $\mathbb{R}^n \setminus S_0$;

(ii) $\operatorname{supp} \theta_\nu = Q_\nu^*$ for all ν;

(iii) $|D^\alpha \theta_\nu| \leq c(k, n)(\operatorname{diam} Q_\nu)^{-|\alpha|}$ on $\mathbb{R}^n$ for $|\alpha| \leq k+1$ and all ν.

The crucial point of induction is the observation that every nonempty compact set $S \cap Q_\nu^*$ has fewer slices than S_0. Therefore, for every such ν one can apply the induction hypothesis (the rescaled version of Theorem 10.107 for fewer than N slices) to the family $\{f(x) - T_x^k(L_0 g) + I(x)\}_{x \in S\cap Q_\nu^*}$ to find a linear operator $L_\nu : C_b^k(\mathbb{R}^n)|_{S\cap Q_\nu^*} \to C_b^k(\mathbb{R}^n)$ with the following properties:

(A_ν) $T_x^k(L_\nu g) \in T_x^k(\theta_\nu) \odot_x [f(x) - T_x^k(L_0 g)] + I(x)$;

(B_ν) $|D^\alpha(L_\nu g)| \leq c(k, n)(\operatorname{diam}\ Q_\nu)^{k-|\alpha|} \|g\|_{C_b^k(\mathbb{R}^n)|_{S\cap Q_\nu^*}}$ on $\mathbb{R}^n$ provided $x \in S \cap Q_\nu^*$, $|\alpha| \leq k$ and $g \in C_b^k(\mathbb{R}^n)|_{S\cap Q_\nu^*}$.

In the remaining case of $S \cap Q_\nu^* = \emptyset$, we simply set $L_\nu = 0$.

Now the operators L_0 and L_ν are combined into

$$L := L_0 + \sum_\nu \theta_\nu L_\nu.$$

Using (A_0), (B_0), (A_ν), (B_ν) and Glaeser stability, one should show that Lf with $f \in C_b^k(\mathbb{R}^n)|_S$ is the required C_b^k selection of the k-bundle $\{f(x)+I(x)\}_{x\in S}$ linearly depending on f.

Unfortunately, the last claim cannot be true unless the linear operator of Assertion I satisfies some stronger (and more difficult to achieve) condition than (A). Specifically, let $N^* \geq \mathcal{F}(C_b^k(\mathbb{R}^n))$ be the integer of Theorem 10.101. As above, $S \subset \mathbb{R}^n$ is compact and $H := \{I(x)\}_{x\in S}$ is a Glaeser stable subbundle of H_0. For $x_0 \in S$, $C > 0$ and $g \in C_b^k(\mathbb{R}^n)|_S$ one defines $\mathcal{K}_H(g; x_0; C)$ as the set of polynomials $p_0 \in g(x_0) + I(x_0)$ with the following property:

Given points $x^j \in S$, $1 \leq j \leq N^*$, there exist polynomials $p_j \in g(x^j) + I(x^j)$ such that

$$\max_{|\alpha|\leq k} \max_{1\leq i\leq N^*} \left\{ |D^\alpha p_j(x^j)| + \frac{|D^\alpha[p_i - p_j](x^j)|}{\|x^i - x^j\|^{k-|\alpha|}} \right\} \leq C.$$

It may be easily seen that $\mathcal{K}_H(g; x_0; C)$ is a compact convex subset of $g(x_0) + I(x_0)$. The point of this definition is that, if F_g is a C_b^k selection of size C of $\{g(x) + I(x)\}_{x\in S}$, then $T_{x_0}^k F_g$ belongs to $\mathcal{K}_H(g; x_0; \gamma(k,n)C)$. (To see this just take $p_j := T_{x^j}^k F_g$ in the definition of $\mathcal{K}_H(g; x_0; \gamma(k,n)C)$. The desired estimates of $p_i - p_j$ follow from the Taylor formula).

The strengthening of condition (A) is given by

(A′) $T_x^k(L_0 g) \in \mathcal{K}_H(g; x_0; C)$ for some $C = C(k,n)$ and for all $x \in S$ and $g \in C_b^k(\mathbb{R}^n)|_S$.

Because of the above discussion condition (A′) is necessary for validity of Theorem 10.107. It was proved in [F-2007a] that this condition is also sufficient, i.e., Assertion I can be improved as follows.

Assertion III. Let S be a one slice compact set. Then there exists a linear operator $L' : C_b^k(\mathbb{R}^n)|_S \to C_b^k(\mathbb{R}^n)$ satisfying conditions (A′) and (B).

Applying this instead of Assertion I, it is relatively easy to prove inequalities (B_ν) and check that $T_x^k(L_0' g) = T_x^k(Lg)$ for $x \in S$. Here L_0' is the operator of Assertion III constructed for the S_0 and L is defined as above but with L_0' in place of L_0.

These facts admit to complete induction and prove the theorem.

10.5 Jet space $J^\ell\Lambda^{2,\omega}(\mathbb{R}^n)$: finiteness and linearity

The next case to study is naturally following Fefferman's solution and relates to the finiteness and linearity problems for the space $C^\ell\Lambda^{2,\omega}(\mathbb{R}^n)$. Unfortunately, the methods of the present chapter allow us to solve the main problems only for $\ell = 0$, see Shvartsman [Shv-1987] (with the sharp finiteness constant) and Yu. Brudnyi and Shvartsman [BSh-1985] and also [BSh-1999] (the linearity problem). Here we present an intermediate result, a solution to the aforementioned problems for the homogeneous jet space $J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)$ (the results for its normed counterpart are easily derived from this case).

Let us recall that $J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)$ consists of ℓ-jets $\vec{f} := (f_\alpha)_{|\alpha|\le\ell}$ on $\mathbb{R}^n$ such that $f_\alpha = D^\alpha f$ for all $|\alpha| \le \ell$ for some (unique) function f from $C^\ell \dot{\Lambda}^\omega(\mathbb{R}^n)$. The seminorm of $\vec{f}$ is given by

$$|\vec{f}|_{J^\ell\Lambda^{2,\omega}(\mathbb{R}^n)} := |f|_{C^\ell\Lambda^{2,\omega}(\mathbb{R}^n)},$$

where f is the function associated to $\vec{f}$ and the right-hand side is recalled to be defined by

$$|f|_{C^\ell\Lambda^{2,\omega}(\mathbb{R}^n)} := \max_{|\alpha|=\ell} \sup_{t>0} \frac{\omega_2(D^\alpha f; t)}{\omega(t)}.$$

Clearly, the 2-majorant ω here is such that $C^\ell\Lambda^{2,\omega}(\mathbb{R}^n)$ is distinct from any space $C^\ell\Lambda^{1,\widetilde{\omega}}(\mathbb{R}^n)\,(:= C^{\ell,\widehat{\omega}}(\mathbb{R}^n))$, since otherwise the problems in question have been solved by the Whitney-Glaeser Theorem 2.19 of Volume I. In particular, if ω is a power function with exponent $\lambda \in (0, 2]$, then only the case $\lambda = 1$ is of interest. However, in what follows we are working with an arbitrary 2-majorant so that this "degenerate" case is covered as well. The results of this section were due to Yu. Brudnyi and Shvartsman [BSh-1998].

10.5.1 Finiteness property

Our main result is

Theorem 10.108. *The space $J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)$ has the Uniform Finiteness Property and its finiteness constant satisfies*

$$\mathcal{F}(J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)) \le 3 \cdot 2^{d(n,\ell)},$$

where $d(n,\ell) := \binom{n+\ell-1}{\ell+1}$.

According to Shvartsman [Shv-1987] this inequality is sharp for $\ell = 0$. Apparently, this is also true for $\ell > 0$.

Proof. We follow the approach used in the proof of Theorem 10.69. So, the space $J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)|_S$ should be first characterized in terms of local polynomial approximation. This has already been done even for the general situation by Theorem 9.8; the required special case is formulated as

Lemma 10.109. *Let $\vec{f} := (f_\alpha)|_{|\alpha|\le\ell}$ be an ℓ-jet on a set $S \subset \mathbb{R}^n$. Then $\vec{f}$ belongs to $J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)|_S$ if and only if there exist a constant $c > 0$ and a family of polynomials $\{p_Q\}_{Q\in\mathcal{K}_S}$ of degree $\ell + 1$ satisfying the conditions:*

(i) *For every* $|\alpha| \le \ell$ *and* $Q \in \mathcal{K}_S$,

$$D^\alpha p_Q(c_Q) = f_\alpha(c_Q).$$

(ii) *For every pair of cubes* $Q' \subset Q$ *from* $\mathcal{K}_S$,

$$\sup_{Q'} |p_Q - p_{Q'}| \le c(\|c_Q - c_{Q'}\| + r_{Q'})^\ell \omega(r_Q).$$

Moreover, the trace norm of $\vec{f}$ *is equivalent to* $\inf c$ *with constants of equivalence depending only on* ℓ *and* n.

Let us recall that $\mathcal{K}_S$ consists of all closed cubes Q whose centers $c_Q \in S$ and radii $r_Q \le 2\,\mathrm{diam}\, S$. All distances hereafter are measured by the ℓ_∞-norm of $\mathbb{R}^n$ denoted by $\|\cdot\|$.

Now we reformulate the above criterion as a selection type result. To this end we reduce every family $\{p_Q\}_{Q\in\mathcal{K}_S}$, retaining only polynomials with a special interpolation property, and prove that the subfamily obtained in this way also characterizes the space $J^\ell\dot{\Lambda}^{2,\omega}(\mathbb{R}^n)$. This result is formulated below while its proof will be presented later.

In what follows we exploit the graph which has already appeared in the proof of Theorem 10.69. Its vertex set $\mathcal{M}_S$ is given by

$$\mathcal{M}_S := \{(x, y) \in S \times S\,;\, x \ne y\} \tag{10.154}$$

and vertices $m := (x, y)$ and $m' := (x', y')$ are joined by a (unique) edge if the sets $\{x, y\}$ and $\{x', y'\}$ have a common point; this edge is denoted by $m \leftrightarrow m'$. We turn this object into a metric graph using a weight ψ_ω defined on the edge set of the graph by

$$\psi_\omega(m \leftrightarrow m') := \int_{min\{|m|,|m'|\}}^{2(|m|+|m'|)} \frac{\omega(t)}{t^2}\, dt; \tag{10.155}$$

here as before we set

$$|m| := \|x - y\| \quad \text{for} \quad m := (x, y). \tag{10.156}$$

Then a (geodesic) metric associated to this weight is defined on the vertex set by

$$d_\omega(m, m') := \inf_{\{m_i\}} \sum_{i=1}^{k} \psi_\omega(m_i \leftrightarrow m_{i+1}), \tag{10.157}$$

where the infimum is taken over all chains $m := m_1 \leftrightarrow m_2 \leftrightarrow \cdots \leftrightarrow m_k =: m'$ and all k.

Now the above mentioned result is as follows:

Proposition 10.110. *An ℓ-jet $\vec{f} := \{f_\alpha\}_{|\alpha|\le\ell}$ on S belongs to the trace space $J^\ell\dot{\Lambda}^{2,\omega}(\mathbb{R}^n)|_S$ if and only if there exist a constant $c > 0$ and a map $V_{\vec{f}} : \mathcal{M}_S \to \mathcal{P}_{\ell+1,n}$ such that for every vertex $m := (x, y)$ the following is true:*

A. The equality

$$(D^\alpha V_{\vec{f}})(z) = f_\alpha(z)$$

holds for all $|\alpha| \le \ell$ if $z = x$ and for all $|\alpha| = \ell$ if $z = y$.

B. The inequality

$$|f_\alpha - (D^\alpha V_{\vec{f}})(m)|(y) \le c\|x - y\|^{\ell-|\alpha|}\omega(\|x - y\|)$$

holds for all $|\alpha| \le \ell$.

C. The function $m \mapsto \big(D^\alpha V_{\vec{f}}(m)\big)_{|\alpha|=\ell+1}$ satisfies for every $m, m' \in \mathcal{M}_S$,

$$|D^\alpha(V_{\vec{f}}(m) - V_{\vec{f}}(m'))| \le c d_\omega(m, m').$$

Moreover, the trace norm of $\vec{f}$ is equivalent to inf c *with constants of equivalence depending only on ℓ and n.*

Now we present this result in a form suitable for the subsequent derivation. To this end we denote by $\mathcal{H}_{\ell,n} \subset \mathcal{P}_{\ell,n}$ the linear space of *homogeneous polynomials* of degree ℓ equipped with the norm

$$\|h\|_{\mathcal{H}_{\ell,n}} := \max_{|\alpha|=\ell} |D^\alpha h|.$$

Further, given an ℓ-jet $\vec{f} := \{f_\alpha\}_{|\alpha|\le\ell}$ on S, a set-valued map $L_{\vec{f}} : (\mathcal{M}_S, d_\omega) \to \mathrm{Aff}(\mathcal{H}_{\ell+1,n})$ is defined at $m := (x, y) \in \mathcal{M}_S$ by

$$L_{\vec{f}}(m) := \{h \in \mathcal{H}_{\ell+1,n}\,;\, (D^\alpha h)(x - y) = f_\alpha(x) - f_\alpha(y) \text{ for all } |\alpha| = \ell\}. \quad (10.158)$$

Proposition 10.111. *An ℓ-jet $\vec{f}$ on S belongs to the space $J^\ell\dot{\Lambda}^{2,\omega}(\mathbb{R}^n)|_S$ if and only if there is a constant $c > 0$ such that the following holds:*

(i) *The restriction of $\vec{f}$ to every two-point subset extends to an ℓ-jet from $J^\ell\dot{\Lambda}^{2,\omega}(\mathbb{R}^n)$ of seminorm bounded by c.*

(ii) *The map $L_{\vec{f}}$ has a Lipschitz selection with Lipschitz constant bounded by c.*

Moreover, the trace seminorm of the $\vec{f}$ is equivalent to $\inf c$ *with constants of equivalence depending only on ℓ and n.*

Proof. (Necessity) Let $\vec{f} \in \dot{X}|_S$, where as above $\dot{X}$ stands for the space $J^\ell\dot{\Lambda}^{2,\omega}(\mathbb{R}^n)$. Then for every two-point subset S' the seminorm of $\vec{f}|_{S'}$ is bounded by that of $\vec{f}$, i.e., (i) holds with $c := |\vec{f}|_{\dot{X}|_S}$.

To prove (ii) we use the family of polynomials $\{V_{\vec f}(m)\}_{m\in\mathcal{M}_S}\subset\mathcal{P}_{\ell+1,n}$ from the necessity part of Proposition 10.110 to define the required selection of the set-valued map $L_{\vec f}$, say s, by setting for $m\in\mathcal{M}_S$,

$$s(m)(z):=\sum_{|\alpha|=\ell+1}\frac{D^\alpha V_{\vec f}(m)}{\alpha!}z^{\ell+1},\quad z\in\mathbb{R}^n.$$

This is clearly a homogeneous polynomial of degree $\ell+1$.

To show that $s(m)\in L_{\vec f}(m)$ we use the identity

$$(D^\alpha V_{\vec f}(m))(z)-(D^\alpha V_{\vec f}(m))(x)=\sum_{i=1}^n D^{\alpha+e_i}V_{\vec f}(m)(z_i-x_i).$$

The right-hand side equals $D^\alpha(s(m))(z-x)$ and setting $z=y$ and using assertion A of Proposition 10.110 we have

$$f_\alpha(y)-f_\alpha(x)=(D^\alpha s(m))(y-x),$$

i.e., s is a selection of $L_{\vec f}$.

To show that s is Lipschitz we use assertion B of Proposition 10.110 to obtain

$$\begin{aligned}\|s(m)-s(m')\|_{\mathcal{H}_{\ell+1,n}}&:=\max_{|\alpha|=\ell+1}|D^\alpha(s(m)-s(m'))|\\&=\max_{|\alpha|=\ell+1}|D^\alpha(V_{\vec f}(m)-V_{\vec f}(m'))|\le c(\ell,n)|\vec f|_{\dot X|_S}d_\omega(m,m').\end{aligned}$$

(Sufficiency) We suppose that assumptions (i) and (ii) hold and define a map $V_{\vec f}:\mathcal{M}_S\to\mathcal{P}_{\ell+1,n}$ by setting for $m:=(x,y)$ and $z\in\mathbb{R}^n$,

$$(V_{\vec f}(m))(z):=s(m)(z-x)+\sum_{|\alpha|\le\ell}\frac{f_\alpha(x)}{\alpha!}(z-x)^\alpha; \tag{10.159}$$

here s is a Lipschitz selection from assumption (ii) with Lipschitz constant $L(s)$.

Let us prove that the family of polynomials $\{V_{\vec f}(m)\}_{m\in\mathcal{M}_S}\subset\mathcal{P}_{\ell+1,n}$ satisfies conditions A–C of Proposition 10.110 with the constant c satisfying

$$c\le O(1)\max\left\{L(s)\,,\ \sup_{\text{card }S'=2}|\vec f|_{X|_{S'}}\right\}.$$

This would imply existence of the required extension of $\vec f$ to an ℓ-jet from $\dot X$ with seminorm bounded by the constant c, and hence, prove this part of the proposition.

First, by definition we have for $|\alpha|\le\ell$ and $m:=(x,y)\in\mathcal{M}_S$,

$$(D^\alpha V_{\vec f})(x)=(D^\alpha s(m))(x-x)+f_\alpha(x)=f_\alpha(x).$$

On the other hand, due to assumption (ii) we have for $|\alpha| = \ell$,

$$(D^\alpha V_{\vec{f}}(m))(y) = (D^\alpha s(m))(y-x) + f_\alpha(x) = f_\alpha(y) - f_\alpha(x) + f_\alpha(x) = f_\alpha(y).$$

Hence, condition A of Proposition 10.110 holds for this family.

To prove the inequality of condition B we apply assertion (i) of Proposition 10.111 to the trace $\vec{f}|_{S'}$, where $S' := \{x, y\} \subset S$. Due to this assertion the trace extends to an ℓ-jet $\vec{f}|_{S'} \in \dot{X}$ with the seminorm bounded by the constant c. Then the necessity part of Proposition 10.110 applied to $\vec{f}|_{S'}$ asserts existence for $m := (x, y) \in \mathcal{M}_S$ a polynomial $p_m \in \mathcal{P}_{\ell+1,n}$ such that

$$(D^\alpha p_m)(x) = f_\alpha(x) \quad \text{and} \quad (D^\alpha p_m)(y) = f_\alpha(y), \tag{10.160}$$

where the former is true for all $|\alpha| \le \ell$ and the latter for all $|\alpha| = \ell$. Moreover, this polynomial also satisfies for all $|\alpha| \le \ell$ the inequality

$$|f_\alpha - D^\alpha p_m|(y) \le c(\ell, n)\|x - y\|^{\ell - |\alpha|}\omega(\|x - y\|).$$

Replacing here the α-derivative of p_m at point y by $(D^\alpha V_{\vec{f}}(m))(y)$ would lead to the required inequality in B. Hence, it remains to prove that for all $|\alpha| \le \ell$,

$$D^\alpha(p_m - V_{\vec{f}}(m))(y) = 0. \tag{10.161}$$

To this end we introduce a polynomial q_m in $z \in \mathbb{R}^n$ given by

$$q_m(z) := V_{\vec{f}}(m)(z + x) - p_m(z + x) := s(m)(z) + \sum_{|\alpha| \le \ell+1} \frac{f_\alpha(x)}{\alpha!} z^\alpha - p_m(z + x).$$

Due to (10.160),

$$p_m(z + x) = \sum_{|\alpha| \le \ell+1} \frac{(D^\alpha p_m)(x)}{\alpha!} z^\alpha = \sum_{|\alpha| \le \ell} \frac{f_\alpha(x)}{\alpha!} z^\alpha + \sum_{|\alpha| = \ell+1} \frac{(D^\alpha p_m)(x)}{\alpha!} z^\alpha.$$

Inserting this into the previous equality and noting that $s(m) \in \mathcal{H}_{\ell+1,n}$ we conclude that q_m is a homogeneous polynomial of degree $\ell + 1$.

Let us show that for $|\alpha| = \ell$,

$$(D^\alpha q_m)(y - x) = 0; \tag{10.162}$$

we then derive from here the same equality also for all $|\alpha| \le \ell$ which is clearly equivalent to the required equality (10.161).

Since $D^\alpha q_m$ with $|\alpha| = \ell$ is a linear function, we have

$$\begin{aligned}&(D^\alpha q_m)(y - x) = (D^\alpha q_m)(y) - (D^\alpha q_m)(x)\\ &= [(D^\alpha s(m))(y) - (D^\alpha s(m))(x)] - [(D^\alpha p_m)(y) - (D^\alpha p_m)(x)].\end{aligned}$$

Due to (10.158) and (10.160) the right-hand side here equals

$$(f_\alpha(y) - f_\alpha(x)) - (f_\alpha(y) - f_\alpha(x)) = 0$$

and (10.162) follows.

We derive from (10.162) that $(D^\alpha q_m)(y - x) = 0$ also for all $|\alpha| < \ell$ using the following fact.

Lemma 10.112. *Let $h(x) := \sum_{|\alpha|=\ell+1} c_\alpha x^\alpha$, $x \in \mathbb{R}^n$, be a homogeneous polynomial such that for some $x_0 \neq 0$ and all $|\alpha| = \ell$,*

$$(D^\alpha h)(x_0) = 0. \tag{10.163}$$

Then this equality holds also for all $|\alpha| < \ell$.

Proof. Since the assertion is invariant under orthogonal transforms of $\mathbb{R}^n$, we may without loss of generality assume that $x_0 = (1, 0, \ldots, 0)$. Then (10.163) implies that all c_α with $\alpha_1 = 0$ are zeros, i.e., h does not depend on x_1. Therefore for $|\alpha| < \ell$,

$$(D^\alpha h)(x_0) = (D^\alpha h)(0, \ldots, 0) = 0,$$

as required. □

The remaining inequality of assumption C of Proposition 10.110 is an immediate consequence of the following from (10.159) identity $D^\alpha V_{\vec{f}} = D^\alpha s$, $|\alpha| = \ell+1$, and the Lipschitz condition for s.

This, finally, proves sufficiency of Proposition 10.111. □

Now we apply the Lipschitz selection Theorem 5.56 of Volume I (more precisely its version given by Proposition 10.74) to the graph $\mathcal{M}_S$ and the set-valued map $L_{\vec{f}} : \mathcal{M}_S \to \mathrm{Aff}(\mathcal{H}_{\ell+1,n})$ given by (10.158). It will be proved, see Corollary 10.115 below, that for all m,

$$\dim L_{\vec{f}}(m) = d(n, \ell) \left(:= \binom{n + \ell - 1}{\ell + 1} \right). \tag{10.164}$$

Therefore Theorem 5.56 of Volume I immediately implies

Proposition 10.113. *If the trace of $L_{\vec{f}}$ to every subset $\mathcal{M}'$ of $\mathcal{M}_S$ with no isolated points[6] and of at most $2^{d(n,\ell)+1}$ points has a selection with Lipschitz constant 1, then $L_{\vec{f}}$ itself has a selection with Lipschitz constant bounded by $c(n, \ell)$.*

Now we are ready to complete the proof of Theorem 10.108. We need to show that

$$\mathcal{F}(J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)) \leq 3 \cdot 2^{d(n,\ell)} \quad \text{and} \quad \gamma(J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)) \leq c(n, \ell).$$

Using scaling we reduce this to the following assertion:

[6] i.e., the subgraph of the graph $\mathcal{M}_S$ with the vertex set $\mathcal{M}'$ is connected.

If the restriction of $\vec{f} \in \dot{X}|_S$ to every subset $S' \subset S$ with card $S' \leq 3 \cdot 2^{d(n,\ell)}$ *admits an extension to an ℓ-jet from $\dot{X}$ of norm at most* 1*, then $\vec{f}$ extends to an ℓ-jet from $\dot{X}$ of norm at most $c(n,\ell)$.*

We prove this by proceeding along the line of the proof of Theorem 10.69. As there we fix a connected subgraph $\mathcal{M}'$ of the graph $\mathcal{M}_S$ consisting of at most $2^{d(n,\ell)+1}$ points and associate to $\mathcal{M}'$ a subset of S given by

$$S_{\mathcal{M}'} := \{z \in S\,;\, z \in \{x, y\} \quad \text{for some vertex} \quad (x, y) \in \mathcal{M}'\}.$$

Then Lemma 10.75 implies that

$$\text{card } S_{\mathcal{M}'} \leq 3 \cdot 2^{d(n,\ell)}$$

and therefore the trace $\vec{f}|_{S_{\mathcal{M}'}}$ (denoted for brevity by $\vec{f'}$) extends to an ℓ-jet from $\dot{X}$ with norm at most 1. Applying the necessity part of Proposition 10.111 to $\vec{f'}$ and the associated set-valued map $L_{\vec{f'}} : \mathcal{M}_{S'} \to \mathrm{Aff}(\mathcal{H}_{\ell+1,n})$ we conclude that $L_{\vec{f'}}$ has a selection with Lipschitz constant bounded by $c(n,\ell)$.

On the other hand, $S' \subset S_{\mathcal{M}'}$; moreover, by the definitions of the latter set and the map $L_{\vec{f}}$, see (10.158), we get $L_{\vec{f'}} = L_{\vec{f}}$ on $\mathcal{M}'$. Hence, $L_{\vec{f}}|_{\mathcal{M}'}$ has a selection with Lipschitz constant bounded by $c(n,\ell)$.

Since $\mathcal{M}'$ is an arbitrary subgraph of $\mathcal{M}_S$ with card $\mathcal{M}' \leq 2^{d(n,\ell)+1}$, Proposition 10.113 implies that $L_{\vec{f}}$ has a selection with Lipschitz constant bounded by $c(n,\ell)$. Then the sufficiency part of Proposition 10.111 implies that $\vec{f}$ extends to an ℓ-jet from $\dot{X}$ with norm at most $c(n,\ell)$.

This proves the above formulated assertion and, hence, Theorem 10.108. □

Proof of Proposition 10.110. (Necessity) Let $\vec{f} \in \dot{X}|_S$ and its trace norm be at most 1 (as above, $\dot{X} = J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)$). We must prove the following statements.

There exist a constant $c = c(n,\ell)$ and a map $V_{\vec{f}} : \mathcal{M}_S \to \mathcal{P}_{\ell+1,n}$ such that for every vertex $m := (x,y)$ of the graph $\mathcal{M}_S$ the following is true:

A. *The equality*

$$(D^\alpha V_{\vec{f}}(m))(z) = f_\alpha(z)$$

holds for all $|\alpha| \leq \ell$ if $z = x$ and only for $|\alpha| = \ell$ if $z = y$.

B. *The inequality*

$$|f_\alpha - D^\alpha V_{\vec{f}}(m)|(y) \leq O(1)\|x-y\|^{\ell-|\alpha|}\omega(\|x-y\|)$$

holds for all $|\alpha| \leq \ell$ (recall that $O(1)$ stands for constants depending only on n and ℓ).

C. *The function $m \mapsto \{D^\alpha V_{\vec{f}}(m)\}_{|\alpha|=\ell+1}$ acting from the metric graph $(\mathcal{M}_S, d_\omega)$ into the space $\mathbb{R}^{\dim \mathcal{P}_{\ell+1,n}}$ is Lipschitz with constant depending only on n and ℓ.*

To construct the desired map $V_{\vec{f}}$ we will use

Lemma 10.114. *For every $z \in \mathbb{R}^n \setminus \{0\}$ there exists a linear operator $T_z : \mathcal{H}_{\ell,n} \to \mathcal{H}_{\ell+1,n}$ such that for $h \in \mathcal{H}_{\ell,n}$ and $|\alpha| = \ell$,*

$$(D^\alpha T_z h)(z) = D^\alpha h \, (\in \mathbb{R}).$$

Moreover, the norm of T_z satisfies

$$\|T_z\| \le O(1)\|z\|^{-1};$$

hereafter $O(1)$ stands for constants depending only on ℓ and n.

Proof. We define the desired operator by setting for $h \in \mathcal{H}_{\ell,n}$ and $x \in \mathbb{R}^n$,

$$(T_z h)(x) := \int_0^{\langle x,z^*\rangle} h(x - tz)\, dt,$$

where $z^* := \frac{z}{\langle z,z\rangle}$ (hence, $\langle x, z^*\rangle = \sum_{i=1}^n x_i z_i / \sum_{i=1}^n z_i^2$).

To simplify subsequent computations we change coordinates in $\mathbb{R}^n$ turning z into a vector $\lambda e_1 = (\lambda, 0, \dots, 0)$. Since an orthogonal transform of $\mathbb{R}^n$ induces an invertible linear operator on $\mathcal{H}_{\ell,n}$, say D, the initial norm of $h(x) = \sum_{|\alpha|=\ell} c_\alpha x^\alpha$, $x \in \mathbb{R}^n$ (equal to $\max_{|\alpha|=\ell} |c_\alpha|$), is equivalent to $\|Dh\|$ with constants of equivalence depending only on $\dim \mathcal{P}_{\ell,n}$. Hence, it suffices to prove the lemma for $z := \lambda e_1$. The straightforward computation then gives

$$(T_z h)(x) = \int_0^{\lambda^{-1}x} h(x - \lambda e_1 t)\, dt = \frac{1}{\lambda} \sum_{|\alpha|=\ell} c_\alpha x_1 x^\alpha \in \mathcal{H}_{\ell+1,n}.$$

Differentiating this identity we then have for $|\alpha| = \ell$,

$$(D^\alpha T_z h)(z) = \frac{\alpha!}{\lambda} c_\alpha x_1|_{x_1=\lambda} = \alpha! c_\alpha = D^\alpha h,$$

as required.

Finally, the same identity yields

$$\|T_z h\|_{\mathcal{H}_{\ell+1,n}} := \frac{1}{|\lambda|} \max_{|\alpha|=\ell} |c_\alpha| := \frac{1}{\|z\|} \|h\|_{\mathcal{H}_{\ell,n}}.$$

The result is proved. □

Corollary 10.115. *For every $m \in \mathcal{M}_S$, the dimension of the affine subspace $L_{\vec{f}}(m) \subset \mathcal{H}_{\ell+1,n}$, see (10.158), equals $\binom{n+\ell-1}{\ell+1}$.*

Proof. By definition

$$L_{\vec{f}}(m) := \{h \in \mathcal{H}_{\ell+1,n}\,;\ (D^\alpha h)(y-x) = f_\alpha(y) - f_\alpha(x) \quad \text{for all} \quad |\alpha| = \ell\},$$

where $m := (x, y)$. Hence, $L_{\vec{f}}(m)$ is isomorphic to the kernel of a linear operator $R : \mathcal{H}_{\ell+1,n} \to \mathcal{H}_{\ell,n}$ defined by the relation

$$D^\alpha(Rh) = (D^\alpha h)(y-x) \quad \text{for all} \quad |\alpha| = \ell.$$

By the previous result R is a surjection, i.e., $\operatorname{Im} R = \mathcal{H}_{\ell,n}$. Hence, $\dim L_{\vec{f}}(m) = \dim \mathcal{H}_{\ell+1,n} - \dim(\operatorname{Im} R) = \binom{n+\ell}{n-1} - \binom{n+\ell-1}{n-1} = \binom{n+\ell-1}{\ell+1}$, as required. □

To construct the required set-valued map $V_{\vec{f}}$ we exploit the family of polynomials $\{p_Q\}_{Q\in\mathcal{K}_S} \subset \mathcal{P}_{\ell+1,n}$ associated to the $\vec{f}$ in accordance with Lemma 10.109. We use only the part of this family indexed by cubes $Q[m]$, where for $m := (x, y) \in \mathcal{M}_S$,

$$Q[m] := Q_{|m|}(x).$$

Since $x, y \in S$ and $|m| := \|x - y\| \le \operatorname{diam} S$, this defines a cube in $\mathcal{K}_S$. Further, we write for brevity $p_m := p_{Q[m]}$ and form the required subfamily $\{p_m\}_{m\in\mathcal{M}_S} \subset \{p_Q\}_{Q\in\mathcal{K}_S}$.

Due to condition (i) of Lemma 10.109 for every $m := (x, y) \in \mathcal{M}_S$ and $|\alpha| \le \ell$,

$$(D^\alpha p_m)(x) = f_\alpha(x). \tag{10.165}$$

The second condition of this lemma yields for every pair $Q[m'] \subset Q[m]$ the inequality

$$\sup_{Q[m']} |p_m - p_{m'}| \le c(n, \ell)(\|x - x'\| + |m'|)^\ell \omega(|m|). \tag{10.166}$$

It would be natural to take p_m as the desired $V_{\vec{f}}(m)$. This choice, in fact, provides fulfillment of conditions B, C but condition A may fail. To recover the situation we add to p_m a shifted homogeneous polynomial $g_m \in \mathcal{H}_{\ell+1,n}$ given for $m := (x, y)$ by

$$\begin{gathered} g_m := T_{y-x} h_m, \text{ where} \\ h_m(z) := \sum_{|\alpha|=\ell} (f_\alpha - D^\alpha p_m)(y) \cdot \frac{z^\alpha}{\alpha!}, \quad z \in \mathbb{R}^n. \end{gathered} \tag{10.167}$$

Now $V_{\vec{f}}$ is introduced for $m := (x, y) \in \mathcal{M}_S$ and $z \in \mathbb{R}^n$ by

$$(V_{\vec{f}}(m))(z) := p_m(z) + g_m(z - y). \tag{10.168}$$

Using (10.168) we get for $|\alpha| \le \ell$,

$$(D^\alpha V_{\vec{f}}(m))(x) = (D^\alpha p_m)(x) + (D^\alpha g_m)(0) = f_\alpha(x).$$

Further, by (10.168), (10.165) and Lemma 10.114 we have for all $|\alpha| = \ell$,

$$\begin{aligned}(D^\alpha p_m)(y) &= (D^\alpha p_m)(y) + (D^\alpha T_{y-x}h_m)(y-x) = (D^\alpha p_m)(y) + D^\alpha h_m \\ &= (D^\alpha p_m)(y) + (f_\alpha - D^\alpha p_m)(y) = f_m(y).\end{aligned}$$

Hence condition A holds for $V_{\vec{f}}$.

To prove B we first estimate the left-hand side of the inequality presented there as follows:

$$|f_\alpha - D^\alpha V_{\vec{f}}(m)|(y) \le |f_\alpha - D^\alpha p_m|(y) + |D^\alpha(p_m - V_{\vec{f}}(m))|(y) =: I_1 + I_2.$$

Due to (10.168) and $(\ell+1)$-homogeneity of $g_m := T_{y-x}h_m$ we then get

$$\begin{aligned}I_2 &:= |D^\alpha g_m|(y-x) \le O(1)\|y-x\|^{\ell+1-|\alpha|} \cdot \max_{|\beta|=\ell+1} |D^\beta g_m| \\ &:= O(1)|m|^{\ell+1-|\alpha|}\|T_{y-x}h_m\|_{\mathcal{H}_{\ell+1,n}}.\end{aligned}$$

By Lemma 10.114 and (10.167) the norm here is bounded by

$$\frac{O(1)}{\|y-x\|}\|h_m\|_{\mathcal{H}_{\ell,n}} := \frac{O(1)}{|m|}\max_{|\alpha|=\ell}|D^\alpha h_m| = \frac{O(1)}{|m|}\max_{|\alpha|=\ell}|f_\alpha - D^\alpha p_m|(y).$$

Combining these inequalities we get

$$I_2 \le O(1)|m|^{\ell-|\alpha|}\max_{|\beta|=\ell}|f_\beta - D^\beta p_m|(y). \tag{10.169}$$

Now we will show that for $|\beta| \le \ell$,

$$|f_\beta - D^\beta p_m|(y) \le O(1)|m|^{\ell-|\beta|}\omega(|m|). \tag{10.170}$$

This and (10.169) therefore would yield

$$I_2 \le O(1)|m|^{\ell-|\alpha|}\omega(|m|). \tag{10.171}$$

Since I_1 is also bounded by the right-hand side of (10.171), we obtain the inequality required by condition B,

$$|f_\alpha - V_{\vec{f}}(m)|(y) \le I_1 + I_2 \le O(1)|m|^{\ell-|\alpha}\omega(|m|).$$

Hence, it remains to prove (10.170). To this end we set $\widetilde{Q}[m] := 2Q_{|m|}(y)$. Then

$$Q[m] := Q_{|m|}(x) \subset \widetilde{Q}[m].$$

Applying subsequently assertions (i), (ii) of Lemma 10.109 with $|\beta| \le \ell$ and $p_m := p_{Q[m]}$, and the Markov inequality we have

$$\begin{aligned}&|f_\beta - D^\beta p_m|(y) = |D^\beta(p_{\widetilde{Q}[m]} - p_{Q[m]})|(y) \le \sup_{Q[m]}|D^\beta(p_{\widetilde{Q}[m]} - p_{Q[m]})| \\ &\le O(1)|m|^{-|\beta|}\sup_{Q[m]}|p_{\widetilde{Q}[m]} - p_{Q[m]}| \le O(1)|m|^{-|\beta|}(\|y-x\| + |m|)^\ell\omega(|m|) \\ &\le O(1)|m|^{\ell-|\beta|}\omega(|m|).\end{aligned}$$

This proves (10.170) and condition B.

To prove C we must establish for every $m, m' \in \mathcal{M}_S$ the inequality

$$\max_{|\alpha|=\ell+1} |D^\alpha(V_{\vec{f}}(m) - V_{\vec{f}}(m'))| \le O(1) d_\omega(m, m').$$

Due to the definitions of $V_{\vec{f}}$ and d_ω, see (10.168) and (10.157), respectively, this inequality is equivalent to

$$\max_{|\alpha|=\ell+1} |D^\alpha(p_m - p_{m'})| \le O(1) \int_{\min(|m|,|m'|)}^{2(|m|+|m'|)} \frac{\omega(t)}{t^2}\, dt \tag{10.172}$$

where the vertices $m := (x, y)$, $m' := (x', y')$ are *joined by an edge* (i.e., $\{x, y\} \cap \{x', y'\} \neq \emptyset$).

The proof of this fact is based on the inequality for the family $\{p_Q\}_{Q\in\mathcal{K}_S}$ of Lemma 10.109 given by

Lemma 10.116. *Let $Q' \subset Q$ be cubes from $\mathcal{K}_S$ with a common center and of radii r' and r. Then for every $|\alpha| = \ell + 1$,*

$$|D^\alpha(p_Q - p_{Q'})| \le O(1) \int_{r'}^{r} \frac{\omega(t)}{t^2}\, dt.$$

Proof. Let $J \ge 1$ be an integer determined by the condition

$$2^{J-1} r' \le r \le 2^J r'.$$

We then define the chain of cubes

$$Q_0 := Q' \subset Q_1 \subset \cdots \subset Q_{J-1} \subset Q_J := Q$$

with a common center where Q_i is of radius $2^i r'$, $1 \le i < J$. By the Markov inequality we have for $|\alpha| = \ell + 1$,

$$|D^\alpha(p_{Q_i} - p_{Q_{i+1}})| \le O(1)(r_{Q_i})^{-(\ell+1)} \max_{Q_i} |p_{Q_i} - p_{Q_{i+1}}|.$$

Estimating the maximum by the inequality in Lemma 10.109 (ii) we bound the left-hand side by

$$c(n,\ell)(r_{Q_i})^\ell \omega(r_{Q_{i+1}})(r_{Q_i})^{-(\ell+1)} \le c(n,\ell)\frac{\omega(2^{i+1}r')}{2^i r'} \le 4c(n,\ell)\frac{\omega(2^i r')}{2^i r'}.$$

Summing the inequalities so obtained and using monotonicity of the 2-majorant ω we, further, have

$$|D^\alpha(p_Q - p_{Q'})| \le \sum_{i=1}^{J-1} |D^\alpha(p_{Q_i} - p_{Q_{i+1}})| \le O(1) \sum_{i=1}^{J-1} \frac{\omega(2^i r')}{2^i r'}$$

$$\le O(1) \sum_{i=1}^{J-1} \int_{2^i r'}^{2^{i+1} r'} \frac{\omega(t)}{t^2}\, dt = O(1) \int_{2r'}^{2r} \frac{\omega(t)}{t^2}\, dt \le O(1) \int_{r'}^{r} \frac{\omega(t)}{t^2}\, dt.$$

This proves the result. □

Returning to the proof of (10.172) we assume for definiteness that

$$|m'| \le |m|$$

and by z denote the common point of the sets $\{x', y'\}$ and $\{x, y\}$. Further, we define cubes $K', K \in \mathcal{K}_S$ by setting

$$K' := 2Q_{|m'|}(z), \quad K := 2Q_{|m|}(z)$$

and add and subtract the polynomials $p_{K'}, p_K$ inside the left-hand side of (10.172). Using then the triangle inequality we estimate the left-hand side of (10.172) by the maximum over $|\alpha| = \ell + 1$ of the sum of the numbers

$$J_\alpha(m') := |D^\alpha(V_{\vec{f}}(m') - p_{K'})|, \quad J_\alpha(m) := |D^\alpha(V_{\vec{f}}(m) - p_K)|,$$
$$J_\alpha(m', m) := |D^\alpha(p_K - p_{K'})|.$$

The last number is estimated by Lemma 10.116 as required in (10.172), i.e.,

$$J_\alpha(m, m') \le O(1) \int_{2|m'|}^{2|m|} \frac{\omega(t)}{t^2}\, dt \le O(1) \int_{|m'|}^{|m|} \frac{\omega(t)}{t^2}\, dt.$$

Now we show that

$$J_\alpha(m') \le O(1) \frac{\omega(|m'|)}{|m'|}; \tag{10.173}$$

the same with m' replaced by m is clearly true for $J_\alpha(m)$. To this end we write

$$J_\alpha(m') \le |D^\alpha(V_{\vec{f}}(m') - p_{m'})| + |D^\alpha(p_{m'} - p_{K'})|.$$

The first term here equals $|D^\alpha g_{m'}|$, see (10.168), and has already been estimated by $O(1)\frac{\omega(|m'|)}{|m'|}$, see (10.171) with $|\alpha| = \ell + 1$.

To estimate the second term we first apply the Markov inequality to bound it by $O(1)\frac{1}{|m'|^{\ell+1}} \sup_{Q[m']} |p_{m'} - p_{K'}|$.
Further, since $z \in \{x', y'\}$, the cube $Q[m'] := Q_{|m'|}(x')$ embeds into $K' := 2Q_{|m'|}(z)$. Therefore Lemma 10.109 (ii) gives

$$\sup_{Q[m']} |p_{Q[m']} - p_{K'}| \le O(1)(\|z - x'\| + |m'|)^\ell \cdot \omega(2|m'|) \le O(1)|m'|^\ell \omega(|m'|).$$

Inserting this in the previous inequality we prove (10.173).

Hence we have proved that

$$\max_{|\alpha|=\ell+1} |D^\alpha(V_{\vec{f}}(m) - V_{\vec{f}}(m'))| \le O(1) \left[\frac{\omega(|m'|)}{|m'|} + \frac{\omega(|m|)}{|m|} + \int_{|m'|}^{2(|m|+|m'|)} \frac{\omega(t)}{t^2}\, dt \right].$$

Since the first two terms are bounded by the third one with factor $O(1)$, this proves (10.172) and the necessity part of Proposition 10.110.

(Suffiency) We are given an ℓ-jet $\vec{f} := \{f_\alpha\}_{|\alpha|\le\ell}$ defined on a set $S \subset \mathbb{R}^n$ and a map $V_{\vec{f}} : \mathcal{M}_S \to \mathcal{P}_{\ell+1,n}$ related to $\vec{f}$ by conditions A–C of the previous part. We should prove that $\vec{f}$ belongs to the trace space $\dot{X}|_S$ $(:= J^\ell\dot{\Lambda}^{2,\omega}(\mathbb{R}^n)|_S)$ and its norm there is bounded by $c(n,\ell)$. This will be achieved by exploiting Lemma 10.109. In accordance with this result we should find a family of polynomials $\{p_Q\}_{Q\in\mathcal{K}_S}$ related to $\vec{f}$ by the next conditions:

(i) For every $Q \in \mathcal{K}_S$ and $|\alpha| \le \ell$,

$$(D^\alpha p_Q)(c_Q) = f_\alpha(c_Q). \tag{10.174}$$

(ii) For every pair $Q' \subset Q$ of cubes from $\mathcal{K}_S$,

$$\max_{Q'} |p_Q - p_{Q'}| \le O(1)(\|c_Q - c_{Q'}\| + r_{Q'})^\ell \cdot \omega(r_Q). \tag{10.175}$$

Using the argument from the just proved necessity part in the opposite direction we associate to some cubes $Q \in \mathcal{K}_S$ vertices $m[Q] := (x,y)$, $x,y \in \mathcal{M}_S$, by setting

$$x := c_Q, \quad y \in S \cap Q \quad \text{and} \quad \frac{1}{4} r_Q < \|x-y\| \le 4r_Q. \tag{10.176}$$

This may be not unique and we fix one of them.

Then we define the required polynomial p_Q for such cubes Q by setting

$$p_Q := V_{\vec{f}}(m[Q]). \tag{10.177}$$

Since $V_{\vec{f}}$ satisfies assumption A, we get for all $|\alpha| \le \ell$,

$$(D^\alpha p_Q)(c_Q) = f_\alpha(c_Q),$$

i.e., condition (i) of Lemma 10.109 holds.

Unfortunately, the set of cubes $Q \in \mathcal{K}_S$ satisfying (10.176) is only a proper part of $\mathcal{K}_S$; e.g., if $\operatorname{diam}(Q \cap S)$ is small enough while the distance $d(Q\cap S, S\setminus Q)$ is large, the inequality in (10.176) does not hold.

To define the polynomials p_Q for the set of the remaining cubes we need some geometric facts. To formulate the corresponding result we decompose $\mathcal{K}_S$ into sets $\mathcal{K}^I, \mathcal{K}^{II}$ by setting

$$\mathcal{K}^I := \{Q \in \mathcal{K}_S\,;\,(10.176)\text{ holds for } Q\}, \quad \mathcal{K}^{II} := \mathcal{K}_S \setminus \mathcal{K}^I. \tag{10.178}$$

As a formality, we add to $\mathcal{K}^I$ the "infinite cube" $\mathbb{R}^n$.

Further, we define an equivalence relation on $\mathcal{K}_S$ by

$$Q' \sim Q \iff Q' \cap S = Q \cap S, \tag{10.179}$$

and by $\widehat{Q}$ denote the class of equivalence for $Q \in \mathcal{K}_S$.

Due to this definition $\widehat{Q}$ is infinite if $Q \in \mathcal{K}^{II}$. We "border" $\widehat{Q}$ for this case by two cubes denoted by $K[\widehat{Q}]$ and $\widetilde{K}[\widehat{Q}]$ that are defined in the following way:

Let $x_{\widehat{Q}}, y_{\widehat{Q}}$ belong to the closed set $Q \cap S$ and satisfy

$$\|x_{\widehat{Q}} - y_{\widehat{Q}}\| = \operatorname{diam}(Q \cap S).$$

Then we set

$$K[\widehat{Q}] := Q_r(x_{\widehat{Q}}), \quad \text{where} \quad r := \operatorname{diam}(Q \cap S). \tag{10.180}$$

If $r = 0$, i.e., $Q \cap S = \{x_{\widehat{Q}}\}$, $K[\widehat{Q}]$ is a single point set (*the first degenerate case*).

The second cube, $\widetilde{K}[\widehat{Q}]$, is the smallest one of center $x_{\widehat{Q}}$ which contains $K[\widehat{Q}]$ and does not belong to the class of equivalence of $K[Q]$. In other words,

$$\begin{aligned} \widetilde{K}[\widehat{Q}] := \min\{&Q_r(x_{\widehat{Q}}) \in \mathcal{K}_S\,;\, K[\widehat{Q}] \subset Q_r(x_{\widehat{Q}}) \\ &\text{and } K[\widehat{Q}] \cap S \ne Q_r(x_{\widehat{Q}}) \cap S\}. \end{aligned} \tag{10.181}$$

For a bounded set S the cube $\widetilde{K}[\widehat{Q}]$ may be undefined, e.g., if $\frac{1}{4}Q \supset S$. In all these cases, we set $\widetilde{K}[\widehat{Q}] := \mathbb{R}^n$ (*the second degenerate case*). If S contains more than one point,[7] then $\widetilde{K}[\widehat{Q}] = \mathbb{R}^n$ implies that $K[\widehat{Q}]$ is nontrivial. Conversely, if $K[\widehat{Q}] = \{x_{\widehat{Q}}\}$, then $\widetilde{K}[\widehat{Q}]$ is bounded and nontrivial.

The basic properties of the geometric objects so introduced are presented in the next result whose proof will be postponed until the final part. In its formulation, we conventionally write for cubes Q from $\mathcal{K}^I$:

$$\widehat{Q} := \{Q\}, \quad K[\widehat{Q}] = \widetilde{K}[\widehat{Q}] = Q. \tag{10.182}$$

Geometric Lemma. *Let Q', Q be cubes from $\mathcal{K}_S$. The following is true.*

(a) *The associated cubes $K[\widehat{Q}], \widetilde{K}[\widehat{Q}]$ belong to the class $\mathcal{K}^I$.*

(b) *If $Q \in \mathcal{K}^{II}$ and $Q' \in \widehat{Q}$, then*

$$K[\widehat{Q}] \subset Q' \subset \frac{1}{2}\widetilde{K}[\widehat{Q}].$$

(c) *If $Q' \subset Q$ and $Q' \notin \widehat{Q}$, then either $\widetilde{K}[\widehat{Q}'] \subset 6K[Q]$ or $K[\widehat{Q}] \subset 5K[\widehat{Q}']$. In the latter case, $Q' \in \mathcal{K}^{II}$ and $Q \in \mathcal{K}^I$ (hence, $K[\widehat{Q}] = Q$).*

[7] we exclude the trivial case of card $S = 1$ from the further consideration

Now we define the required polynomials p_Q also for $Q \in \mathcal{K}^{II}$. Since p_Q should satisfy condition (i), i.e., $(D^\alpha p_Q)(c_Q) = f_\alpha(c_Q)$ for all $|\alpha| \le \ell$, it remains to find its derivatives of order $\ell+1$, i.e., the numbers $D^\alpha p_Q$, $|\alpha| = \ell+1$. To this end we use the already defined polynomials $p_{K[\widehat{Q}]}, p_{\widetilde{K}[\widehat{Q}]}$, since $K[\widehat{Q}], \widetilde{K}[\widehat{Q}]$ belong to $\mathcal{K}^I$. Specifically, we first define an auxiliary polynomial $h_Q \in \mathcal{P}_{\ell+1,n}$ by setting

$$h_Q := (1-\tau_Q)p_{\widetilde{K}[\widehat{Q}]} + \tau_Q p_{K[\widehat{Q}]}, \tag{10.183}$$

where

$$\tau_Q := \int_r^{\widetilde{\rho}} \frac{\omega(t)}{t^2}\,dt \Bigg/ \int_\rho^{\widetilde{\rho}} \frac{\omega(t)}{t^2}\,dt \tag{10.184}$$

and $r := r_Q$, $\rho := r_{K[\widehat{Q}]}$, $\widetilde{\rho} := r_{\widetilde{K}[\widehat{Q}]}$.

In the degenerate cases, i.e., when $K[\widehat{Q}] = \{x_{\widehat{Q}}\}$ or $\widetilde{K}[\widehat{Q}] = \mathbb{R}^n$, we set $\tau_Q := 0$ and $\tau_Q := 1$, respectively.

Finally, we uniquely define the remaining polynomials $p_Q \in \mathcal{P}_{\ell+1,n}$ for $Q \in \mathcal{K}^{II}$ by the conditions

$$\begin{aligned} &(D^\alpha p_Q)(c_Q) = f_\alpha(c_Q) \quad \text{for all} \quad |\alpha| \le \ell, \\ &D^\alpha p_Q = D^\alpha h_Q \quad \text{for all} \quad |\alpha| = \ell+1. \end{aligned} \tag{10.185}$$

By (10.177) and this definition the so-introduced family $\{p_Q\}_{Q\in\mathcal{K}_S}$ satisfies condition (i) of Lemma 10.109. To prove for it the second assertion of this lemma we need the following.

Claim. For every pair $Q' \subset Q$ of cubes in $\mathcal{K}_S$ and every $|\alpha| = \ell+1$,

$$|D^\alpha(p_Q - p_{Q'})| \le O(1)\frac{\omega(r_Q)}{r_{Q'}}. \tag{10.186}$$

Proof is divided into several parts the first of which is

Lemma 10.117. *Let $Q' \subset Q$ be cubes from the class $\mathcal{K}^I$ whose radii r', r satisfy $r' \le r/2$. Then for every $|\alpha| = \ell+1$,*

$$|D^\alpha(p_Q - p_{Q'})| \le O(1)\int_{r'}^{r} \frac{\omega(t)}{t^2}\,dt \left(\le O(1)\frac{\omega(r)}{r'}\right).$$

Proof. We prove under the weaker assumption $Q' \subset 6Q$ on these cubes that

$$|D^\alpha(p_Q - p_{Q'})| \le O(1)\int_{r'}^{16r} \frac{\omega(t)}{t^2}\,dt \tag{10.187}$$

and then derive from here the desired result.

Since $Q', Q \in \mathcal{K}^I$, the polynomials involved are given by $p_{Q'} := V_{\vec{f}}(m[Q'])$, $p_Q := V_{\vec{f}}(m[Q])$. Therefore we need to prove some inequalities related to the centers and radii of $m[Q], m[Q']$. Let us recall, see (10.176), that $m[Q'] := (c_{Q'}, y_{Q'})$ where the point $y_{Q'} \in S$ satisfies

$$\frac{1}{4} r_{Q'} \le \|c_{Q'} - y_{Q'}\| \le 4 r_{Q'} \tag{10.188}$$

and $m[Q]$ is defined similarly.

Without loss of generality we may assume that

$$\|c_Q - c_{Q'}\| \le \|c_Q - y_{Q'}\| \tag{10.189}$$

and set

$$\widetilde{r} := r' + \|c_Q - c_{Q'}\|, \quad \widetilde{m} := (c_Q, y_{Q'})\,(\in \mathcal{M}_S)$$

(if the converse to (10.189) were true, we would set $\widetilde{m} := (c_Q, c_{Q'})$).

By (10.188) and (10.189) we get

$$\widetilde{r} \le 4\|c_{Q'} - y_{Q'}\| + \|c_Q - c_{Q'}\| \le 6\|c_Q - y_{Q'}\|$$

and, moreover,

$$\|c_Q - y_{Q'}\| \le \|c_Q - c_{Q'}\| + \|c_{Q'} - y_{Q'}\| \le 4\widetilde{r}.$$

This implies that

$$\frac{\widetilde{r}}{6} \le |\widetilde{m}| := \|c_Q - y_{Q'}\| \le 4\widetilde{r}. \tag{10.190}$$

Further, the vertex $\widetilde{m}$ is joined with vertices $m[Q']$ and $m[Q]$ by edges, since the set $\{c_Q, y_{Q'}\}$ has common points with $\{c_{Q'}, y_{Q'}\}$ and $\{c_Q, y_Q\}$. Therefore, by assumption C of Proposition 10.110 and the definition of the metric d_ω, see (10.157), we get for $|\alpha| = \ell + 1$,

$$\begin{aligned} |D^\alpha(p_Q - p_{Q'})| &\le |D^\alpha(V_{\vec{f}}(m[Q']) - V_{\vec{f}}(\widetilde{m}))| + |D^\alpha(V_{\vec{f}}(\widetilde{m}) - V_{\vec{f}}(m[Q])| \\ &\le O(1)\left[\psi_\omega(m[Q'] \leftrightarrow \widetilde{m}) + \psi_\omega(\widetilde{m} \leftrightarrow m[Q])\right], \end{aligned}$$

where $\psi_\omega(m \leftrightarrow m')$ is recalled to be defined at an edge $m \leftrightarrow m'$ by

$$\psi_\omega(m \leftrightarrow m') := \int_{\min(|m|,|m'|)}^{2(|m|+|m'|)} \frac{\omega(t)}{t^2}\,dt.$$

To estimate the right-hand side here it suffices to evaluate the limits of the integrals involved. To do this we use (10.188) and (10.190) to get

$$\min\{|m[Q']|, |\widetilde{m}|\} := \min\{\|c_{Q'} - y_{Q'}\|, \|c_Q - y_Q\|\} \ge \min\left\{\frac{1}{4}r', \frac{1}{6}\widetilde{r}\right\} \ge \frac{1}{6}r'.$$

Moreover, since $Q' \subset 6Q$, we also get

$$\widetilde{r} := r' + \|c_Q - c_{Q'}\| \le 6r.$$

Further, using again (10.188) and (10.190) we have

$$|m[Q']| + |\widetilde{m}| \le 4r' + 4\widetilde{r} := 8r' + 4\|c_Q - c_{Q'}\| \le 48r.$$

Combining these estimates we then obtain

$$\psi_\omega(m[Q'] \leftrightarrow \widetilde{m}) \le \int_{\frac{1}{6}r'}^{96r} \frac{\omega(t)}{t^2}\,dt \le 6\int_{r'}^{16r} \frac{\omega(t)}{t^2}\,dt.$$

The same inequality (10.188) for Q in place of Q' similarly gives

$$\min\{|m[Q]|, |\widetilde{m}|\} \ge \min\left\{\frac{1}{6}\widetilde{r}, \frac{1}{4}r\right\} \ge \frac{1}{24}r',$$

and, moreover,

$$|\widetilde{m}| + |m[Q]| \le 4\widetilde{r} + 4r \le 28r.$$

This, in turn, implies that

$$\psi_\omega(\widetilde{m} \leftrightarrow m[Q]) \le \int_{\frac{1}{24}r'}^{96r} \frac{\omega(t)}{t^2}\,dt \le 24\int_{r'}^{4r} \frac{\omega(t)}{t^2}\,dt.$$

Summing these two estimates for the values of ψ_ω we finally get the desired inequality (10.187) with constant 30.

Now it remains to show that if $r' \le r/2$, then

$$\int_{r'}^{16r} \frac{\omega(t)}{t^2}\,dt \le O(1)\int_{r'}^{r} \frac{\omega(t)}{t^2}\,dt. \tag{10.191}$$

It suffices to prove that

$$\int_{r}^{16r} \frac{\omega(t)}{t^2}\,dt = \frac{1}{16}\int_{r/16}^{r} \frac{\omega(16s)}{s^2}\,ds \le O(1)\int_{r'}^{r} \frac{\omega(s)}{s^2}\,ds.$$

Since ω is a 2-majorant, i.e., it and the function $t \mapsto \frac{t^2}{\omega(t)}$ are nondecreasing, the intermediate term here is at most

$$\frac{1}{16}\cdot 16^2 \int_{r/16}^{r} \frac{\omega(s)}{s^2}\,ds \le 16\int_{r'}^{r} \frac{\omega(s)}{s^2}\,ds,$$

provided that $r' \le r/16$.

Otherwise, $r/16 \le r' \le r/2$ and the intermediate term is at most

$$\frac{1}{16}\frac{\omega(r)}{(r/16)^2}\int_{r/16}^{r} ds \le O(1)\frac{\omega(r/16)}{r} \le O(1)\frac{\omega(r')}{r}$$
$$\le O(1)\left(\int_{r'}^{r}\frac{\omega(s)}{s^2}\,ds\right)\left(\frac{1}{r'}-\frac{1}{r}\right)^{-1} \le O(1)\int_{r'}^{r}\frac{\omega(s)}{s^2}\,ds.$$

This proves (10.191) and the lemma. □

Lemma 10.118. *Let $Q' \subset Q$ be cubes in $\mathcal{K}_S$ of radii $r' \le r$. Assume that the cube $Q \in \mathcal{K}^{II}$ and Q' belongs to its class of equivalence $\widehat{Q}$. Then for every $|\alpha| = \ell + 1$,*

$$|D^\alpha(p_Q - p_{Q'})| \le O(1)\int_{r'}^{r}\frac{\omega(t)}{t^2}\,dt. \tag{10.192}$$

Proof. Let first $Q' \in \mathcal{K}^{II}$. Due to the definition of polynomials $p_{Q'}, p_Q$ for this case, see (10.183)–(10.185), we have for $|\alpha| = \ell + 1$,

$$|D^\alpha(p_Q - p_{Q'})| = |D^\alpha(h_Q - h_{Q'})| = (\tau_Q - \tau_{Q'})|D^\alpha(p_{\widetilde{K}[\widehat{Q}]} - p_{K[\widehat{Q}]})|.$$

Here we take into account that the associated K-cubes for $\widehat{Q}'$ and $\widehat{Q}$ coincide, since Q' and Q belong to the same equivalence class.

Further, the first factor in the right-hand side equals by definition

$$\left(\int_r^{\widetilde{\rho}}\frac{\omega(t)}{t^2}\,dt - \int_{r'}^{\widetilde{\rho}}\frac{\omega(t)}{t^2}\,dt\right)\Big/\int_\rho^{\widetilde{\rho}}\frac{\omega(t)}{t^2}\,dt = \int_{r'}^{r}\frac{\omega(t)}{t^2}\,dt\Big/\int_\rho^{\widetilde{\rho}}\frac{\omega(t)}{t^2}\,dt\,,$$

where $\rho := r_{K[\widehat{Q}]}$ and $\widetilde{\rho} := r_{\widetilde{K}[\widehat{Q}]}$.

Next, by assertions (a), (b) of the Geometric Lemma, $K[\widehat{Q}]$ and $\widetilde{K}[\widehat{Q}]$ belong to $\mathcal{K}^I$ and $K[\widehat{Q}] \subset \frac{1}{2}\widetilde{K}[\widehat{Q}]$, so $\rho \le \frac{1}{2}\widetilde{\rho}$. Hence, these cubes satisfy the condition of Lemma 10.117 which yields for all $|\alpha| = \ell + 1$,

$$|D^\alpha(p_{\widetilde{K}[\widehat{Q}]} - p_{K[\widehat{Q}]})| \le O(1)\int_\rho^{\widetilde{\rho}}\frac{\omega(t)}{t^2}\,dt.$$

Combining this with the previous two equalities we obtain (10.192) for this case.

Let now $Q' \in \mathcal{K}^I$. Then $K[\widehat{Q}']$ equals (conventionally) Q' and, moreover, $K[\widehat{Q}] = K[\widehat{Q}']$; therefore for $|\alpha| = \ell + 1$,

$$|D^\alpha(p_Q - p_{Q'})| = |D^\alpha(h_Q - p_{Q'})| = |D^\alpha[(1-\tau_Q)p_{\widetilde{K}[\widehat{Q}]} + \tau_Q p_{K[\widehat{Q}]} - p_{Q'}]|$$
$$= (1-\tau_Q)|D^\alpha(p_{\widetilde{K}[\widehat{Q}]} - p_{Q'})|.$$

The first factor here equals

$$1 - \left(\int_r^{\widetilde{\rho}}\frac{\omega(t)}{t^2}\,dt\Big/\int_\rho^{\widetilde{\rho}}\frac{\omega(t)}{t^2}\,dt\right) = \left(\int_{r'}^{r}\frac{\omega(t)}{t^2}\,dt\Big/\int_{r'}^{\widetilde{\rho}}\frac{\omega(t)}{t^2}\,dt\right),$$

since $\rho := r_{K[\widehat{Q}]} = r_{Q'} := r'$.

As in the previous case, using the Geometric Lemma, we estimate the second factor by $O(1)\int_{r'}^{\rho} \frac{\omega(t)}{t^2}\,dt$ and in this way complete the proof. $\square$

Remark 10.119. In the same fashion, we can derive (10.192) for $Q' = K[\widehat{Q}]$ or $Q = \widetilde{K}[\widehat{Q}]$ (with ρ or $\widetilde{\rho}$ in place of r' or r, respectively). E.g., in the former case,

$$|D^\alpha(p_Q - p_{K[\widehat{Q}]})| = \left(\int_\rho^r \frac{\omega(t)}{t^2}\,dt \Big/ \int_\rho^{\widetilde{\rho}} \frac{\omega(t)}{t^2}\,dt\right) \cdot |D^\alpha(p_{\widetilde{K}[\widehat{Q}]} - p_{K[\widehat{Q}]})|$$
$$\leq O(1)\int_\rho^r \frac{\omega(t)}{t^2}\,dt.$$

Now we prove inequality (10.186) of Claim. So, let $Q' \subset Q$ be cubes of $\mathcal{K}_S$ with radii $r' \leq r$. We begin with the case $Q' \in \widehat{Q}$ and $Q \in \mathcal{K}^{II}$ (the case of $Q \in \mathcal{K}^I$ is trivial since $Q' \in \widehat{Q}$ means that $Q' = Q$, see (10.182)). Lemma 10.118 is valid under these assumptions and we get for $|\alpha| = \ell + 1$,

$$|D^\alpha(p_Q - p_{Q'})| \leq O(1)\int_{r'}^r \frac{\omega(t)}{t^2}\,dt \leq O(1)\frac{\omega(r)}{r'}.$$

If now $Q' \notin \widehat{Q}$ and $Q \in \mathcal{K}^{II}$, then assertion (c) of the Geometric Lemma gives

$$\widetilde{K}[\widehat{Q}'] \subset 6K[\widehat{Q}]. \tag{10.193}$$

Further, for $|\alpha| = \ell + 1$,

$$|D^\alpha(p_Q - p_{Q'})| \leq |D^\alpha(p_{Q'} - p_{\widetilde{K}[\widehat{Q}']})| + |D^\alpha(p_{\widetilde{K}[\widehat{Q}']} - p_{K[\widehat{Q}]})| + |D^\alpha(p_{K[\widehat{Q}]} - p_Q)|,$$

where the extreme terms of the sum have been already estimated in Remark 10.119.

Moreover, by the Geometric Lemma the associated K-cubes of the intermediate term belong to $\mathcal{K}^I$ and in addition satisfy (10.193). Therefore inequality (10.187) holds for this term.

Hence, we get for $|\alpha| = \ell + 1$,

$$|D^\alpha(p_Q - p_{Q'})| \leq O(1)\left\{\int_{r'}^{\widetilde{\rho}} + \int_{\widetilde{\rho}}^{16\rho} + \int_\rho^r\right\}\frac{\omega(t)}{t^2}\,dt;$$

recall that $\rho := r_{K[\widehat{Q}]}$ and $\widetilde{\rho} := r_{\widetilde{K}[\widehat{Q}]}$.

The right-hand side here is bounded by $O(1)\left\{\frac{\omega(\widetilde{\rho})}{r'} + \frac{\omega(\rho)}{\widetilde{\rho}} + \frac{\omega(r)}{\rho}\right\}$ which, in turn, is at most $O(1)\frac{\omega(r)}{r'}$. Actually, due to (10.193) and assertion (b) of the Geometric Lemma,

$$r' \leq 6\min\{\rho, \widetilde{\rho}\} \quad \text{and} \quad r \geq \frac{1}{6}\max\{\rho, \widetilde{\rho}\}$$

and we conclude that

$$\frac{\omega(\widetilde{\rho})}{r'}+\frac{\omega(\rho)}{\widetilde{\rho}}+\frac{\omega(r)}{\rho}\le\frac{7\omega(6r)}{r'}+\frac{6\omega(r)}{r'}\le O(1)\frac{\omega(r)}{r'}.$$

Now we consider cubes $Q'\subset Q$ such that $Q'\notin\widehat{Q}$, $Q'\in\mathcal{K}^{II}$ and $Q\in\mathcal{K}^{I}$. Due to assertions (c) and (b) of Geometric Lemma $Q\subset 5K[Q']$ and $K[\widehat{Q}']\subset K[\widehat{Q}]=:Q\subset Q'$. This leads to the inequality

$$r_{K[\widehat{Q}']}\le r'\le r\le 5r_{K[Q']}. \tag{10.194}$$

Then as before for $|\alpha|=\ell+1$ we write

$$\begin{aligned}&|D^\alpha(p_Q-p_{Q'})|\le|D^\alpha(p_{Q'}-p_{K[\widehat{Q}']})|+|D^\alpha(p_{K[\widehat{Q}']}-p_Q)|\\ &\le O(1)\left(\int_{r_{K[Q']}}^{r'}+\int_{r}^{r_{K[\widehat{Q}']}}\right)\frac{\omega(t)}{t^2}\,dt\le O(1)\left(\frac{\omega(r')}{r_{K[Q']}}+\frac{\omega(r_{K[Q']})}{r}\right).\end{aligned}$$

Applying (10.194) we obtain from here the desired inequality

$$|D^\alpha(p_Q-p_{Q'})|\le O(1)\left(\frac{\omega(r)}{\frac{1}{5}r'}+\frac{\omega(r)}{r'}\right)\le O(1)\frac{\omega(r)}{r'}.$$

Finally, let these cubes belong to $\mathcal{K}^I$. Due to (10.177) we have for $Q\in\mathcal{K}^I$,

$$p_Q:=V_{\vec{f}}(m[Q]),$$

where the coordinates x,y of the vertex $m[Q]$ by (10.176) satisfy

$$x:=c_Q\quad\text{and}\quad\frac{1}{4}r_Q\le\|x-y\|\le 4r_Q.$$

Moreover, by assumption C of Proposition 10.110 the map $m\mapsto(D^\alpha V_{\vec{f}}(m))_{|\alpha|=\ell+1}$ is Lipschitz with the constant bounded by $c(n,\ell)$. Hence for the cubes $Q'\subset Q$ we obtain from here

$$\begin{aligned}\max_{|\alpha|=\ell+1}|D^\alpha(p_Q-p_{Q'})|&\le c(n,\ell)d_\omega(m[Q],m[Q'])\\ &\le c(n,\ell)\int_{\frac{1}{4}r'}^{8(r+r')}\frac{\omega(t)}{t^2}\,dt\le O(1)\frac{\omega(r)}{r'}.\end{aligned}$$

This completes the proof of our Claim.

Now we are ready to prove the sufficiency part of Proposition 10.110. To this end it remains to establish assertion (ii) of Lemma 10.109, i.e., to prove that for a pair $Q'\subset Q$ of cubes in $\mathcal{K}_S$ of radii $r'\le r$,

$$\sup_{Q'}|p_Q-p_{Q'}|\le O(1)(\|c_Q-c_{Q'}\|+r')^\ell\omega(r). \tag{10.195}$$

By the Taylor formula

$$\sup_{Q'} |p_Q - p_{Q'}| \le \sum_{|\alpha|\le \ell} (r')^{|\alpha|} |D^\alpha (p_Q - p_{Q'})(c_{Q'})| + (r')^{\ell+1} \sum_{|\alpha|=\ell+1} |D^\alpha (p_Q - p_{Q'})|. \tag{10.196}$$

Using inequality (10.186) of our CLAIM we then estimate the second sum by $O(1)(r')^{\ell+1}\frac{\omega(r)}{r'} = O(1)(r')^{\ell}\omega(r)$ which, in turn, is bounded by the right-hand side of (10.195).

To obtain the required estimate of the first sum we begin with the case $c_{Q'} \ne c_Q$. Then $m := (c_Q, c_{Q'})$ is a vertex of the graph $\mathcal{M}_S$ and therefore the polynomial $V_{\vec{f}}(m)$ (of degree $\ell + 1$) is defined. Moreover, since $Q' \subset Q$,

$$0 < |m| := \|c_Q - c_{Q'}\| \le r. \tag{10.197}$$

Hence, we can write for this m,

$$|D^\alpha(p_Q - p_{Q'})(c_{Q'})| \le |D^\alpha(p_{Q'} - V_{\vec{f}}(m))(c_{Q'})| + |D^\alpha(V_{\vec{f}}(m) - p_Q)(c_{Q'})| =: I_1(\alpha) + I_2(\alpha). \tag{10.198}$$

Further, by the definition of $p_{Q'}$ we have for all $|\alpha| \le \ell$,

$$(D^\alpha p_{Q'})(c_{Q'}) = f_\alpha(c_{Q'}).$$

Using this, assumption B of Proposition 10.110 and (10.197) we get for $|\alpha| \le \ell$,

$$I_1(\alpha) = |(f_\alpha - V_{\vec{f}}(m))(c_{Q'})| \le O(1)|m|^{\ell-|\alpha|}\omega(r). \tag{10.199}$$

To obtain a similar bound for $I_2(\alpha)$ we first use assumption A to have for $|\beta| \le \ell$,

$$(D^\beta V_{\vec{f}}(m))(c_Q) = (D^\beta p_Q)(c_Q) = f_\beta(c_Q).$$

Therefore the Taylor formula yields

$$(V_{\vec{f}}(m) - p_Q)(c_{Q'}) = \sum_{|\beta|=\ell+1} \frac{D^\beta(V_{\vec{f}}(m) - p_Q)}{\beta!}(c_{Q'} - c_Q)^\beta$$

and this leads to the inequality

$$I_2(\alpha) \le |m|^{\ell+1-|\alpha|} \sum_{|\beta|=\ell+1} |D^\alpha(V_{\vec{f}}(m) - p_Q)|.$$

Now we show that for $|\beta| = \ell + 1$,

$$|D^\beta(V_{\vec{f}}(m) - p_Q)| \le O(1)|m|^{-1}\omega(r). \tag{10.200}$$

If this were true we would obtain for $|\alpha| \le \ell$ the desired inequality

$$I_2(\alpha) \le O(1)|m|^{\ell-|\alpha|}\omega(r).$$

Combining this with (10.199) and (10.198) we then get for $|\alpha| \le \ell$,

$$|D^\alpha(p_Q - p_{Q'})(c_{Q'})| \le O(1)|m|^{\ell-|\alpha|}\omega(r).$$

This, in turn, leads to an upper bound for the first sum in (10.196) given by

$$O(1)\sum_{|\alpha|\le\ell}(r')^{|\alpha|}|m|^{\ell-|\alpha|}\omega(r) \le O(1)(|m|+r')^\ell\omega(r) := O(1)(\|c_Q - c_{Q'}\| + r')^\ell\omega(r),$$

i.e., (10.200) implies the required inequality (10.188) (for $c_Q \ne c_{Q'}$).

To prove (10.200) we apply assumption C of Proposition 10.110 to $V_{\vec f}(\widetilde{m})$ with the corresponding $\widetilde{m}$. To this end we set

$$\widetilde{Q} := Q_{|m|}(c_Q) \quad \text{and} \quad \widetilde{m} := m[\widetilde{Q}].$$

According to (10.176) $m[\widetilde{Q}] := (c_Q, \widetilde{y}) \in \mathcal{M}_S$, where $\widetilde{y}$ is a fixed point of S satisfying

$$\frac{1}{4}|m| \le \|c_Q - \widetilde{y}\| \le 4|m|.$$

Since the cube $\widetilde{Q}$ so defined belongs to $\mathcal{K}^I$, we have $p_{\widetilde{Q}} =: V_{\vec f}(m[\widetilde{Q}]) := V_{\vec f}(\widetilde{m})$. Then we write for $|\beta| = \ell + 1$,

$$|D^\beta(V_{\vec f}(m) - p_Q)| \le |D^\beta(V_{\vec f}(m) - V_{\vec f}(\widetilde{m}))| + |D^\beta(p_{\widetilde{Q}} - p_Q)|$$

and estimate each term on the right-hand side.

Assumption C of Proposition 10.110 implies for the first term

$$\begin{aligned} |D^\beta(V_{\vec f}(m) - V_{\vec f}(\widetilde{m}))| &\le O(1)\int_{\min(|m|,|\widetilde{m}|)}^{2(|m|+|m'|)} \frac{\omega(t)}{t^2}\,dt \\ &\le \int_{\frac{1}{2}|m|}^{10|m|} \frac{\omega(t)}{t^2}\,dt \le O(1)\frac{\omega(|m|)}{|m|} \le O(1)|m|^{-1}\omega(r). \end{aligned}$$

Moreover, since $\widetilde{Q} \subset Q$, inequality (10.188) gives for the second term

$$|D^\beta(p_{\widetilde{Q}} - p_Q)| \le O(1)\frac{\omega(r)}{r_{\widetilde{Q}}} = O(1)|m|^{-1}\omega(r).$$

Together with the previous this proves (10.200).

Finally, let Q' and Q have a common center, say c. Then derivatives of $p_{Q'}$ and p_Q up to order ℓ coincide at c and therefore for all $x \in \mathbb{R}^n$,

$$(p_Q - p_{Q'})(x) = \sum_{|\alpha|=\ell+1} D^\alpha(p_Q - p_{Q'})\frac{(x-c)^\alpha}{\alpha!}.$$

This and inequality (10.186) immediately imply

$$\sup_{Q'} |p_Q - p_{Q'}| \le (r')^{\ell+1} O(1) \frac{\omega(r)}{r'} \le O(1)(r')^{\ell}\omega(r).$$

Hence, the required inequality (10.195) holds in this case as well.

Proof of Proposition 10.110 has been completed. □

Proof of the Geometric Lemma. (a) Let $Q \in \mathcal{K}^{II}$, i.e.,

$$\left(\frac{1}{4}Q\right) \cap S = (4Q) \cap S. \tag{10.201}$$

We should prove that the K-cubes associated to Q belong to $\mathcal{K}^I := \mathcal{K}_S \setminus \mathcal{K}^{II}$. By definition, see (10.180), $K[\widehat{Q}] := Q_r(x_{\widehat{Q}})$, where $x_{\widehat{Q}} \in S \cap Q$, $r := \operatorname{diam}(Q \cap S)$ and $\|x_{\widehat{Q}} - y_{\widehat{Q}}\| = \operatorname{diam}(Q \cap S)$ for some point $y_{\widehat{Q}} \in Q \cap S$. Hence, $y_{\widehat{Q}}$ belongs to the boundary $\partial K[\widehat{Q}]$ of $K[\widehat{Q}]$ and therefore this cube cannot satisfy condition (10.201); hence, $K[\widehat{Q}] \subset \mathcal{K}^I$.

Further, $\widetilde{K}[\widehat{Q}]$ is the smallest cube centered at $x_{\widehat{Q}}$ which contains $K[\widehat{Q}]$ and at least one more point of $S \setminus Q$. Hence, $\partial \widetilde{K}[\widehat{Q}]$ contains a point of S and therefore $\widetilde{K}[\widehat{Q}] \in \mathcal{K}^I$.

(b) Let $Q' \in \widehat{Q}$, i.e., $Q' \cap S = Q \cap S$, and, moreover, $Q \in \mathcal{K}^{II}$. We should prove that

$$K[\widehat{Q}] \subset Q' \subset \frac{1}{2}\widetilde{K}[\widehat{Q}]. \tag{10.202}$$

To this aim we first show that

$$K[\widehat{Q}] \cap S = Q' \cap S. \tag{10.203}$$

By (10.201) the radius r of $K[\widehat{Q}]$ satisfies

$$r := \operatorname{diam}(Q \cap S) = \operatorname{diam}(Q' \cap S) = \operatorname{diam}\left((\frac{1}{4}Q') \cap S \right) \le \frac{1}{2}r' \left(:= \frac{1}{2} r_{Q'} \right).$$

This estimates the distance between the centers of Q' and $K[\widehat{Q}]$ as follows:

$$\|x_{\widehat{Q}} - c_{Q'}\| \le \operatorname{diam}(Q \cap S) = \operatorname{diam}(Q' \cap S) \le \frac{1}{2}r'. \tag{10.204}$$

In particular, these two inequalities immediately imply the left embedding in the formula; in particular, $K[\widehat{Q}] \cap S \subset Q' \cap S$. On the other hand, by definition $K[\widehat{Q}] \supset Q \cap S$ and the latter equals $Q' \cap S$. Equality (10.203) now follows.

It remains to prove the right embedding in (10.202). Since

$$\widetilde{K}[\widehat{Q}] := Q_{\widetilde{r}}(x_{\widehat{Q}}), \quad \text{where} \quad \widetilde{r} := \inf_{x\in S\setminus Q} \|x_{\widehat{Q}} - x\| := d(x_{\widehat{Q}}, S \setminus Q),$$

and, by the assumptions on Q' and Q,

$$S \setminus Q = S \setminus Q' = S \setminus (4Q'),$$

we conclude that

$$\widetilde{r} \geq d(c_{Q'}, S \setminus (4Q')) - \|c_{Q'} - x_{\widehat{Q}}\| \geq 4r' - \frac{1}{2}r' = \frac{7}{2}r'.$$

This and (10.204) imply

$$\|x_{\widehat{Q}} - c_{Q'}\| \leq \frac{1}{2}r' \leq \frac{1}{7}\widetilde{r}.$$

Subsequently,

$$Q' \subset \left(\frac{1}{7} + \frac{r'}{\widetilde{r}}\right) \widetilde{K}[\widehat{Q}] \subset \left(\frac{1}{7} + \frac{2}{7}\right) \widetilde{K}[\widehat{Q}] \subset \frac{1}{2}\widetilde{K}[\widehat{Q}],$$

as required.

(c) In this case, $Q' \subset Q$ and $Q' \subset \widehat{Q}$. We must prove that

$$\widetilde{K}[\widehat{Q}'] \subset 6K[Q] \quad \text{or} \quad K[\widehat{Q}] \subset 5K[\widehat{Q}'], \tag{10.205}$$

and the latter case holds only if $Q' \in \mathcal{K}^{II}$ and $Q \in \mathcal{K}^{I}$.

In the subsequent derivation we set for brevity

$$\widetilde{K}' := \widetilde{K}[\widehat{Q}'], \qquad K := K[\widehat{Q}].$$

Let first Q' and Q belong to $\mathcal{K}^{II}$. It was proved in part (b) that in this case

$$Q' \subset \frac{1}{2}K' \text{ and } K \subset Q. \tag{10.206}$$

Further, the assumptions on Q' and Q and (10.203) imply that

$$Q' \cap S \subset Q \cap S = K \cap S. \tag{10.207}$$

In particular, the center $x_{\widehat{Q}'}$ of $\widetilde{K}'$ belongs to $Q' \cap S\,(\subset K)$ and therefore

$$K \subset Q_{2r_K}(x_{\widehat{Q}'}) \subset 3K.$$

On the other hand,

$$Q' \cap S \subset K \cap S \subset Q_{2r_K}(x_{\widehat{Q}}) \cap S$$

and, subsequently,

$$(\widetilde{K}[\widehat{Q}'] =:) \, K' \subset Q_{2r_K}(x_{\widehat{Q}}) \subset 3K \, (:= K[\widehat{Q}]).$$

This proves the first embedding in (10.205) for $Q, Q' \in \mathcal{K}^{II}$.

Now let us show that this embedding is also true for the case of

$$Q' \in \mathcal{K}^I \text{ and } Q \in \mathcal{K}^{II}.$$

Since in this case the cube $\mathcal{K}' := K[\widehat{Q}']$ conventionally coincides with Q', we must prove the first embedding in (10.205) with Q' in the left-hand side. To this end we will prove for $r' := r_{Q'}$ that

$$r' \leq 4 \operatorname{diam}(Q' \cap S). \tag{10.208}$$

If this were done we would estimate the distance from any point $y \in Q'$ to the center of Q by

$$\|y - x_{\widehat{Q}}\| \leq \|y - c_{Q'}\| + \|c_{Q'} - x_{\widehat{Q}}\| \leq r' + \operatorname{diam}(Q' \cap S) \leq 5 \operatorname{diam}(Q' \cap S).$$

But $\operatorname{diam}(Q' \cap S) \leq \operatorname{diam}(Q \cap S) := r_K$ and therefore $y \in 5K$, i.e.,

$$Q' \subset 5K$$

as required (with the better constant).

It remains to prove (10.208). Due to the condition $Q \in \mathcal{K}^{II}$ equality (10.201) holds and therefore

$$(4Q') \cap S \subset (4Q) \cap S = \left(\frac{1}{4}Q\right) \cap S.$$

On the other hand, $Q' \in \mathcal{K}^I$ and therefore there exists a point z such that

$$z \in \left[(4Q') \setminus \left(\frac{1}{4}Q'\right)\right] \cap S.$$

These two relations imply that

$$z \in Q \cap S \text{ and } z \notin \frac{1}{4}Q'.$$

Since the center $c_{Q'} \in Q' \cap S$, the last relation yields the desired inequality

$$r' \leq 4\|c_{Q'} - z\| \leq 4 \operatorname{diam}(Q' \cap S).$$

Finally, let

$$Q' \in \mathcal{K}^{II} \text{ and } Q \in \mathcal{K}^I.$$

We show that now one of the embeddings in (10.205) holds. As above, from $Q \in \mathcal{K}^I$ we derive existence of a point $z \in [(4Q) \setminus (\frac{1}{4}Q)] \cap S$. Assume, first, that either z or c_Q does not belong to $Q' \cap S$. Then by the definition of $\widetilde{K}' := \widetilde{K}[\widehat{Q}'] := Q_{\widetilde{r}}(x_{\widehat{Q}'})$, see (10.181), its radius satisfies

$$\widetilde{r} \le \min\{\|x_{\widehat{Q}'} - z\|, \|x_{\widehat{Q}'} - c_Q\|\}.$$

Since $x_{\widehat{Q}'} \in Q' \cap S \subset Q$ and $z \in 4Q$, the right-hand side is at most $4r$ $(:= 4r_K)$. Subsequently, $\widetilde{K}' \subset 5Q$ as required.

Now let both points z and c_Q belong to $Q' \cap S$. From here and the choice of z we get, for the radius $\rho := \mathrm{diam}(Q' \cap S)$ of the cube $K[Q']$, the inequality

$$\rho \ge \|c_{Q'} - z\| \ge \frac{1}{4}r.$$

On the other hand,

$$K[Q'] \supset Q' \cap S \ni c_Q$$

which together with the estimate of ρ yields the desired embedding

$$Q \subset 5K[Q'].$$

This proves the Geometric Lemma. □

Theorem 10.108 has been now completely proved. □

Remark 10.120. We briefly discuss the finiteness result for the case of the normed space $X := J^\ell\Lambda^{2,\omega}(\mathbb{R}^n)$; its norm is recalled to be defined by

$$\|\vec{f}\|_X := \max_{|\alpha|\le\ell} \sup_{\mathbb{R}^n} |f_\alpha| + |\vec{f}|_X.$$

The analog of Lemma 10.109 characterizing elements $\vec{f} \in \dot{X}|_S$ includes now the additional condition

(iii) $\max_{|\alpha|\le\ell} \sup_S |f_\alpha| \le c$

and the equivalence $\|\vec{f}\|_{X|_S} \approx \inf c$ with constants depending only on n and ℓ, see Theorem 9.8 and Remark 9.16.

These should be clearly included in the other analogs of the homogeneous results. In particular, the analog of the basic fact, Proposition 10.111, now looks as follows.

Proposition 10.121. *An ℓ-jet $\vec{f}$ defined on a subset $S \subset \mathbb{R}^n$ belongs to $X|_S$ if and only if:*

(i)′ *The restriction of $\vec{f}$ to every two-pointed subset of S extends to an ℓ-jet from X of norm at most c.*

(ii)′ *The map* $L_{\vec{f}}$ *given by* (10.48) *has a selection with the Lipschitz constant at most* c.

Moreover, $\inf c \approx \inf \|\vec{f}\|_{X|_S}$ *with constants of equivalence depending only on* n *and* ℓ.

Following then the argument of the final part of Theorem 10.108 we extend the map $L_{\vec{f}} : (\mathcal{M}_S, d_\omega) \to \mathrm{Aff}(\mathcal{H}_{\ell+1,n})$ to the graph $(\mathcal{M}_S^*, d_\omega^*)$, where we recall that $\mathcal{M}_S^* := \mathcal{M}_S \cup \{\infty\}$ and ∞ is joined by an edge with every vertex of $\mathcal{M}_S$. Moreover,

$$d_\omega^*(m, m') := \begin{cases} d_\omega(m, m'), & \text{if } m, m' \in \mathcal{M}_S, \\ 0, & \text{otherwise.} \end{cases}$$

It is easily seen that Proposition 10.121 remains true with $L_{\vec{f}}^*$ substituted for $L_{\vec{f}}$. From here the proof, line by line, follows that of Theorem 10.108 to conclude that

$$\mathcal{F}(J^\ell \Lambda^{2,\omega}(\mathbb{R}^n)) \le 3 \cdot 2^{d(n,\ell)}, \qquad \gamma(J^\ell \Lambda^{2,\omega}(\mathbb{R}^n)) \le c(n, \ell).$$

10.5.2 Linearity

The main result of this part is

Theorem 10.122. *For every* $S \subset \mathbb{R}^n$ *there exists a linear extension operator* $T_S : J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)|_S \to J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)$ *with the norm satisfying*

$$\|T_S\| \le c(n, \ell).$$

Proof. We derive the result from three basic facts. The first is a linearized version of Proposition 10.111 divided for convenience into the following two parts.

Proposition 10.123. *If an* ℓ*-jet* $\vec{f}$ *belongs to* $\dot{X}|_S$, *then the set-valued map* $L_{\vec{f}} : (\mathcal{M}_S, d_\omega) \to \mathrm{Aff}(\mathcal{H}_{\ell+1,n})$ *given for* $m := (x, y) \in \mathcal{M}_S$ *by*

$$L_{\vec{f}}(m) := \{h \in \mathcal{H}_{\ell+1,n}\,;\, (D^\alpha h)(y - x) = f_\alpha(y) - f_\alpha(x) \text{ for all } |\alpha| = \ell\}, \quad (10.209)$$

see (10.48), *has a selection with Lipschitz constant bounded by* $c(n, \ell)|f|_{\dot{X}|_S}$.

The result is the necessity part of Proposition 10.111 (as above $\dot{X} := J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)$).

Proposition 10.124. *Let a set-valued map* $\mathcal{L} : (\vec{f}, m) \to \mathrm{Aff}(\mathcal{H}_{\ell+1,n})$, *given for* $\vec{f} \in \dot{X}|_S$ *and* $m \in \mathcal{M}_S$ *by*

$$\mathcal{L}(\vec{f}; m) := L_{\vec{f}}(m), \quad (10.210)$$

satisfy the conditions:

(a) *For every* $\vec{f}$ *the map* $\mathcal{L}(\vec{f}; \cdot)$ *has a selection* $s(\vec{f})$ *with Lipschitz constant bounded by* $c(n, \ell)|f|_{\dot{X}|_S}$.

(b) *The function* $\vec{f} \mapsto s(\vec{f})$ *is linear.*

Then there exists a linear extension operator from $\dot{X}|_S$ *into* $\dot{X}$ *with norm bounded by* $c(n,\ell)$.

Proof. We repeat, line by line, the proof of the sufficiency part of Proposition 10.111 with only one change: instead of Lemma 10.109 (i.e., Theorem 9.8 with $k = 2$) we use Theorem 9.14 with this k. We formulate it as

Lemma 10.125. *For every* $\vec{f} \in \dot{X}|_S$ *let there exist a family of polynomials* $\{p_Q(\vec{f})\}_{Q\in\mathcal{K}_S} \subset \mathcal{P}_{\ell+1,n}$ *satisfying the conditions:*

(i) *For every* $\vec{f} \in \dot{X}|_S$, $Q \in \mathcal{K}_S$ *and* $|\alpha| \le \ell$,

$$(D^\alpha p_Q(\vec{f}))(c_Q) = f_\alpha(c_Q).$$

(ii) *There exists a constant* $\gamma > 0$ *such that for every pair of cubes* $Q' \subset Q$ *in* $\mathcal{K}_S$ *and every* $\vec{f} \in \dot{X}|_S$,

$$\sup_{Q'} |p_Q(\vec{f}) - p_{Q'}(\vec{f})| \le \gamma(\|c_Q - c_{Q'}\| + r_{Q'})^\ell \omega(r_Q).$$

(iii) $\vec{f} \mapsto p_Q(\vec{f})$ *is linear for every* Q.

Then there exists a linear extension operator $T_S : \dot{X}|_S \to \dot{X}$ *of norm equivalent to* $\inf\gamma$ *where the constants of equivalence depend only on* n *and* ℓ.

Now we construct the family of polynomials $\{p_Q(\vec{f})\}_{Q\in\mathcal{K}_S}$ of this lemma using the map $V_{\vec{f}} : \mathcal{M}_S \to \mathcal{H}_{\ell+1,n}$ similar to that in (10.49). Specifically, for $m = (x,y) \in \mathcal{M}_S$ and $z \in \mathbb{R}^n$ we set as there

$$(V_{\vec{f}}(m))(z) := s(\vec{f};m) + \sum_{|\alpha|\le\ell} \frac{f_\alpha(z)}{\alpha!}(z-x)^\alpha,$$

where $s(\vec{f};\cdot)$ is the Lipschitz selection of the map $\mathcal{L}(\vec{f};\cdot)$. Since $s(\vec{f};\cdot)$ is linear in $\vec{f}$, the map $V_{\vec{f}}$ also is. Therefore the family of polynomials constructed via $V_{\vec{f}}$ by formulas (10.177) and (10.183)–(10.185) of the sufficiency part of Proposition 10.111 is also linear in $\vec{f}$. Moreover, it was proved there that the family so-defined satisfies conditions (i), (ii) of Lemma 10.109 (the same as those in Lemma 10.125). Hence the assumptions of this lemma are fulfilled and the required linear operator T_S exists. □

Now we should show that the set-valued map $\mathcal{L}$ of Proposition 10.124 possesses the required properties. To this end we exploit Theorem 5.66 of Volume I on simultaneous Lipschitz selections of set-valued maps from a metric space into the set of linear subspaces of fixed dimension. We deal with a special case concerning the metric graph $(\mathcal{M}_S, d_\omega)$ and a set-valued map F on $\mathcal{M}_S$ given for $m := (x, y)$ by

$$F(m) := \{h \in \mathcal{H}_{\ell+1,n}\,;\,(D^\alpha h)(x-y) = 0 \text{ for all } |\alpha| = \ell\}. \tag{10.211}$$

Due to Corollary 10.115,

$$\dim F(m) = d(n,\ell) := \binom{n+\ell-1}{\ell+1}.$$

The associated space $\Sigma_F(\mathcal{M}_S)$ consists of all functions $g : \mathcal{M}_S \to \mathcal{H}_{\ell+1,n}$ such that $g + F$ has a Lipschitz selection. This is equipped with the seminorm

$$|g|_{\Sigma_F(\mathcal{M}_S)} := \inf\{L(h)\,;\, h \text{ is a Lipschitz selection of } g + F\}.$$

Theorem 5.66 of Volume I gives in these settings

Proposition 10.126. *Assume that the linear Lipschitz extension constant (see Definition 7.2) $\lambda(\mathcal{M}_S, d_\omega)$ is finite. Then there exists a linear operator $\mathcal{R}_S : \Sigma_F(\mathcal{M}_S) \to \mathrm{Lip}(\mathcal{M}, \mathcal{H}_{\ell+1,n})$ with the norm satisfying*

$$\|\mathcal{R}_S\| \le c(\ell,n)\lambda(\mathcal{M}_S, d_\omega)^{2d(n,\ell)} \tag{10.212}$$

such that $\mathcal{R}_S g$ is a selection of $g + F$ for every $g \in \Sigma_F(\mathcal{M}_S)$.

We prove later that $\lambda(\mathcal{M}_S, d_\omega) \le c(n)$, while now we derive Theorem 10.122 from this fact and Proposition 10.126, i.e., find a linear extension operator, say $\mathcal{E}_S$, from the trace space $\dot{X}|_S$ into $\dot{X}$ (where $\dot{X} := J^\ell \dot{\Lambda}^{2,\omega}(\mathbb{R}^n)$) with the norm bounded by a constant depending only on ℓ and n.

To this end, given $\vec{f} \in \dot{X}|_S$ and $m := (x, y) \in \mathcal{M}_S$, we set

$$h_{\vec{f},m}(z) := \sum_{|\alpha|=\ell} \frac{f_\alpha(x) - f_\alpha(y)}{\alpha!} z^\alpha, \qquad z \in \mathbb{R}^n.$$

Using the linear operator $T_z : \mathcal{H}_{\ell,n} \to \mathcal{H}_{\ell+1,n}$ of Lemma 10.114 we define an operator $\mathcal{G} : \dot{X}|_S \times \mathcal{M}_S \to \mathcal{H}_{\ell+1,n}$ given for $m := (x, y)$ by

$$\mathcal{G}(\vec{f}, m) := T_{x-y}(h_{\vec{f},m}).$$

This is linear in $\vec{f}$ by definition and, due to Lemma 10.114, satisfies

$$D^\alpha[\mathcal{G}(\vec{f}, m)] := f_\alpha(x) - f_\alpha(y), \qquad |\alpha| = \ell.$$

From here, (10.209)–(10.211) and linearity of $\mathcal{G}$ in $\vec{f}$ we then have

$$\mathcal{G}(\vec{f}, m) \in L_{\vec{f}}(m) =: \mathcal{L}(\vec{f}, m) \text{ and } \mathcal{L}(\vec{f}, m) = F(m) + \mathcal{G}(\vec{f}, m).$$

We conclude from here that the seminorm of $\mathcal{G}(\vec{f}, m)$ in $\Sigma_F(\mathcal{M}_S)$ is bounded by $|s|_{\mathrm{Lip}(\mathcal{M}_S,\mathcal{H}_{\ell+1,n})}$, where s is an arbitrary Lipschitz selection of $L_{\vec{f}}$. Further, according to Proposition 10.123 the map $L_{\vec{f}}$ has a selection whose Lipschitz constant is bounded by $c(n,\ell)|\vec{f}|_{\dot{X}|_S}$. Hence,

$$|\mathcal{G}(\vec{f},\cdot)|_{\Sigma|_F(\mathcal{M}_S)} \le c(n,\ell)|\vec{f}|_{\dot{X}|_S}.$$

Further, we exploit Proposition 10.126 to find a linear operator $\mathcal{R}_S : \Sigma_F(\mathcal{M}_S) \to \mathrm{Lip}(\mathcal{M},\mathcal{H}_{\ell+1,n})$ of norm satisfying inequality (10.212) such that $\mathcal{R}_S g$ is a selection of $g+F$ for every g from the domain. Then composition $\mathcal{R}_S \circ \mathcal{G}(\vec{f},\cdot)$ is a selection of $\mathcal{L}(\vec{f},\cdot) := L_{\vec{f}}$, is linear in $\vec{f}$ and satisfies

$$|\mathcal{R}_S \circ \mathcal{G}(\vec{f},\cdot)|_{\mathrm{Lip}(\mathcal{M}_S,\mathcal{H}_{\ell+1,n})} \le c(n,\ell,\lambda(\mathcal{M}_S,d_\omega))|f|_{\dot{X}|_S}.$$

Hence, the map $\mathcal{L}$ meets conditions (a), (b) of Proposition 10.124; according to this proposition there exists the desired linear extension operator from $\dot{X}|_S$ into $\dot{X}$ whose norm is bounded by a constant depending only on ℓ, n and $\lambda(\mathcal{M}_S, d_\omega)$.

Hence, it remains to prove that

$$\lambda(\mathcal{M}_S, d_\omega) \le c(n,\ell). \tag{10.213}$$

To accomplish this we will embed $(\mathcal{M}_S, d_\omega)$ bi-Lipschitz homeomorphically into a metric space $(\mathcal{K}_n, d_w)$ having the required extension property (which we will introduce below). This would imply the estimate

$$\lambda(\mathcal{M}_S, d_\omega) \le D^2\lambda(\mathcal{K}_n, d_w),$$

where D stands for distortion of the embedding. If D and $\lambda(\mathcal{K}_n, d_w)$ were bounded by constants depending only on n, then the required estimate of $\lambda(\mathcal{M}_S, d_\omega)$ would follow.

The underlying set of the metric space $(\mathcal{K}_n, d_w)$ consists of all nontrivial closed cubes of $\mathbb{R}^n$. The set $\mathcal{K}_n$ is regarded as the vertex set of a graph whose edges are pairs of embedded cubes; the edge determined by a pair Q', Q is denoted by $Q' \leftrightarrow Q$. The graph is equipped with a weight w given at an edge $Q' \leftrightarrow Q$ by

$$w(Q' \leftrightarrow Q) := \frac{\omega(r_{Q'})}{r_{Q'}} + \frac{\omega(r_Q)}{r_Q} + \int_m^{m+d(Q',Q)} \frac{\omega(t)}{t^2}\,dt, \tag{10.214}$$

where $m := \min\{r_{Q'}, r_Q\}$ and d is a metric on $\mathcal{K}_n$ defined at a pair $Q' := Q_{r'}(c')$, $Q'' := Q_{r''}(c'')$ by

$$d(Q',Q'') := |r' - r''| + \|c' - c''\|.$$

The geodesic metric generated by this weight is the required metric d_w on $\mathcal{K}_n$. Hence, see Volume I, subsection 3.3.6, at Q', Q'',

$$d_w(Q',Q'') := \inf \sum_{i=1}^{j} w(Q_i \leftrightarrow Q_{i+1}),$$

where the infimum is taken over all paths $Q' =: Q_1 \leftrightarrow Q_2 \leftrightarrow \cdots \leftrightarrow Q_{j+1} := Q''$ and all j.

Now, let S be a subset of $\mathbb{R}^n$ containing more than one point and

$$\mathcal{K}_S := \{\bar{Q}_r(c)\,;\, c \in S,\, 0 < r \le 2\operatorname{diam} S\} \subset \mathcal{K}_n.$$

In the same fashion as for $\mathcal{K}_n$, we define a graph structure on $\mathcal{K}_S$. Since $\mathcal{K}_S$ can be regarded as the space of parameterized closed balls of the metric subspace $S \subset \ell_\infty^n$, the graph associated to $\mathcal{K}_S$ is connected, see Volume I, subsection 3.3.6. Therefore the weight (10.214) determines a *geodesic metric* on $\mathcal{K}_S$ denoted by d_w^S.

The space $(\mathcal{K}_S, d_w^S)$ is not a metric subspace of $(\mathcal{K}_n, d_n)$ but, as we will see, admits a bi-Lipschitz embedding into the latter with distortion bounded by a numerical constant. In turn, it will be proved that the space $(\mathcal{M}_S, d_\omega)$ admits a similar embedding into $(\mathcal{K}_S, d_w^S)$. These two facts will appear within the proof of the next

Proposition 10.127. *The space $(\mathcal{M}_S, d_\omega)$ admits a bi-Lipschitz embedding into $(\mathcal{K}_n, d_w)$ with distortion bounded by a numerical constant.*

Proof. We use an auxiliary geodesic metric $d_{\widehat{\omega}}^S$ on $\mathcal{K}_S$ determined by a weight $\widehat{\omega}$ given at an edge $Q' \leftrightarrow Q$ by

$$\widehat{\omega}(Q' \leftrightarrow Q) := \frac{\omega(\max\{r_{Q'}, r_Q\})}{\min\{r_{Q'}, r_Q\}}.$$

In the formulation of the result, we also use a function $\widetilde{\omega} : \mathcal{K}_n \times \mathcal{K}_n \to \mathbb{R}_+$ given for $Q', Q \in \mathcal{K}_n$ by

$$\widetilde{\omega}(Q', Q) := \int_m^M \frac{\omega(t)}{t^2}\, dt, \tag{10.215}$$

where $m := \min\{r_{Q'}, r_Q\}$, $M := r_{Q'} + r_Q + \|c_Q - c_{Q'}\|$.

Lemma 10.128. *Uniformly in $Q', Q \in \mathcal{K}_S$,*

$$d_{\widehat{\omega}}^S(Q', Q) \approx \widetilde{\omega}(Q', Q). \tag{10.216}$$

Moreover, uniformly in $Q' \leftrightarrow Q$, where $Q', Q \in \mathcal{K}_n$,

$$w(Q' \leftrightarrow Q) \approx \widetilde{\omega}(Q', Q). \tag{10.217}$$

Both relations hold with some numerical constants of equivalence.

The result is a special case of Proposition 3.127 of Volume I.

Corollary 10.129. *The canonical embedding*

$$(\mathcal{K}_S, d_w^S) \hookrightarrow (\mathcal{K}_n, d_n)$$

is a bi-Lipschitz homeomorphism onto the image with distortion bounded by a numerical constant.

Proof. Since $\widetilde{\omega}$ is independent of S, equivalence (10.216) gives

$$d^S_{\widehat{\omega}} \approx \widetilde{\omega} \approx d^{\mathbb{R}^n}_{\widehat{\omega}}|_{\mathcal{K}_S\times\mathcal{K}_S},$$

with numerical constants of equivalence (here and below all constants of equivalence are numerical).

Further, for every edge $Q' \leftrightarrow Q$ of the graph $\mathcal{K}_S$ we derive from (10.217)

$$w(Q' \leftrightarrow Q) \approx \widetilde{\omega}(Q', Q) \approx d^{\mathbb{R}^n}_{\widehat{\omega}}(Q', Q) := \widehat{\omega}(Q' \leftrightarrow Q).$$

Since the extreme terms of this chain are weights determining the corresponding length metrics, we obtain from here $d^S_w \approx d^{\mathbb{R}^n}_{\widehat{\omega}}|_{\mathcal{K}_S\times\mathcal{K}_S}$.

Applying the same argument to edges of $\mathcal{K}_n$ we then have $d_w \approx d^{\mathbb{R}^n}_{\widehat{\omega}}$.

Combining the results so obtained we finally get the required equivalence

$$d^S_w \approx d_n|_{\mathcal{K}_n\times\mathcal{K}_n}.$$

□

It remains to prove

Lemma 10.130. *The space $(\mathcal{M}_S, d_\omega)$ embeds bi-Lipschitz homeomorphically into $(\mathcal{K}_S, d^S_w)$ with distortion bounded by a numerical constant.*

Proof. The required embedding is given at $m := (x, y) \in \mathcal{M}_S$ by

$$\mathcal{I} : m \mapsto \bar{Q}_{|m|}(x).$$

Since $|m| := \|x - y\|$ and $\mathcal{M}_S$ consists of pairs $x, y \in S$ with $x \neq y$, the cube $\mathcal{I}(m)$ belongs to $\mathcal{K}_S$.

Now we show that for every pair $m', m \in \mathcal{M}_S$ and some numerical constant $c > 0$,

$$d_\omega(m', m) \geq c d^S_w(\mathcal{I}(m'), \mathcal{I}(m)). \tag{10.218}$$

To this end we first recall, see (10.155), that d_ω is generated by the weight ψ_ω given at an edge $m' \leftrightarrow m$ by

$$\psi_\omega(m' \leftrightarrow m) := \int_{\min\{|m'|,|m|\}}^{2(|m|+|m'|)} \frac{\omega(t)}{t^2}\, dt. \tag{10.219}$$

Since an edge $m' := (x', y') \leftrightarrow m := (x, y)$ is defined by the condition $\{x', y'\} \cap \{x, y\} \neq \emptyset$, the centers of cubes $\mathcal{I}(m')$, $\mathcal{I}(m)$ satisfy the condition

$$\|x - x'\| \leq |m'| + |m|.$$

Replacing the upper bound in the integral for ψ_ω by $|m| + |m'| + \|x - x'\|$ we turn it into the expression for $\widetilde{\omega}(\mathcal{I}(m'), \mathcal{I}(m))$, see (10.215). Together with (10.216) this yields for some numerical constant $c > 0$,

$$\psi_\omega(m' \leftrightarrow m) \geq \widetilde{\omega}(\mathcal{I}(m'), \mathcal{I}(m)) \geq c d^S_{\widehat{\omega}}(\mathcal{I}(m'), \mathcal{I}(m)).$$

Now, let $m', m \in \mathcal{M}_S$ and $m' =: m_1 \leftrightarrow m_2 \leftrightarrow \cdots \leftrightarrow m_{j+1} := m$ be a path joining them. Then

$$\begin{aligned} d_\omega(m', m) &:= \inf_{\{m_i\}} \sum_{i=1}^{j} \psi_\omega(m_i \leftrightarrow m_{i+1}) \geq c \inf_{\{m_i\}} \sum_{i=1}^{j} d^S_{\widetilde{\omega}}(\mathcal{I}(m_i), \mathcal{I}(m_{i+1})) \\ &= c d^S_{\widetilde{\omega}}(\mathcal{I}(m'), \mathcal{I}(m)). \end{aligned}$$

Estimating the right-hand side from below by Corollary 10.129 we establish inequality (10.218).

To prove the converse we first establish for $m := (x, y)$, $m' := (x', y') \in \mathcal{M}_S$ the inequality

$$d_\omega(m', m) \leq c \left\{ \frac{\omega(|m'|)}{|m'|} + \frac{\omega(|m|)}{|m|} + \widetilde{\omega}(\mathcal{I}(m'), \mathcal{I}(m)) \right\}. \tag{10.220}$$

We assume for definiteness that $|m'| \leq |m|$ and consider first the case

$$\|x - x'\| \leq \|x' - y\|. \tag{10.221}$$

Setting $m'' := (x', y)$ we obtain the path $m' \leftrightarrow m'' \leftrightarrow m$; hence by definition

$$d_\omega(m', m) \leq \psi_\omega(m', m'') + \psi_\omega(m'', m).$$

Since by the choice of m'',

$$|m'| \leq |m| \leq 2|m''| \qquad \text{and} \qquad |m''| \leq \|x - x'\| + |m|,$$

we derive from here and (10.219) that

$$\psi_\omega(m', m'') + \psi_\omega(m'', m) \leq 2 \int_a^b \frac{\omega(t)}{t^2}\, dt,$$

where

$$a := \frac{1}{2}|m'| \qquad \text{and} \qquad b := |m'| + 2|m| + \|x - x'\|.$$

We present the integral here as the sum $I_1 + I_2 + I_3$ by decomposing the domain of integration into the following intervals:

$$\begin{aligned} &\Delta_1 := \left[\frac{|m'|}{2}, |m'| \right], \quad \Delta_2 := [\, |m'|, |m'| + |m| + \|x - x'\| \,], \\ &\Delta_3 := [\, |m'| + |m| + \|x - x'\|, |m'| + 2|m| + \|x - x'\| \,]. \end{aligned}$$

Then I_2 by definition equals $\widetilde{\omega}(\mathcal{I}(m'), \mathcal{I}(m))$. Further,

$$I_1 \leq \omega(|m'|) \int_{\Delta_1} \frac{dt}{t^2} \leq \frac{\omega(|m'|)}{|m'|}.$$

Moreover, since $t \mapsto \frac{\omega(t)}{t^2}$ is nonincreasing,

$$I_3 \le \int_{|m|}^{2|m|} \frac{\omega(t)}{t^2}\,dt \le \frac{\omega(|m|)}{|m|}.$$

Combining these inequalities we obtain the desired result (10.220).

It remains to note that

$$\frac{\omega(|m'|)}{|m'|}, \frac{\omega(|m|)}{|m|} \le 2\widetilde{\omega}(\mathcal{I}(m'), \mathcal{I}(m)),$$

i.e., (10.220) leads to the inequality

$$d_\omega(m', m) \le c\widetilde{\omega}(\mathcal{I}(m'), \mathcal{I}(m))$$

which together with Corollary 10.129 gives the desired converse result:

$$d_\omega(m', m) \le c\widetilde{\omega}(\mathcal{I}(m'), \mathcal{I}(m)) \le c_1 d_w^S(\mathcal{I}(m'), \mathcal{I}(m)).$$

Hence, the lemma has been proved under condition (10.221). If, however, the converse to (10.221) holds, i.e.,

$$\|x - x'\| > \|x' - y\|,$$

we set $m'' := (x', x)$ and easily check that the very same argument leads to the required equivalence for this case. □

Comparing the maps of Lemmas 10.128 and 10.130 we obtain the required bi-Lipschitz embedding $(\mathcal{M}_S, d_\omega)$ into $(\mathcal{K}_n, d_w)$ and complete the proof of Proposition 10.127. □

At the last stage we should estimate $\lambda(\mathcal{K}_n, d_w)$ by a constant depending only on n. This is given by

Proposition 10.131. *It is true that*

$$\lambda(\mathcal{K}_n, d_w) \le cn^{3/2},$$

where c is a numerical constant.

Proof. We will show that

$$d_w := D_0 + D_1, \tag{10.222}$$

where the metric D_0 is given for $Q' \ne Q$ by

$$D_0(Q', Q) := \frac{\omega(r_{Q'})}{r_{Q'}} + \frac{\omega(r_Q)}{r_Q}$$

and by zero for $Q' = Q$, while the metric D_1 is given at $(Q_i := Q_{r_i}(c^i))_{i=1,2}$ by

$$D_1(Q_1, Q_2) := \int_{\min\{r_i\}}^{\min\{r_i\}+|r_1-r_2|+\|c^1-c^2\|} \frac{\omega(t)}{t^2}\,dt. \tag{10.223}$$

Actually, if $Q' \leftrightarrow Q$, then

$$(D_0 + D_1)(Q', Q) = w(Q' \leftrightarrow Q)$$

and therefore the geodesic metric determined by $D_0 + D_1$ regarded as a weight coincides with d_w. But the former length metric equals $D_0 + D_1$ (by the triangle inequality) and therefore (10.222) follows.

Covering the graph $\mathcal{K}_S$ by stars (sets of vertices joined with a fixed one) and using the corresponding partition of unity it is relatively easy to show that $\lambda(\mathcal{K}_n, D_0)$ is bounded by a numerical constant. The same fact with a constant $c(n)$ will be established below for $\lambda(\mathcal{K}_n, D_1)$. However, we cannot straightforwardly derive from here the desired estimate for $\lambda(\mathcal{K}_n, D_0+D_1) = \lambda(\mathcal{K}_n, d_w)$. Fortunately, a general result of this kind, Proposition 5.68 of Volume I, had been already proved. We use a special case of this result with $\widetilde{\mathcal{M}} = \mathcal{M} = \mathcal{K}_n$, the map $\varphi : \widetilde{\mathcal{M}} \to \mathcal{M}$ to be the identity and the metric $\widetilde{d}$ on $\widetilde{\mathcal{M}}$ given by $\widetilde{d} := D_0 + D_1$.

In these settings, Proposition 5.68 of Volume I yields.

Lemma 10.132. *It is true that*

$$(\lambda(\mathcal{K}_n, d_w) :=)\, \lambda(\mathcal{K}_n, D_0 + D_1) \le c\lambda(\mathcal{K}_n, D_1),$$

where c is a numerical constant.

It remains to estimate the right-hand side. This will be done by

Lemma 10.133. *It is true that*

$$\lambda(\mathcal{K}_n, D_1) \le cn^{3/2},$$

where c is a numerical constant.

Proof. We first prove the result under the next assumption on the 2-majorant ω which later will be eliminated.

For every $r > 0$,

$$\int_r^\infty \frac{\omega(t)}{t^2}\,dt = \infty. \tag{10.224}$$

Let $\widehat{\omega} : t \mapsto \frac{\omega(t)}{t^2}$, $t > 0$. It was proved in Theorem 4.58 of Volume I that the space $(\mathcal{K}_n, D_1)$ is bi-Lipschitz homeomorphic to the generalized hyperbolic space $\mathcal{H}^n_{\widehat{\omega}}$ with distortion at most 3, see (4.98) there. The proof of this fact exploits the condition for $\widehat{\omega}$ which coincides with (10.224).

In turn, due to Corollary 7.38 with $N = 1$ we have

$$\lambda(\mathcal{H}^n_{\hat{\omega}}) \le cn^{3/2}$$

with some numerical constant c.

This yields the required estimate

$$\lambda(\mathcal{K}_n, D_1) \le 9cn^{3/2}.$$

It remains to discard the restriction (10.224). To this end we replace ω by a 2-majorant ω_ε, $\varepsilon > 0$, given for $t > 0$ by

$$\omega_\varepsilon(t) := \max\{\omega(t), \varepsilon t^2\}.$$

Clearly we assume that the nonincreasing function $t \mapsto \frac{\omega(t)}{t^2}$ tends to 0 as $t \to +\infty$, since otherwise (10.224) holds for ω.

The length metric on $\mathcal{K}_n$ obtained by replacing in (10.223) ω by ω_ε is denoted by D_1^ε. By definition $D_1 \le D_1^\varepsilon$ and ω_ε satisfies condition (10.224). Hence, for every subset $S \subset \mathcal{K}_n$ there exists a linear extension operator T_S^ε from $\mathrm{Lip}(S, D_1^\varepsilon|_{S\times S})$ into $\mathrm{Lip}(\mathcal{K}_n, D_1^\varepsilon)$ such that for every f,

$$|T_S^\varepsilon f|_{\mathrm{Lip}(\mathcal{K}_n, D_1^\varepsilon)} \le 9cn^{3/2}|f|_{\mathrm{Lip}(S, D_1^\varepsilon|_{S\times S})} \le 9cn^{3/2}|f|_{\mathrm{Lip}(S, D_1|_{S\times S})}.$$

We get from here for every pair $Q', Q \in \mathcal{K}_n$ and every $f \in \mathrm{Lip}(S, D_1|_{S\times S})$:

$$|(T_S^\varepsilon f)(Q) - (T_S^\varepsilon f)(Q')| \le \gamma_n(f) D_1^\varepsilon(Q', Q), \tag{10.225}$$

where for brevity we set

$$\gamma_n(f) := 9cn^{3/2}|f|_{\mathrm{Lip}(S, D_1|_{S\times S})}.$$

We would obtain from here the desired result by letting ε to 0 if $T_\varepsilon f$ were pointwise converging to some Tf where T is linear. To prove this claim we fix a dense countable subset of $\mathcal{K}_n$ denoted by $\mathcal{K}_n^o$. Let, further, S be a *finite* subset of $\mathcal{K}_n^o$. Then, for sufficiently small $\varepsilon > 0$, it is true that

$$\mathrm{Lip}(S, D_1^\varepsilon|_{S\times S}) = \mathrm{Lip}(S, D_1|_{S\times S}). \tag{10.226}$$

Actually, let t_ε be a solution of the equation

$$\frac{\omega(t)}{t^2} = \varepsilon.$$

Since $\frac{\omega(t)}{t^2}$ is monotone and tends to zero, t_ε exists and tends to $+\infty$ as $\varepsilon \to 0$. Now we choose $\varepsilon = \varepsilon(S)$ so that $t_\varepsilon = 4\,\mathrm{diam}\, S$. Then the upper bound of the integral for D_1^ε, see (10.223), is bounded by t_ε and therefore $\omega(t) \ge \varepsilon t^2$ for t in the domain of integration. Hence, $\omega_\varepsilon(t) = \omega(t)$ and $D_1^\varepsilon = D_1$, and therefore the operator T_S^ε acts from $\mathrm{Lip}(S, D_1|_{S\times S})$ into $\mathrm{Lip}(\mathcal{K}_n, D_1^\varepsilon)$.

Further, by N we denote dimension of $\mathrm{Lip}(S, D_1|_{S\times S})$ (equals $\mathrm{card}\, S$) and choose a basis, say $\{f_j\}_{1\le j\le N}$, of this space. Applying inequality (10.225) to f_j we then write for $Q', Q \in \mathcal{K}_n^o$,

$$|(T_S^\varepsilon f_j)(Q) - (T_S^\varepsilon f_j)(Q')| \le \gamma_n(f_j) D_1^\varepsilon(Q', Q), \quad 1 \le j \le N. \tag{10.227}$$

Without loss of generality we may and will assume that for every $f \in \mathrm{Lip}(S, D_1|_{S\times S})$ and $\varepsilon > 0$, and some fixed cube $Q_0 \in S$,

$$f(Q_0) = (T_\varepsilon f)(Q_0) = 0.$$

Then for each $Q \in \mathcal{K}_n^o$ and $1 \le j \le N$ the set $\{(T_\varepsilon f_j)(Q)\}_{0<\varepsilon\le\varepsilon(S)}$ is bounded. Therefore for some sequence $\varepsilon_n \to 0$ the limit of $\{(T_{\varepsilon_n} f_j)(Q)\}_{n\ge1}$ exists. Using countability of $\mathcal{K}_n^o$ we apply the Cantor diagonal procedure to find an infinite subsequence $\{\eta_n\} \subset \{\varepsilon_n\}$ such that for every $Q \in \mathcal{K}_n^o$ and every $1 \le j \le N$ the limit of $\{(T_{\eta_n} f_j)(Q)\}_{n\ge1}$ exists as $n \to \infty$. Denoting this limit by $(T_S f_j)(Q)$ we extend T_S to the space $\mathrm{Lip}(S, D_1|_{S\times S})$ by setting for $f = \sum_{j=1}^N \lambda_j f_j$ and $Q \in \mathcal{K}_n^o$

$$(T_S f)(Q) := \sum_{j=1}^N \lambda_j (T_S f_j)(Q).$$

Since T_S^ε is linear, we have, for $Q \in \mathcal{K}_n^o$ and $f, g \in \mathrm{Lip}(S, D_1|_{S\times S})$, by passing to limit

$$T_S(f+g)(Q) = (T_S f)(Q) + (T_S g)(Q),$$

i.e., T_S is a linear operator acting on $\mathrm{Lip}(S, D_1|_{S\times S})$.

Further, passing to limit in (10.225) we also have for $Q', Q \in \mathcal{K}_n^o$,

$$|(T_S f)(Q) - (T_S f)(Q')| \le \gamma(n)(f) D_1(Q', Q).$$

Hence $T_S f$ is uniformly continuous on the dense subset of $\mathcal{K}_n$ and therefore can be (uniquely) continuously extended to $\mathcal{K}_n$. Preserving for the extended operator the same symbol, we conclude that $T_S : \mathrm{Lip}(S, D_1|_{S\times S}) \to \mathrm{Lip}(\mathcal{K}_n, D_1)$ and its norm is bounded by $9cn^{3/2}$. In accordance with Definition 7.2, this implies for every finite subset $S \subset \mathcal{K}_n^o$ the inequality

$$\lambda(S, D_1|_{S\times S}) \le 9cn^{3/2}.$$

Further, by Corollary 7.13,

$$\lambda(\mathcal{K}_n, D_1) = \sup \lambda(S, D_1|_{S\times S}),$$

where S runs over all finite subsets of $\mathcal{K}_n^o$. Hence, we get the desired inequality

$$\lambda(\mathcal{K}_n, D_1) \le 9cn^{3/2}$$

also for 2-majorants ω that do not obey restriction (10.224).

This completes the proof of the lemma. □

Now subsequently applying Lemmas 10.132 and 10.133, we prove a similar inequality for $\lambda(\mathcal{K}_n, d_w)$ and therefore establish Proposition 10.131. □

Since the latter inequality gives the final part of the proof for Theorem 10.122, the theorem has been proved. □

Remark 10.134. A simultaneous extension result analogous to Theorem 10.122 holds also for the normed space $J^\ell \Lambda^{2,\omega}(\mathbb{R}^n)$. For its proof the metric graph $(\mathcal{M}_S, d_\omega)$ should be replaced by that of $(\mathcal{M}_S^*, d_\omega^*)$ introduced for the similar aim in Remark 10.120. Then the proof with small changes is similar to that for the homogeneous case.

Comments

The seminal 1934 Whitney papers opened and shaped the area and essentially influenced its further development, justifying our choice of Whitney's name for the problems studied in this chapter. The extraordinary difficulty of their solution, even for the classical case of $C_b^k(\mathbb{R}^n)$ spaces, was a compelling argument to their reformulation in more accessible related subproblems. Such a reformulation was proposed in the Yu. Brudnyi research program at the beginning of the 1980s. Its final goal was solutions to the finiteness, linearity and divided difference characteristic problems for the spaces $C^\ell \Lambda^{k,\omega}(\mathbb{R}^n)$. The basic tools for the research program are: the local approximation criteria, the modified Whitney extension method and the Lipschitz selection conjecture (see Chapters 9 and 5 of Volume I, respectively). A few known results of that time confirming validity of the program were Whitney's extension theorems [Wh-1934a], [Wh-1934b] for C^k spaces and Yu. Brudnyi's extension theorem [Br-1970b] for $C^{k,\omega}(\mathbb{R}^n)|_S$ with $\omega(t) := t^\lambda$, $0 < \lambda \le 1$, and n-dimensional Markov subsets of $\mathbb{R}^n$.

The first breakthrough was due to Shvartsman who proved the Lipschitz selection Theorem 5.56 of Volume I and the related sharp finiteness theorem for $\Lambda^{2,\omega}(\mathbb{R}^n)$. These outstanding results were excluded for a bureaucratic reason from his 1984 PhD thesis and published only in a highly succinct form in the 12 pages paper [Shv-1987]; this fact may well illustrate the unfriendly scientific environment in the last ten years of the former Soviet Union.

The next step was the proof of the linearity conjecture for $\Lambda^{2,\omega}(\mathbb{R}^n)$ due to Yu. Brudnyi and Shvartsman in 1984. By the same "environmental" reason the result was only announced at that time (in [BSh-1985]); the complete account was published later in [BSh-1999].

For the time being, the next advanced results along the line marked by the aforementioned research program are the finiteness and linearity theorems for $J^\ell \Lambda^{2,\omega}(\mathbb{R}^n)$ due to Yu. Brudnyi and Shvartsman [BSh-1998] (they are presented in Section 9.5). Further progress in this direction requires much more complicated selection results dealing with set-valued functions on a kind of "multimetric" spaces.

Meanwhile, Fefferman's recent series of papers solving the $C_b^\ell(\mathbb{R}^n)$ and $C^{\ell,\omega}(\mathbb{R}^n)$ finiteness and linearity problems open a new line of research with function theory methods essentially dominating those of Geometric Analysis. The interaction of these two trends, as we hope, will lead to new extension methods solving both old and new problems related to the area, e.g., for smoothness spaces over L_p, BMO and the likes.

Now we discuss several conjectures related to the (counter-) examples of Section 10.1. We begin with the Shvartsman sharp lower bound result for $\mathcal{F}(\Lambda^{2,\omega}(\mathbb{R}^n))$, see [Shv-1987], and its version for $C^{1,\omega}(\mathbb{R}^n)$ given by Theorem 10.16. These two results are a shaky ground for a plausible conjecture. Nevertheless, we risk claiming the inequality

$$\mathcal{F}(C^{2,\omega}(\mathbb{R}^n)) \geq 2\mathcal{F}(C^{1,\omega}(\mathbb{R}^n))$$

which, being true, might be the first step of induction.

The counterexample given by Theorem 10.32 leads to the next apparently very difficult

Question. Under what conditions on geometric structure of a subset $S \subset \mathbb{R}^n$ is there no simultaneous extension from $C_u^\ell(\mathbb{R}^n)|_S$ into $C_u^\ell(\mathbb{R}^n)$?

The counterexample suggests a possible direction of study related to the paratangent set of S. In Theorem 10.32, this set coincides with $w^{\lim}$ and condition (A) (for 0 to be a nonisolated point of $w^{\lim}$) is also sufficient.

Extending the notion of depth by using the set of linear functionals $L_x f := \sum_{|\alpha|\leq\ell} c_\alpha (D^\alpha f)(x)$, $x \in S$, Fefferman [F-2007b] proved the following result strikingly contrasted with his counterexample given by Theorem 10.39.

For every finite subset $S \subset \mathbb{R}^n$ there exists a linear extension operator $T \in Ext(S; C_b^\ell(\mathbb{R}^n))$ of (extended) depth and norm bounded by a constant depending only on n and ℓ.

It seems to be plausible that this nonlocal phenomenon holds for other smoothness spaces of $n > 1$ variables (in particular, for $C^{\ell,\omega}(\mathbb{R}^n)$).

The trace and extension theorems for Markov sets were proved by the authors of the book. Surprisingly, the local interpolation property established in Proposition 10.55, which is basic for Markov sets, has never appeared in this relatively vast area.

We conjecture that all results of Section 10.2 may be generalized to the spaces $C^\ell\Lambda^{k,\omega}(\mathbb{R}^n)$ with $\ell \geq 1$ (this extension, in particular, requires a $C^\ell(\mathbb{R}^n)$ analog of the Remez-Shnirelman finiteness Theorem 10.42 and the related analog of divided difference operator, Definition 10.64).

The sharp result for $\mathcal{F}(C^{\ell,\omega}(\mathbb{R}^n))$ given by Whitney for $\ell \geq 0, n = 1$ and by Theorem 10.69 for $\ell = 1$, $n \geq 1$ give no base for formulating reliable conjectures. We single out only one of them formulated by Shvartsman in his lecture at the

Second Workshop on Whitney Problems (August, 2009). It claims that

$$\mathcal{F}(C^{\ell,\omega}(\mathbb{R}^n)) = \prod_{j=0}^{\ell} (\ell - j + 2)^{\binom{n+j-2}{j}}.$$

The improvement of Fefferman's upper bound for $\mathcal{F}(C^{\ell,\omega}(\mathbb{R}^n))$ given in Corollary 10.79 was due independently and by different methods to Bierstone and Milman [BM-2007] and Shvartsman [Shv-2008]; the former authors proved the result for $C_b^\ell(\mathbb{R}^n)$ and noted that their method works for $C^{\ell,\omega}(\mathbb{R}^n)$. It is important to emphasize that the norm of an extension operator from $C^{\ell,\omega}(\mathbb{R}^n)|_S$ into $C^{\ell,\omega}(\mathbb{R}^n)$ in both proofs tends to infinity along with card S.

A seemingly natural conjecture due to Shvartsman [Shv-2009] claims that the finiteness constant $\mathcal{F}_\Phi(C^\ell\Lambda^{k,\omega}(\mathbb{R}^n))$, where Φ denotes the class of finite subsets in $\mathbb{R}^n$, see Definition 10.10, is bounded by $2^{\dim \mathcal{P}_{k+\ell-1,n}}$. The cited paper contains several new ideas and results that may lead to the proof of the conjecture.

The survey of the Ch. Fefferman results in Sections 10.3 and 10.4 is based on the papers cited there and his handwritten survey of the $C^{\ell,\omega}$ results kindly presented to the authors. Our formulation of the Fefferman theorems in some places slightly deviates from his to make further expose the set-valued selection results crucial for his approach.

The strong finiteness property of $C_b^\ell(\mathbb{R}^n)$ given in Theorem 10.102 is due to the authors.

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Index